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Functional identities on prime rings with involution

Abu Zaid Ansari[1], **Faiza Shujat**[2]

[1]Department of Mathematics
Faculty of Science
Islamic University of Madinah
Madinah, K.S.A.

[2]Department of Mathematics
Faculty of Science
Taibah University
Madinah, K.S.A.

email: ansari.abuzaid@gmail.com, ansari.abuzaid@iu.edu.sa,
faiza.shujat@gmail.com

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Abstract

In the present manuscript, our intention is to prove the following: Suppose that R is a prime ring with involution. If $d, G : R \to R$ are two additive mappings satisfying the identity $G(r^{2l}) = G(r^l)(r^\star)^l + r^l d(r^l)$, $\forall\ r \in R$, then d will be a X-inner and G is of the form $G(r) = qr^\star + d(r)$ for all $r \in R$ and $q \in Q_s(R)$.
These theorems have been validated by providing instances demonstrating that they are not inconsequential.

1 Introduction

In this work, an associative ring with an identity e is referred to as R. $Q_s(R)$ treated as symmetric Martindale quotients ring, the maximal symmetric ring

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of quotients as $Q_{ms}(R)$ of a ring R and $\mathcal{C}$ is indicated as extended centroid. A ring R is considered to be a l-torsion free, if $lr = 0$ entails $r = 0$ for all $r \in R$ and l is a positive integer. Note that R is termed as prime ring if $rRs = \{0\}$ yields that either $r = 0$ or $s = 0$ and known as semiprime ring if $rRr = \{0\}$ entails $r = 0$. A mapping $\star : R \to R$ is termed as an involution if it satisfies $(r + s)^\star = r^\star + s^\star$, $(rs)^\star = s^\star r^\star$ and $(s^\star)^\star = s$ for all $r, s \in \mathcal{A}$. A ring having an involution mapping is known as a ring with involution or $\star$-ring. A mapping $d : R \to R$ is known as a derivation if it is additive and $d(rs) = d(r)s + rd(s)$, $\forall\ r, s \in R$ and particularly, termed as a Jordan derivation if $d(r^2) = d(r)r + rd(r)$ is satisfied $\forall\ r \in R$. Every derivation is a Jordan derivation, as is readily apparent, however this does not have to be the case in all contexts. According to a classical conclusion of Herstein [4], every Jordan derivation with a characteristic other than 2 is derivation on a prime ring. Cusack [3] has generalized this theorem concerning to a semiprime ring. Further, a mapping $d : R \to R$ with additivity is known as a $\star$-derivation if $d(rs) = d(r)s^\star + rd(s)$ for all $r, s \in R$ and particularly, termed as a Jordan $\star$-derivation if $d(r^2) = d(r)r^\star + rd(r)$ $\forall\ r \in R$. It is easy to verify that if d is a $\star$-derivation, then d is recognised as X-inner and it is of the form $d(r) = ra - ar^\star$, for all $r \in R$ and $a \in Q_{ms}(R)$. Next, point out a general case of $\star$-derivation as follows: an additive mapping G from R to itself is termed as a generalized $\star$-derivation associated with d, a $\star$-derivation such that $G(rs) = G(r)s^\star + rd(s)$, $\forall\ r, s \in R$. Particularly, an additive mapping G from $R \to R$ is termed to be a generalized Jordan $\star$-derivation if $G(r^2) = G(r)r^\star + rd(r)$, $\forall\ r \in R$, where d is an associated Jordan $\star$-derivation. If G is a generalized Jordan $\star$-derivation on a semiprime ring, then for all $q \in Q_s$, G is of the form $G(r) = qr^\star + d(r)$ for all $r \in R$. It is simple to demonstrate that every generalized $\star$-derivation is indeed a generalized Jordan $\star$-derivation, however the converse is not truly the case. One can find more generalizations in [2, 6, 7] and the references there in. If G is a generalized Jordan $\star$-derivation associated with Jordan $\star$-derivation d on R, then it satisfied the algebraic identity $G(r^{2l}) = G(r^l)(r^\star)^l + r^l d(r^l)$ for all $r \in R$ but the converse does not hold generally. So, the question is, under what condition the converse of this statement is also true? This manuscript is dedicated to the study of converse. Specifically, with some conditions on R, G is a generalized Jordan $\star$-derivation having associated with a Jordan $\star$-derivation d if it satisfies the algebraic identity $G(r^{2l}) = G(r^l)(r^\star)^l + r^l d(r^l)$ for all $r \in R$.

Lemma 1.1 ([5, Lemma 2.1]). *If R is a ring with involution $*$, then $e^* = e$.*

2 Main results

Now, let us proceed with the following result:

Theorem 2.1. *Suppose that R is a $(2l-1)!$ torsion free prime ring ald l is a fixed positive integer. If $G, d : R \to R$ are two additive mappings satisfying the algebraic identity $G(r^{2l}) = G(r^l)(r^\star)^l + r^l d(r^l),\ \forall r \in R$, then G will be of the form $G(r) = qr^\star + d(r)$ for all $r \in R$ and $q \in Q_s(R)$, where d is a $\star$-derivation.*

Proof. Given that

$$G(r^{2l}) = G(r^l)(r^\star)^l + r^l d(r^l),\ \forall\ r \in R. \tag{2.1}$$

Replacing r by $r + ks$ in the above identity, where k is positive integer and using the fact that $d(e) = 0$, we obtain
$G\Big(r^{2l}+\binom{2l}{1}r^{2l-1}ks+\binom{2l}{2}r^{2l-2}k^2s^2+...+k^{2l}s^{2l}\Big) = G\Big(r^l+\binom{l}{1}r^{l-1}ks+\binom{l}{2}r^{l-2}k^2s^2+ ... + k^ls^l\Big)\Big((r^\star)^l + \binom{l}{1}ks^\star(r^\star)^{l-1} + \binom{l}{2}k^2(s^\star)^2(r^\star)^{l-2} + ... + k^l(s^\star)^l\Big)^\star + \Big(r^l + \binom{l}{1}r^{l-1}ks + \binom{l}{2}r^{l-2}k^2s^2 + ... + k^ls^l\Big)d\Big(r^l + \binom{l}{1}r^{l-1}ks + \binom{l}{2}r^{l-2}k^2s^2 + ... + k^ls^l\Big)$.
Rewrite the above expression using (2.1) as

$$kg_1(r, s) + k^2g_2(r, s) + ... + k^{2l-1}g_{2l-1}(r, s) = 0,$$

where the coefficients of k^j for all $j = 1, 2, ..., 2l-1$ are denoted by $g_j(r, s)$. A system of $(2l-1)$ homogeneous equations can be found if k is replaced by $1, 2, ..., 2l-1$ in tern. A Vander Monde matrix V_M of order $2l-1 \times 2l-1$ is provided. That is, all coefficients power of k vanishes. i.e., $g_i(r, s) = 0$ for all $i = 1, 2, .., 2l-1$. In particular, We obtain $g_1(r, s) = \binom{2l}{1}G(r^{2l-1}s) - \binom{l}{1}G(r^l)s^\star(r^\star)^{l-1} - \binom{l}{1}G(r^{l-1}s)(r^\star)^l - \binom{l}{1}r^l d(r^{l-1}s) - \binom{l}{1}r^{l-1}sd(r^l) = 0$. Put $r = e$ and applying Lemma 1.1 and $d(e) = 0$ to appear $2lG(s) = lG(e)s^\star + lG(s) + ld(s)$. Using torsion restriction on R, we find

$$G(s) = G(e)s^\star + d(s) \text{ for all } s \in R. \tag{2.2}$$

Next observe that $g_2(r, s) = \binom{2l}{2}G(r^{2l-2}s^2) - \binom{l}{2}G(r^l)(s^\star)^2(r^\star)^{l-2} - \binom{l}{1}\binom{l}{1}G(r^{l-1}s)s^\star(r^\star)^{l-1} - \binom{l}{2}G(r^{l-2})(s^\star)^2(r^\star)^l - \binom{l}{2}d(r^{l-2}s^2) - \binom{l}{1}\binom{l}{1}r^{l-1}sd(r^{l-1}s) - \binom{l}{2}r^{l-2}s^2d(r^2) = 0$. Rewrite the above expression by substituting e for r to obtain $\binom{2l}{2}G(s^2) = \binom{l}{2}G(e)(s^\star)^2 + \binom{l}{1}\binom{l}{1}G(s)s^\star + \binom{l}{2}G(s^2) + \binom{l}{2}d(s^2) +$

$\binom{l}{1}\binom{l}{1}sd(s)$. This implies that $2l(2l-1)G(s^2) = (l-1)lG(e)(s^\star)^2 + 2l^2G(s)s^\star + (l-1)lG(s^2) + (l-1)ld(s^2) + 2l^2sd(s)$. A simple manipulation give us

$$(3l^2 - l)G(s^2) = (l-1)lG(e)(s^\star)^2 + 2l^2G(s)s^\star + (l-1)ld(s^2) + 2l^2sd(s).$$

Making use of l -torsion freeness of R to get

$$(3l-1)G(s^2) = (l-1)G(e)(s^\star)^2 + 2lG(s)s^\star + (l-1)d(s^2) + 2lsd(s).$$

Applying (2.2) leads to in that

$$(3l-1)\Big[G(e)s^2 + d(s^2)\Big] = (l-1)G(e)(s^\star)^2 + 2l\Big[G(e)s^\star + d(s)\Big]s^\star + (l-1)d(s^2) + 2lsd(s).$$

On simplifying above expression, we obtain $2ld(s^2) = 2ld(s)s^\star + 2lsd(s)$. Due to the fact that R is $2l$-torsion free, we can formulate the final equation as follows $d(s^2) = d(s)s^\star + sd(s)$. That is Jordan $\star$-derivation on R. Then d is X-inner. Consider (2.2) again such that $G(s^2) = G(e)(s^\star)^2 + d(s^2) = \big[G(e)s^\star + d(s)\big]s^\star + sd(s) = G(s)s^\star + sd(s)$, $\forall s \in R$. As a conclusion, G is indeed a generalized Jordan $\star$-derivation on R. Utilizing the essential theorem from [1], we find the required conclusion. □

The above theorem has direct ramifications:

Corollary 2.2. *Suppose that R is a $(2l-1)!$ torsion free prime ring. If an additive mapping $G : R \to R$ satisfies $G(r^{2l}) = G(r^l)(r^\star)^l$, $\forall r \in R$, then there exists $q \in Q_s(R)$ such that $G(r) = qr^\star$ for all $r \in R$, where l is any positive fixed integer.*

Proof. Using $d = 0$ in the above theorem, we obtain the desired outcome.

Corollary 2.3. *Suppose that R is a $(2l-1)!$ torsion free prime ring. If an additive mapping $d : R \to R$ satisfies the identity $d(r^{2l}) = d(r^l)(r^\star)^l + r^ld(r^l)$, $\forall r \in R$, then d will be a Jordan $\star$-derivation, where l is any positive fixed integer.*

Proof. We get the required result by taking d as G and following the identical procedures that is used in the first main theorem. □

The following illustration supports our finding:

Examine a ring $R = \left\{ \begin{pmatrix} 2r_1 & 0 \\ 0 & 2r_2 \end{pmatrix} \mid r_1, r_2 \in \mathbb{Z}_8 \right\}$ with involution $I_v : R \to R$ by $I_v(r) = \begin{pmatrix} 2r_2 & 0 \\ 0 & 2r_1 \end{pmatrix}$, $\mathbb{Z}_8$ has its usual meaning. Define maps $d, G : R \to R$ by $G(r) = \begin{pmatrix} 0 & 0 \\ 0 & 2r_2 \end{pmatrix}$, $d(r) = \begin{pmatrix} 2r_1 & 0 \\ 0 & 0 \end{pmatrix}$ for all $r \in R$. It is evident that G is not generalized derivation in a generic way on R but G, d satisfy the algebraic conditions in main theorem, which shows that primeness and torsion restrictions on R are the essential conditions in the key result.

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