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On the k^{th} metallic ratio and the Diophantine equations

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Abstract

Metallic ratios define a set of related ratios that are among the most significant real numbers and are used, in this paper, to solve several Diophantine Equations. The method used to solve the supplied problems is simple and provides direction for solving countless other analogous equations.

1 Introduction

In this paper, we shall be mainly concerned with the integer solutions of the Diophantine equations

$$x^2 - 5kxy + (6k^2 - 1)y^2 = \pm 1 \tag{1.1}$$

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It should be noted that solving (1.1) in its current form is extremely difficult; that is, we cannot determine how many solutions (1.1) has and what they are. As a result, depending on the methods used in [1, 2], we must convert (1.1) into a suitable Diophantine equation that can be solved easily.

We present a pair of equations and provide their solutions using successive convergence of continued fraction expansion corresponding to the k^{th} metallic ratio. Then we transform the Diophantine equations (1.1) into the provided equation form.

2 The Diophantine equations $x^2-kxy-y^2=\pm 1$

The main goal of this section is to solve the equations

$$x^2 - kxy - y^2 = \pm 1, \tag{2.2}$$

where x and y are both positive integers.

The pair of Diophantine Equations (2.2) can be solved using a variety of methods. However, we will adopt the which is continued fraction expansion method.

Theorem 2.1. *The continued fraction expansion of the k^{th} metallic ratio is:* $\dfrac{\sqrt{k^2+4}+k}{2} = [k, \overline{k}]$.

Proof. Let $\dfrac{\sqrt{k^2+4}+k}{2}$ be the k^{th} metallic ratio, the largest root of the equation $x^2 - kx - 1 = 0$. In the following, we will derive the continued fraction expansion of this number. Recall that, the k^{th} metallic ratio is the reciprocal of the real number $\dfrac{\sqrt{k^2+4}-k}{2}$, which we can derive its continued fraction expansion easily.

Let

$$\frac{\sqrt{k^2+4}-k}{2} = \frac{1}{\frac{\sqrt{k^2+4}+k}{2}}$$

$$= \frac{1}{k+\frac{\sqrt{k^2+4}-k}{2}}$$

$$= \frac{1}{k+\frac{1}{k+\frac{\sqrt{k^2+4}-k}{2}}}$$

$$= \frac{1}{k+\frac{1}{k+\frac{1}{k+\frac{1}{k+\frac{1}{k+\cdots}}}}}$$

Then the desired continued fraction expansion of the k^{th} metallic ratio ratio is:

$$\frac{\sqrt{k^2+4}+k}{2} = 4+\frac{1}{k+\frac{1}{k+\frac{1}{k+\frac{1}{k+\frac{1}{k+\cdots}}}}} \tag{2.3}$$

$$= [k, \overline{k}].$$

□

Theorem 2.2. *Let* $\frac{p_{2n}}{q_{2n}}$ *and* $\frac{p_{2n+1}}{q_{2n+1}}$ *be the even and the odd convergence of the continued fraction expansion of the k^{th} metallic ratio ratio, respectively. Then*

$$p_{2n+1}^2 - kp_{2n+1}q_{2n+1} - q_{2n+1}^2 = 1 \quad \textit{and} \quad p_{2n}^2 - kp_{2n}q_{2n} - q_{2n}^2 = -1$$

Proof. It is clear that

$$\begin{aligned}\frac{p_{n+2}}{q_{n+2}} &= [\underbrace{k, k, \ldots, k}_{n\ times}, k, k] \\ &= [k, k, \underbrace{k, k, \ldots, k}_{n\ times}] \\ &= k + \cfrac{1}{k + \cfrac{1}{\cfrac{p_n}{q_n}}} \\ &= \frac{(k^2+1)p_n + kq_n}{kp_n + q_n}\end{aligned}.$$

Then

$$\begin{aligned}&p_{n+2}^2 - kp_{n+2}q_{n+2} - q_{n+2}^2 \\ &= ((k^2+1)p_n + kq_n)^2 - k((k^2+1)p_n + kq_n)(kp_n + q_n) \\ &-(kp_n + q_n)^2 \\ &= (k^2+1)^2p_n^2 + 2k(k^2+1)p_nq_n + k^2q_n^2 - k^2(k^2+1)p_n^2 \\ &-k^3p_nq_n - k(k^2+1)p_nq_n - k^2q_n^2 - k^2p_n^2 - 2kp_nq_n - q_n^2 \\ &= p_n^2 - kp_nq_n - q_n^2.\end{aligned}$$

Since $c_0 = \frac{p_0}{q_0} = \frac{k}{1}$ and $c_1 = [k, k] = \frac{p_1}{q_1} = \frac{k^2+1}{k}$, $p_0 = k,\ q_0 = 1,\ p_1 = k^2 + 1,\ q_1 = k$. Therefore,

$$p_0^2 - kp_0q_0 - q_0^2 = -1 \quad \text{and} \quad p_1^2 - kp_1q_1 - q_1^2 = 1.$$

We get

$$p_{2n+1}^2 - kp_{2n+1}q_{2n+1} - q_{2n+1}^2 = 1$$

and

$$p_{2n}^2 - kp_{2n}q_{2n} - q_{2n}^2 = -1.$$

Using the successive convergence of the continued fraction expansion of the k^{th} metallic ratio, we had determined all possible solutions in the positive integers for these two Diophantine equations described in (1.1).

In fact, let

$$\begin{aligned}
c_0 = &\ \frac{p_0}{q_0} = \frac{k}{1}, \\
c_1 = &\ [k,k] = \frac{p_1}{q_1} = k + \frac{1}{k} = \frac{k^2+1}{k}, \\
c_2 = &\ [k,k,k] = \frac{p_2}{q_2} = k + \cfrac{1}{k + \cfrac{1}{k}} = \frac{k^3+2k}{k^2+1}, \\
c_3 = &\ [k,k,k,k] = \frac{p_3}{q_3} = k + \cfrac{1}{k + \cfrac{1}{k + \cfrac{1}{k}}} = \frac{k^4+3k^2+1}{k^3+2k},
\end{aligned}$$

$$\begin{aligned}
c_4 = &\ [k,k,k,k,k] = \frac{p_4}{q_4} = \frac{k^5+4k^3+3k}{k^4+3k^2+1}, \\
c_5 = &\ [k,k,k,k,k,k] = \frac{p_5}{q_5} = \frac{k^6+5k^4+6k^2+1}{k^5+4k^3+3k}, \\
&\ \ldots etc.
\end{aligned}$$

We can see that $c_0,\ c_2,\ c_4,\ \ldots,\ c_{2n},\ \ldots$ provides solutions to

$$x^2 - kxy - y^2 = -1,$$

and $c_1,\ c_3,\ c_5,\ \ldots,\ c_{2n+1},\ \ldots$ provides solutions to

$$x^2 - kxy - y^2 = 1.$$

Moreover, by induction, we can prove the following result using induction:

Theorem 2.3. *For all* $n \geq 1$, *the above solutions for both sequences* $c_0,\ c_2,\ c_4,\ \ldots,\ c_{2n},\ \ldots$ *and* $c_1,\ c_3,\ c_5,\ \ldots,\ c_{2n+1},\ \ldots$, *fulfill the recurrence relations:*

$$\begin{aligned}
x_{n+2} &= (k^2+2)x_{n+1} - x_n \\
y_{n+2} &= (k^2+2)y_{n+1} - y_n.
\end{aligned}$$

3 The Diophantine equations $x^2 - 5kxy + (6k^2 - 1)y^2 = \pm 1$

The principal goal of this section is to solve the Diophantine equations

$$x^2 - 5kxy + (6k^2-1)y^2 = \pm 1 \tag{3.4}$$

Note that the Diophantine equations (3.4) can be written as

$$x^2 - 4kxy + 4k^2y^2 - kxy + 2k^2y^2 - y^2 = \pm 1$$

which give

$$(x - 2ky)^2 - k(x - 2ky)y - y^2 = \pm 1.$$

Let $z = x - 2ky$. Thus (3.4) becomes

$$z^2 - kzy - y^2 = \pm 1, \tag{3.5}$$

which takes the same form as equations (2.2).

As a result, the solutions of (3.4) are deduced from the solutions of (3.5) as follows:

Let (x, y) be solution of (3.5). Then $(x + 2ky, y)$ is a solution of (3.4).

So $(p_0 + 2kq_0, q_0)$ $(p_2 + 2kq_2, q_2)$, $(p_4 + 2kq_4, q_4)$, $\ldots$, $(p_{2n} + 2q_{2n}, q_{2n})$, $\ldots$ provides solutions to

$$x^2 - 5kxy + (6k^2 - 1)y^2 = -1$$

and (p_1+2kq_1, q_1) (p_3+2kq_3, q_3), (p_5+2kq_5, q_5), $\ldots$, $(p_{2n+1}+2kq_{2n+1}, q_{2n+1})$, $\ldots$ provides solutions to

$$x^2 - 5kxy + (6k^2 - 1)y^2 = 1,$$

where $\frac{p_0}{q_0}$, $\frac{p_1}{q_1}$, $\frac{p_2}{q_2}$, $\ldots$, $\frac{p_{2n}}{q_{2n}}$, $\frac{p_{2n+1}}{q_{2n+1}}$, $\ldots$ are the successive convergence of continued fraction expansion corresponding to the k^{th} metallic ratio.

Furthermore, we see that for $n \geq 1$ that the above solutions for both sequences fulfill the same recurrence relations:

$$\begin{aligned} x_{n+2} &= (k^2 + 2)x_{n+1} - x_n \\ y_{n+2} &= (k^2 + 2)y_{n+1} - y_n. \end{aligned}$$

4 The Diophantine equations $x^2 - (k + 4)xy + (2k + 3)y^2 = \pm 1$

Let us solve the Diophantine equations

$$x^2 - (k + 4)xy + (2k + 3)y^2 = \pm 1 \tag{4.6}$$

We note that the Diophantine equations (4.6) can be written as

$$x^2 - kxy + 4y^2 - kxy + 2ky^2 - y^2 = \pm 1$$

which gives

$$(x-2y)^2 - k(x-2y)y - y^2 = \pm 1$$

Let $z = x - 2y$. Thus (4.6) becomes

$$z^2 - kzy - y^2 = \pm 1,$$

which is the same equations of the previous section.

Then, the solutions of (4.6) are deduced as follows:

Let (x, y) be a solution of (3.5). Then $(x + 2y, y)$ is a solution of (4.6).

So $(p_0 + 2q_0, q_0)$ $(p_2 + 2q_2, q_2)$, $(p_4 + 2q_4, q_4)$, $\ldots$, $(p_{2n} + 2q_{2n}, q_{2n})$, $\ldots$ provides solutions to

$$x^2 - (k+4)xy + (2k+3)y^2 = -1$$

and (p_1+2q_1, q_1) (p_3+2q_3, q_3), (p_5+2q_5, q_5), $\ldots$, $(p_{2n+1}+2q_{2n+1}, q_{2n+1})$, $\ldots$ provides solutions to

$$x^2 - (k+4)xy + (2k+3)y^2 = 1,$$

where $\frac{p_0}{q_0}$, $\frac{p_1}{q_1}$, $\frac{p_2}{q_2}$, $\ldots$, $\frac{p_{2n}}{q_{2n}}$, $\frac{p_{2n+1}}{q_{2n+1}}$, $\ldots$ are the successive convergence of continued fraction expansion corresponding to the k^{th} metallic ratio.

5 Examples

Example 1

Let us solve the Diophantine equations

$$x^2 - 20xy + 95y^2 = \pm 1. \tag{5.7}$$

Note that the Diophantine equations (5.7) are particular cases of the equation (3.4) with $k = 4$, that can be solved using the 4^{th} metallic ratio continued fraction expansion.

Using (2.3), it is clear now that the desired continued fraction expansion of the 4^{th} metallic ratio ratio is provided by the equation:

$$\frac{\sqrt{20}+4}{2} = [4, \overline{4}].$$

Using the successive convergence of the continued fraction expansion of the 4^{th} metallic ratio, one can determine all possible solutions in the positive integers of these two Diophantine equations described in (5.7).

In fact, let

$$\begin{aligned} c_0 = & \ \frac{p_0}{q_0} = \frac{4}{1}, \\ c_1 = & \ [4,4] = \frac{p_1}{q_1} = \frac{17}{4}, \\ c_2 = & \ [4,4,4] = \frac{p_2}{q_2} = \frac{72}{17}, \\ c_3 = & \ [4,4,4,4] = \frac{p_3}{q_3} = \frac{305}{72}, \\ c_4 = & \ [4,4,4,4,4] = \frac{p_4}{q_4} = \frac{1292}{305}, \\ c_5 = & \ [4,4,4,4,4,4] = \frac{p_5}{q_5} = \frac{5473}{1292}, \dots \end{aligned}$$

We can see that $c_0,\ c_2,\ c_4,\ \dots,\ c_{2n},\ \dots$ provides solutions to

$$x^2 - 4xy - y^2 = -1,$$

and $c_1,\ c_3,\ c_5,\ \dots,\ c_{2n+1},\ \dots$ provides solutions to

$$x^2 - 4xy - y^2 = 1.$$

Specifically, $(4,1),\ (72,17),\ (1292,305),\dots$ are solutions of

$$x^2 - 4xy - y^2 = -1,$$

and $(17,4),\ (305,72),\ (5473,1292),\dots$ are solutions of

$$x^2 - 4xy - y^2 = 1.$$

Then $(12,1),\ (208,17),\ (3732,305),\dots,(p_{2n}+8q_{2n},q_{2n}),\dots$ are solutions of

$$x^2 - 20xy + 95y^2 = -1,$$

and $(49,4),\ (881,72),\ (15809,1292),\dots,(p_{2n+1}+8q_{2n+1},q_{2n+1}),\dots$ are solutions of

$$x^2 - 20xy + 95y^2 = 1.$$

Example 2 Let us solve the Diophantine equations

$$x^2 - 8xy + 11y^2 = \pm 1. \tag{5.8}$$

Note that the Diophantine equations (5.8) are particular cases of the equation (4.6) with $k = 4$, that can be solved using the 4^{th} metallic ratio continued fraction expansion.

Then

$$(6,1),\ (106,17),\ (1902,305),\ldots,(p_{2n}+2q_{2n},q_{2n}),\ldots$$

are solutions of

$$x^2 - 8xy + 11y^2 = -1,$$

and

$$(17,4),\ (305,72),\ (5473,1292),\ldots,(p_{2n+1}+2q_{2n+1},q_{2n+1}),\ldots$$

are solutions of

$$x^2 - 8xy + 11y^2 = 1.$$

□

References

[1] A. Chandoul, D. Marques, S.S. Albrbar, The Quadratic Diophantine Equations $x^2P(t)y^2 - 2P(t)x + 4P(t)y + (P(t))^2 - 4P(t) - 1 = 0$, Journal of Mathematics Research, **11,** no. 2, (2019), 30–38.

[2] A. Chandoul, The Pell Equation $X^2 - Dy^2 = \pm k^2$, Advances in Pure Mathematics, **1,** no. 2, (2011), 16.