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Poisson approximation for a sum of beta geometric random variables

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Abstract

We use the Stein-Chen method and the beta geometric w-functions to give a bound for the total variation distance between the distribution of a sum of independent beta geometric random variables and a Poisson distribution with mean $\sum_{i=1}^{n} \frac{\beta_i}{\alpha_i - 1}$, where α_i and β_i are parameters of each beta geometric distribution. With this bound, the Poisson distribution with this mean can be used as a good estimate when all β_i are small and all α_i are large.

1 Introduction

Let $X_1, ..., X_n$ be independently distributed beta geometric random variables, each with probability mass funtion $P(X_i = k) = \frac{\alpha_i \Gamma(\beta_i + k)\Gamma(\alpha_i + \beta_i)}{\Gamma(\beta_i)\Gamma(\alpha_i + \beta_i + k + 1)}$, $k \in \mathbb{N} \cup \{0\}$, mean $\mu_i = \frac{\beta_i}{\alpha_i - 1}$ and variance $\sigma_i^2 = \frac{\alpha_i \beta_i(\alpha_i + \beta_i - 1)}{(\alpha_i - 2)(\alpha_i - 1)^2}$, where $\alpha_i > 2$. Let $\mathcal{X} = \sum_{i=1}^{n} X_i$ and let $\mathcal{Z}_\lambda$ denote the the Poisson random variable with mean $\lambda = \sum_{i=1}^{n} \mu_i$. In this paper, we give a bound for approximating the distribution of $\mathcal{X}$ by a Poisson distribution with mean λ, in the form of total variation

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distance

$$d_{TV}(\mathcal{X}, \mathcal{Z}_\lambda) = \sup_{A \subseteq \mathbb{N} \cup \{0\}} |P(\mathcal{X} \in A) - P(\mathcal{Z}_\lambda \in A)|. \tag{1.1}$$

The tools we use are the Stein-Chen method and w-functions, which are in Section 2. In Section 3, we obtain the result using these tools. In Section 4, we conclude our paper.

2 Method

In the following lemma, we present each w-function associated with the corresponding beta geometric random variable.

Lemma 2.1. [2] *For $1 \leq i \leq n$, let w_i be the w-function associated with beta geometric random variable X_i. Then*

$$w_i(k) = \frac{(k+1)(\beta_i + k)}{(\alpha_i - 1)\sigma_i^2}, \; k \in \mathbb{N} \cup \{0\}. \tag{2.2}$$

The Stein-Chen method can be applied [1] for every $\lambda > 0$, every subset A of $\mathbb{N} \cup \{0\}$ and the bounded real valued function $g = g_A : \mathbb{N} \cup \{0\} \to \mathbb{R}$. Thus, Stein's equation for approximating the distribution of $\mathcal{X}$ by the Poisson distribution with mean λ can be written as

$$P(\mathcal{X} \in A) - P(\mathcal{Z}_\lambda \in A) = \mathbb{E}[\lambda g(\mathcal{X}+1) - \mathcal{X} g(\mathcal{X})],$$

which gives

$$d(\mathcal{X}, \mathcal{Z}_\lambda) = |\mathbb{E}[\lambda g(\mathcal{X}+1) - \mathcal{X} g(\mathcal{X})]|, \tag{2.3}$$

where $d(\mathcal{X}, \mathcal{Z}_\lambda) = |P(\mathcal{X} \in A) - P(\mathcal{Z}_\lambda \in A)|$.

For any subset A of $\mathbb{N} \cup \{0\}$ and for every $x \in \mathbb{N}$, Barbour et al. [1] showed that

$$\sup_A |\Delta g(x)| = \sup_A |g(x+1) - g(x)| \leq \frac{1}{x}. \tag{2.4}$$

3 Result

In the following theorem, we present a bound for the total variation distance between the distribution of a sum of independent beta geometric random variables, $P(\mathcal{X} \in A)$, and a Poisson distribution with mean $\lambda = \sum_{i=1}^{n} \mu_i$, $P(\mathcal{Z}_\lambda \in A)$ for every $A \subseteq \mathbb{N} \cup \{0\}$.

Theorem 3.1. *With the definitions mentioned above, we have the following inequality:*

$$d_{TV}(\mathcal{X}, \mathcal{Z}_\lambda) \leq \sum_{i=1}^{n} \frac{\alpha_i \beta_i (\beta_i + 2)}{(\alpha_i - 1)^2 (\alpha_i + \beta_i)}. \tag{3.5}$$

Proof. From (2.3), we have

$$\begin{aligned} d(\mathcal{X}, \mathcal{Z}_\lambda) &= |\mathbb{E}[\lambda g(\mathcal{X}+1) - \mathcal{X} g(\mathcal{X})]| \\ &= |\lambda \mathbb{E}[\Delta g(\mathcal{X})] - \mathbb{C}ov(\mathcal{X}, g(\mathcal{X}))| \\ &= \left| \sum_{i=1}^{n} \mu_i \mathbb{E}[\Delta g(\mathcal{X})] - \sum_{i=1}^{n} \sigma_i^2 \mathbb{E}[w_i(X_i) \Delta g(\mathcal{X})] \right| \text{ (by following [3]).} \end{aligned}$$

By applying (1.1), we have

$$\begin{aligned} d_{TV}(\mathcal{X}, \mathcal{Z}_\lambda) &= \sup_{A \subseteq \mathbb{N} \cup \{0\}} \left| \sum_{i=1}^{n} \mathbb{E}\left\{ [\mu_i - \sigma_i^2 w_i(X_i)] \Delta g(\mathcal{X}) \right\} \right| \text{by (1.1)} \\ &\leq \sum_{i=1}^{n} \sum_{x=1}^{\infty} \left| \frac{\beta_i}{\alpha_i - 1} - \frac{(x+1)(\beta_i + x)}{(\alpha_i - 1)} \right| \frac{1}{x} P(X_i = x) \text{by (2.2) and (2.4)} \\ &= \sum_{i=1}^{n} \sum_{x=1}^{\infty} \frac{x + \beta_i + 1}{\alpha_i - 1} P(X_i = x) \\ &= \sum_{i=1}^{n} \frac{\mu_i + (\beta_i + 1)(1 - P(X_i = 0))}{\alpha_i - 1} \\ &\leq \sum_{i=1}^{n} \frac{\alpha_i \beta_i (\beta_i + 2)}{(\alpha_i - 1)^2 (\alpha_i + \beta_i)}. \end{aligned}$$

which yields the result in (3.5). □

The following Corollary follow immediately from Theorem 3.1:

Corollary 3.1. *If $\alpha_i = \alpha$ and $\beta_i = \beta$ for every $i \in \{1, ..., n\}$, then*

$$d_{TV}(\mathcal{X}, \mathcal{Z}_\lambda) \leq \frac{n\alpha\beta(\beta + 2)}{(\alpha - 1)^2 (\alpha + \beta)}. \tag{3.6}$$

4 Conclusion

The bound for the total variation distance between the distribution of a sum of independent beta geometric random variables and a Poisson distribution was given by using the Stein-Chen method and the beta geometric w-functions. The Poisson distribution with this mean can be used as a good estimate when all β_i are small and all α_i are large.

References

[1] A. D. Barbour, L. Holst, S. Janson, Poisson Approximation. Oxford Studies in Probability 2, 1992.

[2] K. Teerapabolarn, Non uniform bounds on geometric approximation via Stein's method and w-functions. Commun. Stat. Theory Methods, **40**, (2011), 145–158.

[3] P. Kongkam, K. Teerapabolarn, A non-uniform bound on Poisson approximation for a sum of independent non-negative integer-valued random variables. Int. J. Math. Comput. Sci., **18**, no. 2, (2023), 557–562.