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Option Pricing with Fuzzy-TGARCH Volatility Clustering

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Abstract

An option is a financial derivative that can help investors hedge risk or speculate by taking on more risk for more profit. Therefore, option pricing models have played an important role in supporting investors. The option price is influenced by the volatility of an underlying asset

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return, which is impacted by both positive and negative information. The volatility of the option price is considered an important factor for approximating option, especially in short-term option trading. In this research, a fuzzy-TGARCH model is constructed to estimate volatility, which is used to calculate an option price in the stock market with a short-term maturity date. This proposed approach is described and analyzed by comparing the numerical results with those of other methods. The data in the SET50 market are used for observation. With this data, the proposed method performs well for ITM cases when time to maturity is 20 and 30 days.

1 Introduction

An option is a financial contract between holder and writer, providing the right to buy or sell an underlying asset at an exercise price and a specific date. Apart from being a tool for reducing risk in an investment, this contract can also be used as an investment instrument for making a profit in both rising and falling markets. Therefore, this contract is widely used in speculative trading, where speculators are attempting to make exceptionally high returns from bets. Different information measuring the nature of stock prices have been studied in order to find an appropriate duration and price for purchasing. Volatility, a measure of stock price fluctuation, has also been estimated by researchers using various methods. The volatility of a stock price in the stock market may not be a constant. A GARCH model [1] is widely used to estimate a nonconstant volatility, depending on time, in various markets [2, 3, 4].

The GARCH model was established by Bollerslev in 1986 [1], and has been widely used as an instrument for analyzing the daily returns of stock prices [5, 6]. A financial asset's price responds to both positive and negative elements, often with an asymmetric impact [7, 8]. However, the GARCH model cannot capture the volatility of an asset return in response to asymmetric information [9, 10]. Therefore, many researchers have emphasized the need for an improved approximation of asset volatility with an asymmetric impact using modified GARCH models, such as the EGARCH [11], GJR-GARCH [12] and TGARCH [13] models, to overcome the restrictions of the standard GARCH model. Unfortunately, these models do not adequately simulate a stock fluctuation with volatility clustering [14, 15]. To further study volatility clustering, a fuzzy inference system with the GARCH model has been applied for this circumstance.

The fuzzy-GARCH model was proposed by Hung [16], and combines a functional fuzzy inference system with the GARCH model. He demonstrated that this model offers a significant improvement over other GARCH-family models in forecasting the Taiwan market index and NASDAQ index volatilities. Additionally, Hung [10], [14] developed a new adaptive fuzzy-GARCH model for predicting stock volatility. The fuzzy system method with the GARCH model is used to analyze volatility clustering. Based on their simulation results, the estimations of both in-sample and out-of-sample volatility performance are considerably improved.

In this work, an option pricing in a short-term trading period, according to option speculator behavior, is investigated and analyzed. The TGARCH model and fuzzy system are used to analyze stock price volatility by considering the asymmetric impact of positive and negative information. The numerical study of the proposed framework is evaluated using real data from the Stock Exchange of Thailand. This dataset is divided into two parts: before and after the occurrence of the COVID-19 pandemic.

The rest of this article is divided into four sections. In Section 2, we provide a literature review of the TGARCH and fuzzy-TGARCH models. In Section 3, we examine the research methodology. In Section 4, we provide a discussion of the data and numerical performance. In Section 5, we conclude our paper.

2 Literature Review

There are various strategies for trading in an option investment, such as bull, bear and butterfly strategies, in order to reduce risk and maximize return. Asset return volatility estimation is an important factor in designing investment strategies. Bollerslev established the GARCH model in 1986. This model is commonly used to estimate nonconstant volatility, depending on time, in various sectors such as energy price [2, 17], agriculture price [3, 18] and insurance [19, 20]. In the financial market, many researchers have investigated the effectiveness of the GARCH model in describing the volatility of emerging stock markets, [21, 22]. However, the GARCH model works under an assumption of a symmetric response between volatility and returns. Thus, some factors determining asset return cannot be sufficiently described by this model [23, 24] in the event of an asymmetric response. Therefore, many researchers have attempted to develop modified GARCH models to overcome these limits and more accurately capture time-varying volatility and asymmetric responses. For example, Nelson [11] introduced an EGARCH model in 1991, and Glosten et. al [12] proposed a GJR-GARCH model to consider the positive and negative effects of the news on variance. A shock would occur if the shock sign was separated from the variable of the reaction. For similar purposes, Zakoian [13] introduced the TGARCH model, which concentrated on conditional standard deviation (volatility) instead of conditional variance. He analyzed the performance of his model by estimating the volatility of a stock index in France (CAC index). Building on this research, a series of modified GARCH models have been designed to explain the regularity of fluctuations, and they are effective in describing volatility in the financial market.

Zakoian [13] described the difference between GARCH and TGARCH models in a Financial Times series, saying that the negative part of crisis has more impact on volatility than the positive part. Following Zakoian's 1994 findings [13], some researchers have investigated and applied the TGARCH model to approximate the volatility of different underlying assets, including carbon [25], crude oil, ethanol and corn [26], crude oil, natural gas and coal [27] and sugar [28]. For financial market research, the TGARCH model is applied to approximate volatility in several dimensions, such as trading volume index, Forex market, arbitrage trading and option pricing. For example, Sabiruzzman [29] used both GARCH and TGARCH models, to evaluate the pattern of volatility in the Hong Kong stock exchange's daily trading volume index. The TGARCH specification was found to be superior to the GARCH specification. The TGARCH model is also used by Olweny and Omondi [30] to investigate the impact of macroeconomic factors on the volatility of Kenyan stock market returns, including the effects of foreign exchange rates, interest rates, and inflation variability. In addition, the TGARCH model has been applied to establish empirical evidence of calendar effects on individual finance stocks in Malaysia [31]. Alternatively, an investigation of the option price by an analytical approximation has been studied using traditional and modified GARCH models [32, 34, 35]. These analytical approximations are constructed by combining the first four moments of cumulative asset return. Compared to the Black-Scholes formula [36], the formula obtained from a GARCH model is adjusted by adding the skewness and kurtosis of the cumulative asset return. This technique reduces the assumptions of the Black-Scholes formula related to a lognormal distribution of stock price and a constant volatility [32, 33].

Various methods have been developed to determine stock price and its effect on the market. A fuzzy inference system with the GARCH model is an approach that effectively explains stock fluctuations [14, 15]. Therefore, many researchers have devoted their studies to applying a fuzzy logic approach to modeling a stock price index in an attempt to solve financial problems [37], [38]. The fuzzy logic model is also used to forecast stock market volatility. Hung [9] adopted fuzzy logic systems to modify the threshold values for an asymmetric GARCH model. Based on the simulations, he found that the predicted performance significantly improved when the leverage effect of clustering was combined with the fuzzy systems and the GARCH technique. Furthermore, Hung recently proposed the hybrid fuzzy-GARCH model [16]. This model was founded based on asymmetric information and volatility clustering. Maciel [15] also suggested a fuzzy GJR-GARCH model in 2012 to estimate the volatility of the S&P 500 and Ibovespa indices. To evaluate option pricing, Thavaneswaran et al. [39] presented a fuzzy call option model by clustering levels of various factors, such as an underlying price and its volatility, via a fuzzy weighted possibilistic model. In their work, the volatility was set as a constant. This model is sufficiently flexible and can be easily adjusted to find an optimal solution. Wang and Lee [40] computed the call option price via the Black-Scholes model when the underlying stock price, volatility and risk-free interest rate are modeled as fuzzy numbers. As the aforementioned research illustrates, a family of fuzzy-GARCH model is a new and effective method to describe and map stock fluctuations.

3 Research Methodology

In this research, the option pricing model, based on clustering levels of non-constant volatility, is constructed and computed via a Monte Carlo simulation. The asset return volatility is estimated by a fuzzy-TGARCH approach. The content of this section is organized into two parts; an asset return volatility via a fuzzy-TGARCH model, and option pricing via a Monte Carlo simulation.

3.1 Asset Return Volatility via Fuzzy-TGARCH Model

In order to compute asset return volatility with a fuzzy-TGARCH model, we will divide the estimation into two steps. In the first step, we construct the fuzzy-TGARCH model. In the second step, we approximate the optimal parameters of fuzzy-TGARCH model.

3.1.1 Fuzzy-TGARCH Model

The fuzzy-TGARCH model is a combination of the TGARCH approach and a fuzzy inference system. The fuzzy-TGARCH model is constructed by IF-THEN rules, which are used to formulate the conditional statements. The process for constructing the fuzzy-TGARCH model is shown in the following figure.

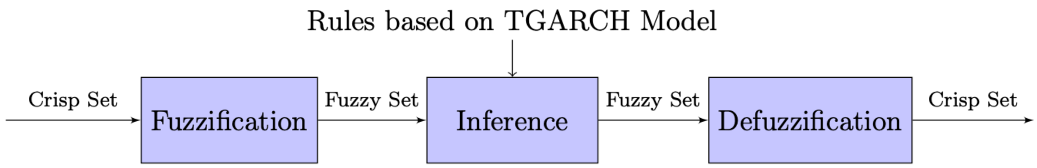


Figure 1: Fuzzy Inference Systems.

The process of fuzzy inference systems is divided into three steps. First, a membership function is employed to convert a crisp set of input data into a fuzzy set. This is referred as a fuzzification step. Then, an inference step is formed based on a set of rules for mapping a given input to an output. Lastly, in the defuzzification step, the fuzzy output from inference step is converted to a crisp output by using a defuzzification method.

In this research, fuzzy logic is described as follows. For the first step, a set of data $\bar{x}_t = [x_{1,t},\ x_{2,t},\ \ldots,\ x_{n,t}]$ is transformed to a crisp set by using the Gaussian membership formula defined as:

$$F_i(x_{i,t}) = \exp\Big(-\frac{1}{2}\Big(\frac{x_{i,t}-c_i}{s_i}\Big)^2\Big),$$

where c_i and s_i are the center and the spread of the membership function corresponding to the ith premise variable, respectively. The value $F_i(x_{i,t})$ is the grade of the membership of $x_{i,t}$ in F_i.

For the inference step, the inputs are applied to a set of IF-THEN control rules, in which the TGARCH model is applied for setting these rules. Instead of conditional variance, the TGARCH model specifies conditional standard deviation. The TGARCH model with the parameters p and q, denoted by TGARCH (p,q), is given by

$$y_t = \sqrt{h_t}\epsilon_t,$$

$$\sqrt{h_t} = \alpha_0 + \sum_{i=1}^{q}(\alpha_i^+ y_{t-i}^+ - \alpha_i^- y_{t-i}^-) + \sum_{j=1}^{p}\beta_j\sqrt{h_{t-j}}, \quad t \geq 1, \qquad (3.1)$$

where $\{\epsilon_t\}$ is a sequence of independent standard normal random variables, $y_t^+ = \max(y_t,\ 0)$ and $y_t^- = \min(y_t,\ 0)$ are the positive and negative parts of y_t, h_t is the volatility, which is $E[y_t^2|\phi_{t-1}]$. α_0 is a positive real number; α_i^+, α_i^- and β_j are non-negative real numbers for $i = 1, \ldots, q$ and $j = 1, \ldots, p$. The notation ϕ_t represents a set of information up to time t. The model in (3.1) can be rewritten in the form

$$\sqrt{h_t} = \alpha_0 + \sum_{i=1}^{q} \alpha_i \Big[|y_{t-i}| - \gamma_i y_{t-i}\Big] + \sum_{j=1}^{p} \beta_j \sqrt{h_{t-j}}, \quad \text{for } t \geq 1,$$

where α_i and γ_i are non-negative real numbers for all $i = 1, \ldots, q$. For describing stock market fluctuation, the proposed fuzzy-TGARCH model is represented by a collection of fuzzy rules in the form of IF-THEN statements. Therefore, for a fixed number L of IF-THEN rules, the k^{th} rule of the fuzzy-TGARCH(p, q), $k = 1, 2, \ldots, L$, is written as:

$$\text{Rule}^{(k)}: \text{ IF } x_{1,t} \text{ is } F_{k1} \text{ AND } \cdots \text{ AND } x_{n,t} \text{ is } F_{kn}, \text{ THEN}$$

$$y_t = \sqrt{h_t}\epsilon_t \text{ and } \sqrt{h_t} = \alpha_k + \sum_{i=1}^{q} \alpha_{ki} \Big[|y_{t-i}| - \gamma_{ki} y_{t-i}\Big] + \sum_{j=1}^{p} \beta_{kj} \sqrt{h_{t-j}}, \qquad (3.2)$$

where y_t is the output of the system, and the output of k^{th} rule is the output from the fuzzy-TGARCH model in equation (3.2).

For the defuzzification step, all outputs will be combined to obtain a final output. From the rules in the inference step, the output of k^{th} rule in equation (3.2) is defuzzied by using a centroid method. By this method, the output of the fuzzy-TGARCH model can be computed as follows:

$$\sqrt{h_t} = \sum_{k=1}^{L} (\text{weight of } k^{th} \text{ rule}) \cdot (\text{output of } k^{th} \text{ rule})$$

where the weight of k^{th} rule is defined by

$$w_k(\bar{x}_t) = \frac{u_k(\bar{x}_t)}{\sum_{k=1}^{L} u_k(\bar{x}_t)},$$

$$u_k(\bar{x}_t) = \prod_{i=1}^{n} F_{ki}(x_{i,t}) = \prod_{i=1}^{n} \exp\Big(-\frac{1}{2}\Big(\frac{x_{i,t} - c_{ki}}{s_{ki}}\Big)^2\Big),$$

where c_{ki} and s_{ki} are the center and the spread of the k^{th} rule membership function corresponding to the ith premise variable. Therefore, we have constructed the fuzzy-TGARCH volatility model using the fuzzy inference system as follows:

$$\sqrt{h_t} = \sum_{k=1}^{L} w_k(\bar{x}_t)\Big\{\alpha_k + \sum_{i=1}^{q}\Big[\alpha_{ki}\Big(|y_{t-i}| - \gamma_{ki} y_{t-i}\Big)\Big] + \sum_{j=1}^{p}\beta_{kj}\sqrt{h_{t-j}}\Big\}.$$

This equation can be rearranged in the matrix form as

$$\begin{aligned}\sqrt{h_t} =& \sum_{k=1}^{L} w_k(\bar{x}_t)\alpha_k + \sum_{k=1}^{L}\sum_{i=1}^{q} w_k(\bar{x}_t)\alpha_{ki}|y_{t-i}| - \sum_{k=1}^{L}\sum_{i=1}^{q} w_k(\bar{x}_t)\alpha_{ki}\gamma_{ki} y_{t-i}\\ &+ \sum_{k=1}^{L}\sum_{j=1}^{p} w_k(\bar{x}_t)\beta_{kj}\sqrt{h_{t-j}}\\ =& \boldsymbol{\Lambda}^T \mathbf{z}_t, \quad \text{where } \boldsymbol{\Lambda}^T = \begin{bmatrix} \sum_{k=1}^{L} w_k(\bar{x}_t)\alpha_k \\ \sum_{k=1}^{L} w_k(\bar{x}_t)\alpha_{ki} \\ -\sum_{k=1}^{L} w_k(\bar{x}_t)\alpha_{ki}\gamma_{ki} \\ +\sum_{k=1}^{L} w_k(\bar{x}_t)\beta_{kj} \end{bmatrix} \text{ and } \mathbf{z}_t = \begin{bmatrix} 1 \\ \sum_{i=1}^{q} |y_{t-i}| \\ \sum_{i=1}^{q} y_{t-i} \\ \sum_{j=1}^{p} \sqrt{h_{t-j}} \end{bmatrix}. \end{aligned} \quad (3.3)$$

The parameters of the fuzzy-TGARCH volatility model in equation (3.3) can be computed using the Genetic Algorithm (GA), which will be described in the next section.

3.1.2 Optimal Parameters of Fuzzy-TGARCH Model

Various iterative techniques can be used to approximate the parameters of the membership functions and TGARCH models such as Genetic Algorithm (GA), Particle Swarm Optimization (PSO) and Differential Evolution (DE). An algorithm based on the genetic algorithm (GA) will be used in this study. The fuzzy-TGARCH model parameters can be computed by minimizing a mean squared error criteria. As a result, an objective function of this optimization problem is defined as follows:

$$E[\boldsymbol{\Lambda}] = \frac{1}{N}\sum_{t=1}^{N}[y_t - \hat{y}_t]^2,$$

where N is the number of in-sample stock market data, $\hat{y}_t$ is the in-sample stock market data and y_t is the output obtained from the fuzzy-TGARCH model. That is,

$$y_t = \sqrt{h_t}\epsilon_t = \mathbf{\Lambda}^T \boldsymbol{z}_t \epsilon_t. \tag{3.4}$$

Therefore, the parameters of the fuzzy-TGARCH model can be estimated by solving the following optimization problem:

$$\min_{\Lambda \in \Theta} E(\mathbf{\Lambda}), \tag{3.5}$$

$$\text{where } y_t = \frac{\sum_{k=1}^{L} \prod_{i=1}^{N} \exp\Big[-\frac{1}{2}\Big(\frac{x_{i,t}-c_{ki}}{s_{ki}}\Big)^2 \times [\alpha_k + \alpha_{ki}(|y_{t-1}| - \gamma_{ki} y_{t-1}) + \beta_{ki}\sqrt{h_{t-1}}]}{\sum_{k=1}^{L} \prod_{i=1}^{N} \exp\Big[-\frac{1}{2}\Big(\frac{x_{i,t}-c_{ki}}{s_{ki}}\Big)^2\Big]} \epsilon_t. \tag{3.6}$$

The fuzzy-TGARCH model in equation (3.4) is a general nonlinear time varying equation. The behaviors of nonlinear dynamic systems have been modeled on this specific type of equation. For this reason, minimizing the mean square error, which is an objective function defined in equation (3.5), would be an effective way to ensure increased success of this model. The Genetic Algorithm (GA) is implemented to solve different types of complex optimization problems. Based on the results, GA was proven to perform better than Differential Evolution (DE) and Particle Swarm Optimization (PSO) in obtaining the highest numbers of best minimum fitness and worked faster. The fuzzy-TGARCH model in (3.6) is solved by using the fitness function defined as $E(\mathbf{\Lambda})$. We then consider a set of solutions for the problem and select the set of best examples. The flowchart representing each step of GA is shown below:

Table 1: The steps of optimizing the parameters of fuzzy-TGARCH via GA.

Procedure Genetic Algorithm (GA)

```
Input the received data x̄_t (ie. x_{1,t} = |y_{t-1}|)
Randomly create the initial population
Evaluate each individual in the population
while (termination conditions are not met) do
  for each individual repeat do
    Select parents for reproducing
    Crossover
    Mutation
    Generate new population
    Evaluate the new population
    Select the best solution from the population.
  end for
end while
Return the best solution.
```

After computing the appropriate parameters via GA, we will use these values to predict the volatility of the asset return, and then use the obtained volatility to approximate option prices. This process will be described in the next section.

3.2 Option Pricing via Monte Carlo Simulation

An option price can be estimated using various factors, such as current stock price, strike price, time to maturity date, interest rate, underlying asset price and volatility. The payoff of a call option can be considered as $\max(S_T - K, 0)$, where S_T and K are the stock price at expiration date and the strike price, respectively. The computing option price via Monte Carlo simulation results from averaging all possible option prices at the expiration date, and then discounting the value of the payoff back to the current time. The general form of the option price at current time t is as follows:

$$C_t = e^{-r(T-t)} E[\max(S_T - K, 0)|\phi_t], \tag{3.7}$$

where C_t is a call premium at the time t, r is a risk-free rate, $T - t$ is time duration until expiration date and ϕ_t is the set (σ-field) of all information up to time t. It can be seen that the expression $E[\max(S_T - K, 0)|\phi_t]$ depends on the asset price at the expiration time T. In this research, the S_T will be computed by following Duan's equations [41]. The equation is

$$S_T = S_t \exp\Big[r(T-t) - \frac{1}{2}\sum_{s=t+1}^{T} h_s + \sum_{s=t+1}^{T} \xi_s\Big], \tag{3.8}$$

where h_t is the volatility of the asset return obtained from the fuzzy-TGARCH model given in section 3.1, and ξ_s is a normal random variable. In computing S_T, the value for ξ_s is generated and the S_T is estimated via a Monte Carlo simulation. The step of computing call option prices for the proposed method is shown below:

Option Pricing via Fuzzy-TGARCH Model
Step 1: Finding the asset volatility via fuzzy-TGARCH model in section 3.1
Step 2: Simulating the asset price S_T in (3.8) and computing the option price C_t in (3.7) by using the Monte Carlo method.

4 Numerical Performance and Discussion

From the previous section, it can be seen that approximating the option prices depends on the value of the underlying asset price and volatility. Therefore, our study is focused on two main elements. Section 4.1 investigates estimating volatility from a fuzzy-TGARCH model. Section 4.2 studies and discusses the option prices estimated from the method proposed in section 3. These results are compared to those obtained from other methods, such as GARCH and TGARCH models, Black-Scholes formula (BS), an analytical approximation via GARCH (A_GARCH) and TGARCH (A_TGARCH) models and an analytical approximation with a Monte Carlo simulation (A_GARCH with MC). The steps of each method are summarized in the following table:

Table 2: Methods for computing option prices.

Method for computing option prices	**Method for computing**		
	Volatility	S_T	C_t
GARCH [41]	GARCH	Monte Carlo Method	
TGARCH [43]	TGARCH		
BS [36]	Historical Volatility	-	Black-Scholes formula
A_GARCH [32]	GARCH	Using parameters from GARCH Model	C_{approx}[32]
A_GARCH with MC [32]	GARCH	Monte Carlo	
A_TGARCH [35]	TGARCH	Using parameters from TGARCH Model	
Proposed Method	Fuzzy-TGARCH	Monte Carlo Method	

This section will investigate the accuracy of these techniques with the real data. The dataset used in this study is focused on the period between January 2020 and December 2020, during the COVID-19 pandemic in Thailand. All real data are collected from the Stock Exchange of Thailand website (http://www.setsmart.com). The SET50 index is considered as the asset price and the contract prices of SET50 index option, namely, S50H20C, S50M20C and S50Z20C, are considered as observation data. The S50H20C contract is a SET50 index option contract that expired at the end of March 2020. During this period, the SET50 index rapidly decreased, caused by the first wave of the COVID-19 outbreak in Thailand. This situation was negative information to investors. For the S50M20C and S50Z20C contracts, the SET50 index option contracts expired at the end of June 2020 and at the end of December 2020. The movement of SET50 index in these two periods continuously increased due to the first and second waves of the COVID-19 outbreak settling down. This is positive information to investors. The closing SET50 index during the periods of contracts S50H20C, S50M20C and S50Z20C is shown in Figure 2, where the yellow, green and purple shaded areas represent the durations of S50H20C, S50M20C and S50Z20C, respectively.

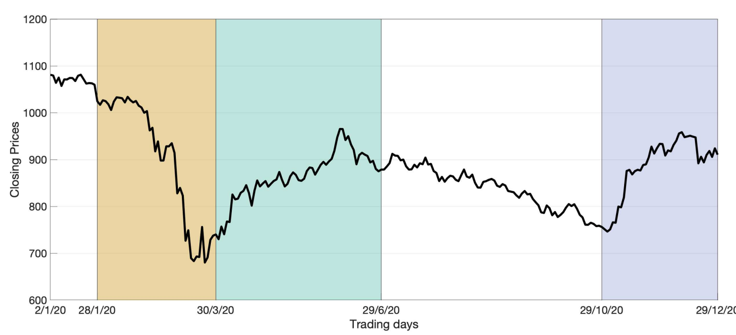


Figure 2: Daily SET 50 Index from May 27, 2019 to June 29, 2020

To approximate the option price in equation (3.7), the values of K, $T-t$, r and S_t are needed. These values are specified by the value in the real market. In this study, the risk-free interest rate is set to 1.99% per year, the Thai government bond rate. Also, due to the market fluctuation, this research focuses on short-term trading for speculators. Therefore, the time to expiration $(T-t)$ is considered in three cases; 20, 30 and 40 days, according to speculate behavior[42]. The value of the underlying asset price, (S_t), is the SET50 index at time t and the strike price K is the strike price in each option contract.

4.1 Estimating Volatility from Fuzzy-TGARCH Model

Here, we assume that the volatility of observation data follows the fuzzy-TGARCH model with $p = q = 1$. From equation (3.3), the fuzzy-TGARCH(1,1) model is

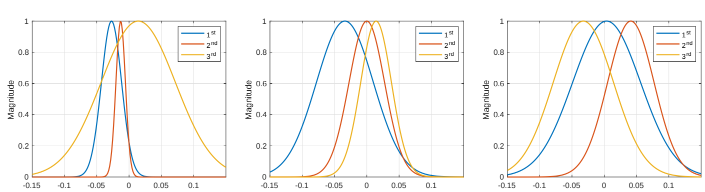
$$\sqrt{h_t} = \begin{bmatrix} w_1(\bar{x}_t)\alpha_1 + w_2(\bar{x}_t)\alpha_2 + \cdots + w_L(\bar{x}_t)\alpha_L \\ w_1(\bar{x}_t)\alpha_{11} + w_2(\bar{x}_t)\alpha_{21} + \cdots + w_L(\bar{x}_t)\alpha_{L1} \\ -w_1(\bar{x}_t)\alpha_{11}\gamma_{11} - w_2(\bar{x}_t)\alpha_{21}\gamma_{21} - \cdots - w_L(\bar{x}_t)\alpha_{L1}\gamma_{L1} \\ w_1(\bar{x}_t)\beta_1 + w_2(\bar{x}_t)\beta_2 + \cdots + w_L(\bar{x}_t)\beta_L \end{bmatrix}^T \begin{bmatrix} 1 \\ |y_{t-1}| \\ y_{t-1} \\ \sqrt{h_{t-1}} \end{bmatrix}.$$

Setting up the values of the initial parameters of the fuzzy-TGARCH model based on observation data, the number of rule (L) is set to be three as laid out by [10]. Given the initial parameters of the fuzzy-TGARCH model, the appropriate parameters of the fuzzy-TGARCH model can be optimized by solving equation (3.5) via GA. However, there are a number of parameters in GA that need to be set, such as mutation rate, crossover rate, replacement factor, selection rate and population size. The following table shows all parameter values that are set in this work.

Table 3: Selected values of parameters in GA estimation approach[10].

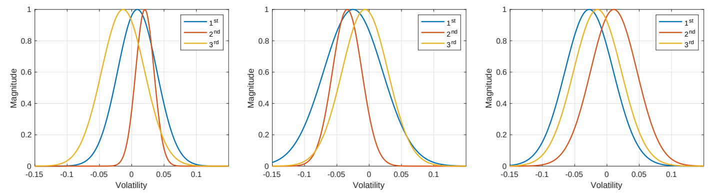
Name of Parameters	Parameter values
Mutation rate	0.01
Crossover rate	0.95
Replacement factor	0.5
Selection rate	0.5
Population size	100

All outputs of the dataset, such as membership functions of or fuzzy sets of volatility in contracts S50H20C, S50M20C and S50Z20C, are shown in Figure 3 - 5, respectively. The appropriate models obtained from the fuzzy-TGARCH are shown in Table 4. The weight fuzzy-TGARCH models for the volatility in each contract are provided. The models is obtained by averaging the results from the three rules.



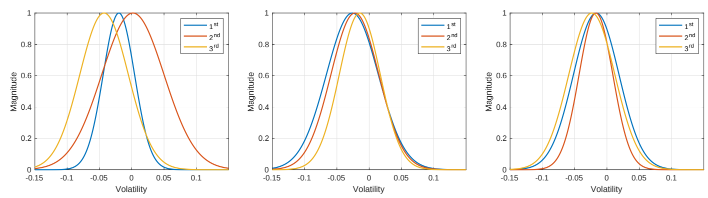
(a) $T-t = 20$ days. (b) $T-t = 30$ days. (c) $T-t = 40$ days.

Figure 3: The Gaussian membership functions of volatility in S50H20C.



(a) $T-t=20$ days. (b) $T-t=30$ days. (c) $T-t=40$ days.

Figure 4: The Gaussian membership functions of volatility in S50M20C.



(a) $T-t=20$ days. (b) $T-t=30$ days. (c) $T-t=40$ days.

Figure 5: The Gaussian membership functions of volatility in S50Z20C.

Table 4: The appropriate parameters of fuzzy-TGARCH model in each rule of S50H20C, S50M20C and S50Z20C.

Contract no.	$T-t$ **(days)**	**Parameters in fuzzy-TGARCH model**	**Values**	**Weight fuzzy-TGARCH model**
S50H20C	20	$(\alpha_1,\ \alpha_{11},\ \gamma_{11},\ \beta_{11})$ $(\alpha_2,\ \alpha_{21},\ \gamma_{21},\ \beta_{21})$ $(\alpha_3,\ \alpha_{31},\ \gamma_{31},\ \beta_{31})$	(0.0245, 0.2143, 0.0260, 0.0157) (0.0457, 0.1510, 0.0193, 0.0333) (0.0269, 0.0322, 0.0209, 0.0322)	$\sqrt{h_t} = 0.0324 + 0.1325\lvert y_{t-1}$ $-(0.1325)(0.0221)y_{t-1}$ $+0.0271\sqrt{h_{t-1}}$
	30	$(\alpha_1,\ \alpha_{11},\ \gamma_{11},\ \beta_{11})$ $(\alpha_2,\ \alpha_{21},\ \gamma_{21},\ \beta_{21})$ $(\alpha_3,\ \alpha_{31},\ \gamma_{31},\ \beta_{31})$	(0.0329, 0.3176, 0.0338, 0.0374) (0.0281, 0.3305, 0.0389, 0.0316) (0.0336, 0.3363, 0.0272, 0.0238)	$\sqrt{h_t} = 0.0315 + 0.3281\lvert y_{t-1}\rvert$ $-(0.3281)(0.0333)y_{t-1}$ $+0.0309\sqrt{h_{t-1}}$
	40	$(\alpha_1,\ \alpha_{11},\ \gamma_{11},\ \beta_{11})$ $(\alpha_2,\ \alpha_{21},\ \gamma_{21},\ \beta_{21})$ $(\alpha_3,\ \alpha_{31},\ \gamma_{31},\ \beta_{31})$	(0.0237, 0.2859, 0.0321, 0.0364) (0.0264, 0.2043, 0.0280, 0.0359) (0.0327, 0.2117, 0.0287, 0.0267)	$\sqrt{h_t} = 0.0276 + 0.2340\lvert y_{t-1}\rvert$ $-(0.2340)(0.0296)y_{t-1}$ $+0.0330\sqrt{h_{t-1}}$
S50M20C	20	$(\alpha_1,\ \alpha_{11},\ \gamma_{11},\ \beta_{11})$ $(\alpha_2,\ \alpha_{21},\ \gamma_{21},\ \beta_{21})$ $(\alpha_3,\ \alpha_{31},\ \gamma_{31},\ \beta_{31})$	(0.0314, 0.1521, 0.0313, 0.0207) (0.0288, 0.2772, 0.0257, 0.0321) (0.0228, 0.1264, 0.0237, 0.0301)	$\sqrt{h_t} = 0.0277 + 0.1852\lvert y_{t-1}\rvert$ $-(0.1852)(0.0269)y_{t-1}$ $+0.0276\sqrt{h_{t-1}}$
	30	$(\alpha_1,\ \alpha_{11},\ \gamma_{11},\ \beta_{11})$ $(\alpha_2,\ \alpha_{21},\ \gamma_{21},\ \beta_{21})$ $(\alpha_3,\ \alpha_{31},\ \gamma_{31},\ \beta_{31})$	(0.0320, 0.3180, 0.0169, 0.0122) (0.0260, 0.3941, 0.0183, 0.0310) (0.0416, 0.2052, 0.0337, 0.0353)	$\sqrt{h_t} = 0.0332 + 0.3058\lvert y_{t-1}\rvert$ $-(0.3058)(0.0230)y_{t-1}$ $+0.0262\sqrt{h_{t-1}}$
	40	$(\alpha_1,\ \alpha_{11},\ \gamma_{11},\ \beta_{11})$ $(\alpha_2,\ \alpha_{21},\ \gamma_{21},\ \beta_{21})$ $(\alpha_3,\ \alpha_{31},\ \gamma_{31},\ \beta_{31})$	(0.0201, 0.2807, 0.0283, 0.0311) (0.0259, 0.3387, 0.0307, 0.0352) (0.0336, 0.3678, 0.0401, 0.0308)	$\sqrt{h_t} = 0.0265 + 0.3291\lvert y_{t-1}\rvert$ $-(0.3291)(0.0330)y_{t-1}$ $+0.0324\sqrt{h_{t-1}}$
S50Z20C	20	$(\alpha_1,\ \alpha_{11},\ \gamma_{11},\ \beta_{11})$ $(\alpha_2,\ \alpha_{21},\ \gamma_{21},\ \beta_{21})$ $(\alpha_3,\ \alpha_{31},\ \gamma_{31},\ \beta_{31})$	(0.0161, 0.3483, 0.0261, 0.0226) (0.0331, 0.3390, 0.0385, 0.0273) (0.0248, 0.1481, 0.0201, 0.0374)	$\sqrt{h_t} = 0.0247 + 0.2785\lvert y_{t-1}\rvert$ $-(0.2785)(0.0282)y_{t-1}$ $+0.0295\sqrt{h_{t-1}}$
	30	$(\alpha_1,\ \alpha_{11},\ \gamma_{11},\ \beta_{11})$ $(\alpha_2,\ \alpha_{21},\ \gamma_{21},\ \beta_{21})$ $(\alpha_3,\ \alpha_{31},\ \gamma_{31},\ \beta_{31})$	(0.0301, 0.2232, 0.0331, 0.0359) (0.0223, 0.2723, 0.0280, 0.0172) (0.0324, 0.21911, 0.0359, 0.0311)	$\sqrt{h_t} = 0.0282 + 0.2289\lvert y_{t-1}\rvert$ $-(0.2289)(0.0323)y_{t-1}$ $+0.0281\sqrt{h_{t-1}}$
	40	$(\alpha_1,\ \alpha_{11},\ \gamma_{11},\ \beta_{11})$ $(\alpha_2,\ \alpha_{21},\ \gamma_{21},\ \beta_{21})$ $(\alpha_3,\ \alpha_{31},\ \gamma_{31},\ \beta_{31})$	(0.0297, 0.2562, 0.0249, 0.0368) (0.0184, 0.2104, 0.0346, 0.0323) (0.0344, 0.2308, 0.0356, 0.0291)	$\sqrt{h_t} = 0.0275 + 0.2324\lvert y_{t-1}\rvert$ $-(0.2324)(0.0317)y_{t-1}$ $+0.0328\sqrt{h_{t-1}}$

4.2 Approximation of Option Price via Fuzzy-TGARCH Model

In order to approximate option prices via a Monte Carlo simulation in eq. (3.7), we have to simulate an underlying asset price, S_t, by following eq. (3.8). For this simulation, 500,000 samples of ξ_s in (3.8) are generated with 500,000 path simulations.

In order to ensure accurate studies, several methods are used to measure approximated results such as, correlation coefficient (CORR), mean absolute error (MAE), mean absolute percentage error (MAPE), and root mean square error (RMSE). Their formulas are as follows:

$$\text{CORR} = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum_{i=1}^{n}(x_i - \bar{x})^2 \sum_{i=1}^{n}(y_i - \bar{y})^2}}, \ \text{MAE} = \sum_{i=1}^{n}\left|\frac{x_i - y_i}{n}\right|,$$

$$\text{MAPE} = \frac{100\%}{n} \times \sum_{i=1}^{n}\left|\frac{x_i - y_i}{x_i}\right| \text{ and } \text{RMSE} = \sqrt{\sum_{i=1}^{n}\frac{(x_i - y_i)^2}{n}},$$

where the x_i is an observed price, y_i is an estimated price, $\bar{x}$ is the mean of observed prices, $\bar{y}$ is the mean of estimated prices and n is the number of observations. Based on these measures, the results in Table 5 illustrate that, for all ITM cases (37 cases) with $K/S_t \leq 1$, the proposed method has superior performance compared with other approaches.

Additionally, the accuracy results of the seven methods in Table 5 are shown in Figure 6. For the CORR, the results of these seven methods are close to 1. As shown by the MAE, MAPE and RMSE, the proposed method provides more accurate results than the other approaches. In particular, the MAPE from our method is significantly less than the others. Moreover, it can be seen that the proposed method has enhanced performance when we focus on cases when the time to maturity date equals 20 and 30 days.

The discussion of the results is separated in two directions. The first direction illustrates the discussion the dimension of time. The absolute percentage error(APE) of all estimated prices in the seven approaches are plotted in Figure 7. According to the Figure 7, it is clear that the APE of the proposed method is less than 20% where $T - t = 20$ and 30 days. These results reveal that the option prices obtained from the proposed method is effective for short periods of time to maturity date ($\approx$ 1 month).

Table 5: The performance of option price via fuzzy-TGARCH compared with other methods.

T-t	S_t	K	K/S_t	Observed	GARCH [41]	TGARCH [43]	BS [36]	A_GARCH [32]	A_GARCH with MC [32]	A_TGARCH [35]	Proposed Method
20	897.96	850	0.9466	60.00	52.2768	49.2304	50.2344	56.4782	34.2538	34.3284	54.3501
		875	0.9744	36.30	32.7388	24.2700	28.7948	35.9798	49.2311	49.3014	35.5991
	896.01	775	0.8649	119.40	128.4753	122.1667	131.7913	141.4448	122.1624	122.2332	120.10321
		800	0.8928	100.00	108.0879	97.1993	111.8597	120.2831	97.1943	97.2775	95.64541
		825	0.9207	77.00	88.3437	72.2376	93.5850	99.8612	72.2545	72.5761	72.22107
		850	0.9487	56.90	71.8459	47.2809	77.1448	80.4872	47.2770	49.6602	51.22455
	933.60	700	0.7498	232.00	234.3580	234.3110	234.4032	237.5685	234.5672	234.3630	233.7031
		725	0.7766	207.00	209.5361	209.3423	209.5228	211.5994	209.7555	209.3903	208.7425
		750	0.8033	181	184.5376	184.3606	184.7955	185.7236	185.6798	184.4175	183.7820
		775	0.8301	153	160.1407	159.3948	160.3933	160.6329	160.7556	159.4448	158.8219
		800	0.8569	130.2	136.1210	134.4198	136.5956	137.0894	136.3639	134.4720	133.8703
		825	0.8837	112.9	113.0502	109.4448	113.7930	96.1286	92.3791	109.4993	108.9556
		850	0.9105	89.7	91.3239	84.4769	92.4571	78.4106	72.5792	84.5269	84.3113
		900	0.9640	45	54.6586	34.5311	56.0543	62.0386	54.6642	41.9300	39.2255
		925	0.9908	28.3	39.9120	9.5561	41.6650	49.7710	43.0165	0.9496	22.2189
30	1025.18	975	0.9511	47.60	55.4279	52.6966	55.5485	59.1919	55.1542	52.4853	52.5197
		1000	0.9754	28.70	35.3150	27.7559	35.5600	38.5719	35.0661	33.5203	29.2535
	854.57	725	0.8484	128.00	131.2508	131.2161	131.3552	131.1378	130.7139	130.7609	131.53948
		750	0.8776	112.00	106.4840	106.2757	106.6926	105.4840	105.7555	105.8020	107.60083
		800	0.9361	63.00	59.5801	56.3938	60.3041	60.6365	58.9086	55.8841	64.24824
		825	0.9654	46.00	39.5358	31.4534	40.7336	42.6728	39.3243	30.9279	46.34986
		850	0.9947	29.00	23.8472	6.5120	25.1467	26.7049	23.5181	7.3866	31.73768
	868.27	700	0.9511	163.00	170.5580	169.3644	170.9073	172.5745	171.9434	169.4145	167.4730
		725	0.9754	143.00	147.1127	144.4093	147.4475	149.5375	146.9764	144.4553	142.5905
		750	0.8484	122.6	124.4231	119.4498	125.0129	128.3376	124.9040	119.4962	117.9257
		775	0.8776	96.5	103.1182	94.4861	103.9829	108.9554	104.0137	94.5371	93.8442
		800	0.9361	78	83.5841	69.5283	84.7273	91.1593	84.1852	69.6010	71.0973
		825	0.9654	58	66.2595	44.5703	67.5530	74.6772	66.7159	49.2199	50.6271
		850	0.9947	43	51.1808	19.6161	52.6602	59.3376	51.4278	30.5125	33.4848
40	1018.08	975	0.9577	46.10	51.4246	46.0742	51.6541	55.8503	51.6553	46.0850	61.2164
		1000	0.9822	27.70	33.1983	21.1538	33.5263	36.1710	33.1221	23.0878	45.7070
	854.34	800	0.9364	93.00	97.7283	56.7986	99.0442	111.9559	98.7330	70.0993	68.55778
		825	0.9657	73.00	83.9523	31.8680	85.0048	94.5720	83.7424	56.2364	52.51664
		850	0.9949	57.90	71.1971	6.9859	72.4069	78.4583	71.3111	0.0000	39.08713
	751.14	700	0.9319	54.50	71.1923	52.6229	72.0716	79.1203	71.5693	52.9092	55.9693
		725	0.9652	36.00	55.3261	27.6816	56.4419	63.4549	55.9280	51.9057	37.2559
		750	0.9985	18.9	41.8921	2.7486	43.2054	49.0000	42.2088	-56.5217	22.6659
				CORR	0.9912	0.9819	0.9882	0.9776	0.9804	0.9652	0.9889
				MAE	7.7645	10.3937	8.9818	12.9535	9.1368	11.0168	5.8643
				MAPE(%)	17.7022%	20.8266%	18.5156%	24.9259%	18.5473%	28.0505%	10.4899%
				RMSE	9.2629	15.3404	10.7869	15.2740	10.9812	18.5937	8.2402

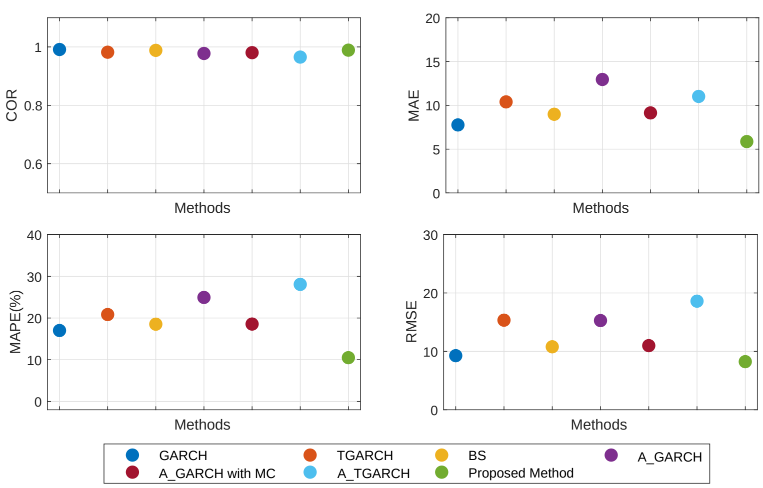


Figure 6: CORR, MAE, MAPE and RMSE of call option price in SET50

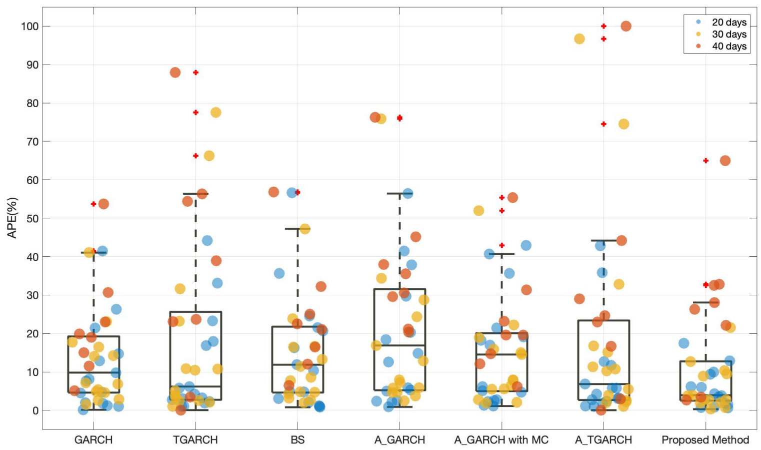


Figure 7: Boxplot of call option price in SET50

Remark 4.1. *The symbol "+" means that the corresponding points are considered as outliers of the box plot.*

In the second direction, the impact of K/S_t corresponding to call option prices is described. For the ITM cases when $K/S_t \leq 0.96$, in Figure 8 shows that the APEs of approximated prices from each technique have similar patterns. For the ITM cases when K/S_t approaches 1, the APE of GARCH, TGARCH, A_GARCH, A_GARCH with MC and A_TGARCH do not perform well. However, it can be seen that the APE of the proposed method still displays a small error for ITM cases when K/S_t approaches 1. This shows a better performance in comparison to the other methods.

In Figure 9, looking closer at a short time to maturity date, 20 days (15 cases) and 30 days (14 cases), the MAEs are 4.1088 and 3.8394 and the MAPEs are 6.4658% and 6.0303%, respectively. Based on these results, the proposed method provides an excellent estimated price for all ITM cases.

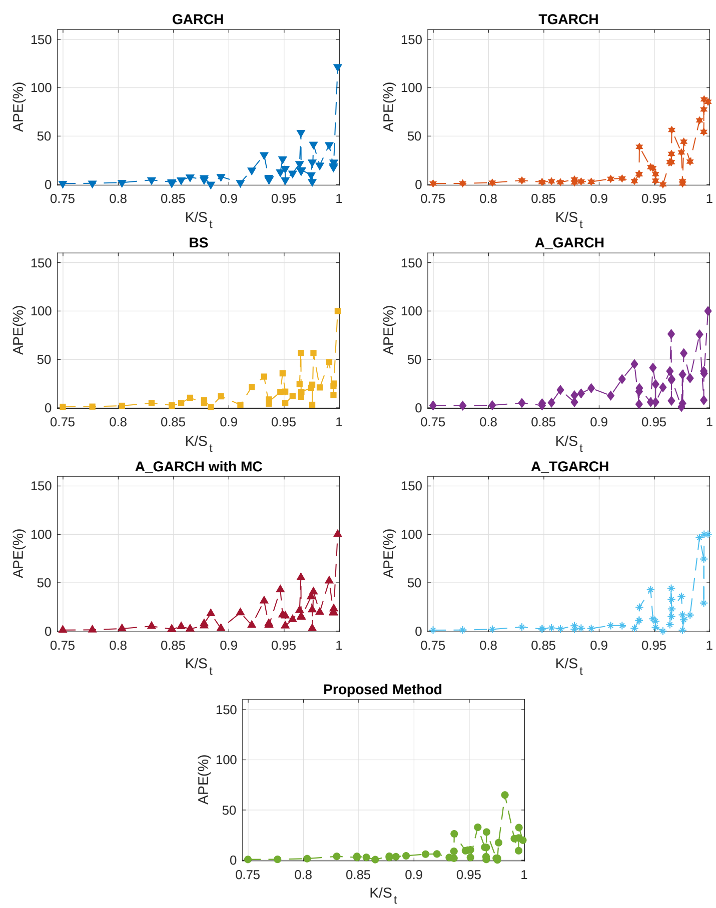


Figure 8: Impact of K/S_t correspondence on call option price in SET50 data

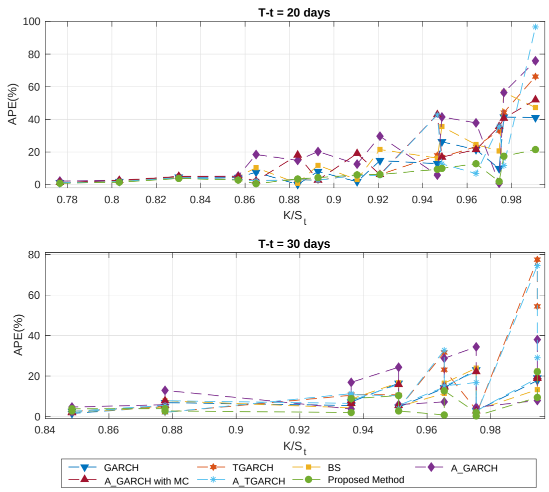


Figure 9: Impact of time to maturity and K/S_t corresponding to option price in SET50.

5 Conclusion

In this study, we developed Option pricing via a fuzzy-TGARCH model. The fuzzy-TGARCH model was mainly used to construct asset volatility models by clustering volatility. These volatility models were used to compute option prices through a Monte Carlo simulation. The numerical results of 37 ITM cases of option pricing were investigated and analyzed. We used data from the Stock Exchange of Thailand from January 2020 to December 2020, in which the market was affected by positive and negative information. These estimated prices show that the fuzzy-TGARCH model outperforms other methods with 10.4899% MAPE. Looking deeper at cases with 20 and 30 days before maturity, the proposed method still significantly outperforms the others by estimating call option prices with lower errors (MAPE equals 6.4658% and 6.0303%, respectively). Based on these numerical results, option pricing via a fuzzy-TGARCH model is one of the better tools for speculator decision in short-term trading. This study contributes both academic knowledge and real-world practice. In terms of academic knowledge, although the Black-Scholes formula is commonly used in option markets, it requires several unrealistic assumptions. The new approach proposed here is more realistic because it relaxes some of those unrealistic conditions. In terms of empirical practice, this approach shows effective performance when applied to real data. It can be seriously considered as an alternative approach for specular investors to determine option pricing.

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