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The set chromatic numbers of the middle graph of tree families

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Abstract

The neighborhood color set of each vertex v in a vertex-colored graph G is defined as the collection of the colors of all the neighbors of v. If there are no two adjacent vertices that have equal neighborhood color sets, then the coloring is called a set coloring of G. The set coloring problem on G refers to the problem of determining its set chromatic number, which refers to the fewest colors using which a set coloring of G may be constructed. In this work, we consider the set coloring problem on graphs obtained from applying middle graph, a unary graph operation. The middle graph of G is the graph whose vertex set is the union of $V(G)$ and $E(G)$ and whose edge set is $\{\{u, uv\} : u \in V(G) \text{ and } uv \in E(G)\} \cup \{\{uv_1, uv_2\} : uv_1, uv_2 \in E(G) \text{ and } v_1 \neq v_2\}$. We consider the set coloring problem on the middle graph of different tree families such as brooms, double brooms and caterpillars. We construct set colorings of such graphs using algorithms or explicit formulas. By proving the optimality of these set

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colorings, we obtain the set chromatic number for these different graph families.

1 Introduction

We consider the set coloring problem on the middle graph of different tree families such as brooms, double brooms, and caterpillars. We denote by $\mathbb{N}_k$, where k is some positive integer, the set $\{1, 2, ..., k\}$. We denote by $N_G(v)$ the collection of neighbors of v in a graph/subgraph G. We begin by defining set colorings.

Definition 1.1 ([2]). *Suppose $c : V(G) \to \mathbb{N}$ is a coloring of a graph G. For each vertex v in G, the neighborhood color set $NC(v)$ of v is defined to be the collection of all colors of all neighbors of v. If there are no two adjacent vertices that have equal neighborhood color sets (NC), we say that c is a set coloring. Moreover, the set chromatic number $\chi_s(G)$ of G is defined to be the minimum number of colors using which a set coloring of G may be constructed.*

As stated in [2], the set chromatic number of a graph G is at most its chromatic number. There have been different studies focused on set colorings: [6] dealt mainly with perfect graphs while [3] considered random graphs. There have also been studies in which the set coloring problem is studied in the context of different graph operations. For example, previous works have studied set coloring in relation to corona [2], join [5, 11], comb product [5], total graph [14], and middle graph [4].

Thus, in line with these recent works, this paper aims to continue the work done in [4], in particular, by considering the set coloring problem on the middle graph of different tree families. The graph operation middle graph was introduced in [7] and was defined using the notion of intersection graph. In this paper, we adopt the following equivalent definition: Given a graph G, its *middle graph* $M(G)$ can be obtained by taking the vertex set of $M(G)$ to be the union of $V(G)$ and $E(G)$ and its edge set to be $E(M(G)) = \{\{u, uv\} : u \in V(G) \text{ and } uv \in E(G)\} \cup \{\{uv_1, uv_2\} : uv_1, uv_2 \in E(G) \text{ and } v_1 \neq v_2\}$. In [10], it was established that $\chi(M(G)) = \Delta(G) + 1$, where $\Delta(G) := \max\{\deg v : v \in V(G)\}$. Consequently, $\Delta(G) + 1 \geq \chi_s(M(G))$ as well. A lower bound for $\chi_s(M(G))$, when G has pendant vertices, has also been obtained previously.

Lemma 1.2 ([4]). *Let G be a graph that has at least one vertex with degree 1. For each vertex v in G, set $S(v) = \{w : vw \in E(G) \text{ and } \deg w = 1\}$. Then $\chi_s(M(G)) \geq 1 + \max\{|S(v)| : v \in V(G)\}$.*

Aside from set coloring, there have also been other graph colorings with which middle graph has been studied. For instance, there have been studies on sigma coloring [9], equitable coloring [12], harmonious coloring [1], r-dynamic vertex coloring [8], and irregular coloring [13].

In this paper, we will consider the middle graph of different tree families such as brooms, double brooms and caterpillars. As different references may have different formulations and notations for these graph families, our definitions are given hereunder.

Definition 1.3. *Let s, t, t_1, t_2 be positive integers and $x_1, x_2, ..., x_s$ be non-negative integers. Let $P_s = v_1 v_2 \cdots v_s$ be a path graph with s vertices.*

1. *The broom $B_{s,t}$ is the graph obtained by identifiying an endvertex of the P_s and the central vertex of the star $K_{1,t}$.*
2. *For $t_1 \geq t_2$, the double broom DB_{s,t_1,t_2} is the graph obtained by identifying one endvertex of P_s, where $s \geq 2$, to the central vertex of the star K_{1,t_1} and identifying the other endvertex of P_s to the central vertex of K_{1,t_2}.*
3. *The caterpillar graph $P_s(x_1, x_2, ..., x_s)$ is the tree obtained by appending, for each $i \in \{1, 2, ..., s\}$, x_i pendant vertices to the vertex v_i of P_s.*

2 The set chromatic numbers of the middle graph of brooms and double brooms

We consider the set coloring problem on the middle graph of brooms and double brooms. To this end, we will construct optimal set colorings of these graphs using explicit formulas. Figure 1 and Figure 2 show the middle graph of the broom $B_{s,t}$ and of the double broom DB_{s,t_1,t_2}, respectively.

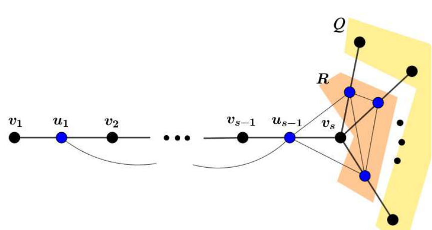

Figure 1: $M(B_{s,t})$

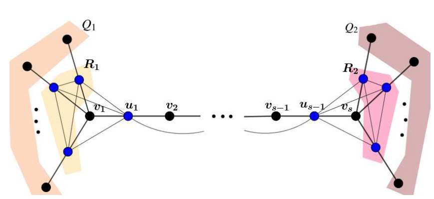

Figure 2: $M(DB_{s,t_1,t_2})$

First, we note that optimal set colorings of the middle graph of path graphs P_n, star graphs $K_{1,m}$, and double-star graphs S_{t_1,t_2}, where $n, m, t_1, t_2 \in \mathbb{N}$, have already been constructed in [4]. Thus, we can already set aside brooms and double brooms that are isomorphic to any of these previously studied graphs. Since the brooms $B_{1,t}, B_{2,t}$, and $B_{s,1}$ are isomorphic to $K_{1,t}, K_{1,t+1}, P_{s+1}$, respectively, we are left to consider brooms $B_{s,t}$ where $s \geq 3$ and $t \geq 2$. Similarly, the double brooms $DB_{2,t_1,t_2}, DB_{s,1,1}$, and $DB_{s,t_1,1}$ are isomorphic to S_{t_1,t_2} (a double-star graph), P_{s+2}, and B_{s+1,t_1}, respectively, so we only need to consider double brooms DB_{s,t_1,t_2} where $s \geq 3$ and $t_1 \geq t_2 \geq 2$.

Our main result on brooms is as follows.

Theorem 2.1. *If $s \geq 3$ and $t \geq 2$, then $\chi_s(M(B_{s,t})) = t + 1$.*

Proof. Refer to Figure 1 for the middle graph of $B_{s,t}$ and the labels we will use to refer to the vertices of $M(B_{s,t})$. Note also the sets R and Q of vertices shown in Figure 1. Observe that in $B_{s,t}$, the vertex v_s has one nonpendant neighbor v_{s-1} and $|Q| = t$ pendant neighbors. Hence, by Lemma 1.2, we have $\chi_s(M(B_{s,t})) \geq t + 1$.

We will now prove that the coloring $c : V(M(B_{s,t})) \to \mathbb{N}_{t+1}$ defined by the following procedure is a set coloring.

1. Set $c(Q) = \mathbb{N}_{t+1} \setminus \{1\}$.
2. Set $c(R \cup \{v_s\}) = \{1\}$.
3. (a) If $s \equiv 0 \pmod 3$, set

$$c(v_k) = \begin{cases} 3, & i \equiv 0 \pmod 3 \text{ \& } i < s, \\ 2, & i \equiv 2 \pmod 3, \\ 1 & i \equiv 1 \pmod 3, \end{cases} \qquad c(u_k) = \begin{cases} 3, & i \equiv 1 \pmod 3, \\ 2, & i \equiv 0 \pmod 3, \\ 1, & i \equiv 2 \pmod 3. \end{cases}$$

(b) If $s \equiv 1 \pmod 3$, set

$$c(v_i) = \begin{cases} 3, & i \equiv 1 \pmod 3 \text{ \& } i < s, \\ 2, & i \equiv 0 \pmod 3, \\ 1, & i \equiv 2 \pmod 3, \end{cases} \qquad c(u_i) = \begin{cases} 3, & i \equiv 2 \pmod 3, \\ 2, & i \equiv 1 \pmod 3, \\ 1, & i \equiv 0 \pmod 3. \end{cases}$$

(c) If $s \equiv 2 \pmod 3$, set

$$c(v_i) = \begin{cases} 3, & i \equiv 2 \pmod 3 \text{ \& } i < s, \\ 2, & i \equiv 1 \pmod 3, \\ 1, & i \equiv 0 \pmod 3, \end{cases} \qquad c(u_i) = \begin{cases} 3, & i \equiv 0 \pmod 3, \\ 2, & i \equiv 2 \pmod 3, \\ 1, & i \equiv 1 \pmod 3. \end{cases}$$

It is clear that c uses exactly $t + 1$ colors. Observe that for any $s \geq 3$, we have $c(u_{s-1}) = 1, c(v_{s-1}) = 2$, and $c(u_{s-2}) = 3$. Moreover, for $i \in \{2, 3, ..., s-2\}$, we have $c(u_{i-1}) = c(v_{i+1})$ while for $i \in \{1, 2, ..., s-2\}$, we have $c(v_i) = c(u_{i+1})$. Using these properties, we construct Table 1, which presents the NCs of the vertices of $M(B_{s,t})$.

Table 1: The NC of each vertex of $M(B_{s,t})$

Vertex	Neighbors	NC
$v \in S$	$r \in R$ w/ $vr \in E(M(B_{s,t}))$	$\{1\}$
$r \in R$	$v \in S$ w/ $vr \in E(M(B_{s,t}))$; $R \setminus \{r\}$; v_s; u_{s-1}	$\{c(v), 1\}$
v_s	$u_{s-1}; R$	$\{1\}$
u_{s-1}	$v_{s-1}; u_{s-2}; v_s; R$	$\{1, 2, 3\}$
$v_i, i \in \{2, ..., s-1\}$	$u_{i-1}; u_i$	$\{c(u_{i-1}), c(u_i)\}$
$u_i, i \in \{2, ..., s-2\}$	$u_{i-1}; v_i; v_{i+1}; u_{i+1}$	$\{c(v_i), c(v_{i+1})\}$
u_1	$v_1; v_2; u_2$	$\{c(v_1), c(v_2)\}$
v_1	u_1	$\{c(u_1)\}$

From Table 1, we see that there are no two adjacent vetices of $M(B_{s,t})$ that have equal neighborhoood color sets. □

Our main result on double brooms is as follows.

Theorem 2.2. *If $s \geq 3$ and $t_1 \geq t_2 \geq 2$ such that $t_1 \geq 3$, then*

$$\chi_s(M(DB_{s,t_1,t_2})) = t_1 + 1.$$

Proof. Refer to Figure 2 for the middle graph of DB_{s,t_1,t_2} and the labels we will use to refer to the vertices of $M(DB_{s,t_1,t_2})$. Note also the sets R_1, R_2, Q_1, and Q_2 of vertices shown in Figure 2. Observe that in DB_{s,t_1,t_2}, the vertex v_1 has one nonpendant neighbor v_2 and $|Q_1| = t_1$ pendant neighbors. Hence, by Lemma 1.2, we have $\chi_s(M(DB_{s,t_1,t_2})) \geq t_1 + 1$.

To complete the proof, we will show that the coloring $c : V(M(DB_{s,t_1,t_2})) \to \mathbb{N}_{t_1+1}$ defined by the following procedure is a set coloring.

1. Set $c(Q_2) = \mathbb{N}_{|S_2|+1} \setminus \{1\}$.
2. Set $c(R_2 \cup \{v_s\}) = \{1\}$.
3. (a) If $s \equiv 0 \pmod 3$, set

$$c(v_i) = \begin{cases} 3, & i \equiv 0 \pmod 3 \ \&\ i < s, \\ 2, & i \equiv 2 \pmod 3, \\ 1, & i \equiv 1 \pmod 3 \ \&\ i > 1, \end{cases} \qquad c(u_i) = \begin{cases} 3, & i \equiv 1 \pmod 3 \ \&\ i > 1, \\ 2, & i \equiv 0 \pmod 3, \\ 1, & i \equiv 2 \pmod 3. \end{cases}$$

(b) If $s \equiv 1 \pmod 3$, set

$$c(v_i) = \begin{cases} 3, & i \equiv 1 \pmod 3 \ \&\ 1 < i < s, \\ 2, & i \equiv 0 \pmod 3, \\ 1, & i \equiv 2 \pmod 3, \end{cases} \qquad c(u_i) = \begin{cases} 3, & i \equiv 2 \pmod 3, \\ 2, & i \equiv 1 \pmod 3 \ \&\ i > 1, \\ 1, & i \equiv 0 \pmod 3. \end{cases}$$

(c) If $s \equiv 2 \pmod 3$, set

$$c(v_i) = \begin{cases} 3, & i \equiv 2 \pmod 3 \ \&\ i < s, \\ 2, & i \equiv 1 \pmod 3 \ \&\ i > 1, \\ 1, & i \equiv 0 \pmod 3, \end{cases} \qquad c(u_i) = \begin{cases} 3, & i \equiv 0 \pmod 3, \\ 2, & i \equiv 2 \pmod 3, \\ 1, & i \equiv 1 \pmod 3 \ \&\ i > 1. \end{cases}$$

4. Set $c(\{v_1, u_1\}) = \{t_1 + 1\}$.
5. Fix one vertex $y \in R_1$ and set $c(y) = t_1$. Set $c(R_1 \setminus \{y\}) = \{t_1 + 1\}$.
6. For $z \in Q_1$ such that $yz \in E(M(DB_{s,t_1,t_2}))$, set $c(z) = 1$. Then set $c(Q_1 \setminus \{z\}) = \mathbb{N}_{t_1-1}$.

It is clear that c uses exactly $t_1 + 1$ colors. Moreover, observe that $c(u_{s-1}) = 1$ for all $s \geq 3$, that $c(u_{i-1}) = c(v_{i+1})$ for all $i \in \{3, 4, ..., s-2\}$, and that $c(v_i) = c(u_{i+1})$ for all $i \in \{2, 3, ..., s-2\}$. Using these properties, we construct Table 2, which presents the NCs of the vertices of $M(DB_{s,t_1,t_2})$. From Table 2, we see that there are no two adjacent vetices of $M(DB_{s,t_1,t_2})$ that have equal neighborhoood color sets. $\square$

Table 2: The NC of each vertex of $M(DB_{s,t_1,t_2})$

Vertex	Neighbors	NC
$v \in S_2$	$r \in R_2$ w/ $vr \in E(M(DB_{s,t_1,t_2}))$	$\{1\}$
$r \in R_2$	$v \in S_2$ w/ $vr \in E(M(DB_{s,t_1,t_2}))$; $R_2 \setminus \{r\}$; $v_s; u_{s-1}$	$\{c(v), 1\}$
v_s	$R_2; u_{s-1}$	$\{1\}$
u_{s-1}	$u_{s-2}; v_{s-1}; v_s; R_2$	$\{1, 2, 3\}$
v_i, $i \in \{2, ..., s-1\}$	$u_{i-1}; u_i$	$\{c(u_{i-1}), c(u_i)\}$
u_i, $i \in \{3, ..., s-2\}$	$u_{i-1}; v_i; v_{i+1}; u_{i+1}$	$\{c(v_i), c(v_{i+1})\}$
u_2	$u_1; v_2; v_3; u_3$	$\{t_1 + 1, c(v_2), c(v_3)\}$
v_1	$R_1; u_1$	$\{t_1, t_1 + 1\}$
u_1	$R_1; v_1; v_2; u_2$	$\{t_1, t_1 + 1, c(v_2), c(u_2)\}$
y	$z; R_1 \setminus \{y\}; v_1; u_1$	$\{1, t_1 + 1\}$
z	y	$\{t_1\}$
$r' \in R_1 \setminus \{y\}$	$v' \in S_1 \setminus \{z\}$ w/ $r'v' \in E(M(DB_{s,t_1,t_2}))$; $R_1 \setminus \{r'\}$; $v_1; u_1$	$\{t_1, t_1 + 1, c(v')\}$
$v' \in S_1 \setminus \{z\}$	$r' \in R_1 \setminus \{y\}$ w/ $r'v' \in E(M(DB_{s,t_1,t_2}))$	$\{c(r')\}$

Figures 3 and 4 show examples of the set colorings constructed in the proofs of Theorems 2.1 and 2.2, respectively.

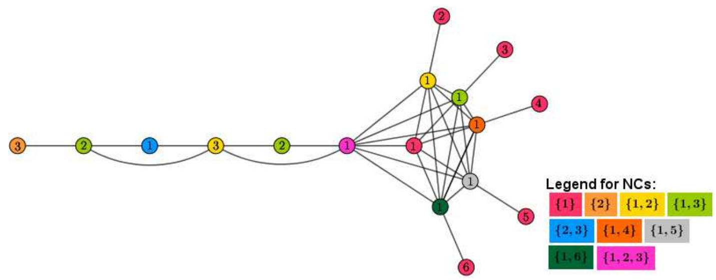

Figure 3: $M(B_{4,5})$ with a set coloring constructed as in the proof of Theorem 2.1

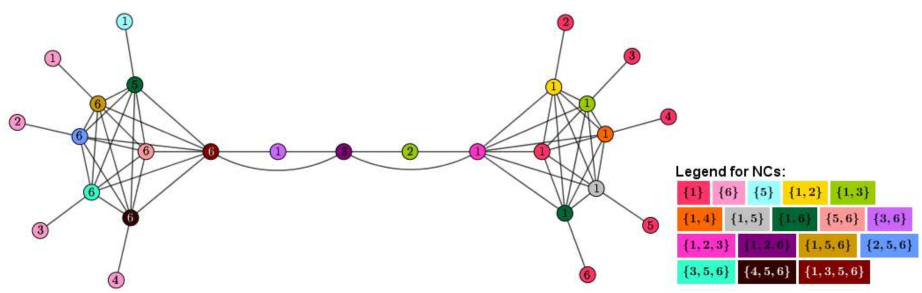

Figure 4: $M(DB_{4,5,5})$ with a set coloring constructed as in the proof of Theorem 2.2

3 The set chromatic numbers of the middle graph of caterpillar graphs $P_4(x_1, x_2, x_3, x_4)$

We now consider caterpillar graphs $G = P_4(x_1, x_2, x_3, x_4)$. We first discuss some special cases. If $x_2 = x_3 = 0$ and $x_1, x_4 \in \{0, 1\}$, then G is isomorphic to a path graph. If $x_1 = x_4 = 0$, $x_2 \geq 1$, and $x_3 \geq 1$, then G is isomorphic to a double-star graph. If $x_1 = x_2 = x_3 = 0$ and $x_4 \geq 2$, then G is a broom that is not a path. If $x_2 = x_3 = 0$ and $x_1 \geq x_4 \geq 2$, then G is a double-broom that is neither a broom nor a path. We now state our main result for a general family of caterpillar graphs $P_4(x_1, x_2, x_3, x_4)$.

Theorem 3.1. *Let $G = P_4(x_1, x_2, x_3, x_4)$. Let $i \in \mathbb{N}_4$ such that $x_i = \max\{x_1, x_2, x_3, x_4\}$. If $x_1 \geq 1, x_4 \geq 1, x_i > x_j$ for all $j \neq i$, and $x_i \geq 4$, then $\chi_s(M(G)) = x_i + 1$.*

Proof. Let $S(v)$ be as introduced in Lemma 1.2. Clearly, $|S(v_j)| = x_j$ for each $j \in \{1, 2, 3, 4\}$. By Lemma 1.2, we must have $\chi_s(M(G)) \geq x_i + 1$.

We now show that $\chi_s(M(G)) \leq x_i + 1$ by algorithmically constructing a set coloring of $M(G)$ for which the number of colors used is $x_i + 1$. We present two algorithms, one each for two of the following cases. For Case 1, we may verify that c is indeed a set coloring using the NCs given in Table 3. A similar table may be constructed for Case 2. In either case, the colorings constructed are set colorings that use $x_i + 1$ colors.

Therefore, we must have $\chi_s(M(G)) = x_i + 1$.

Case 1. Suppose $i \in \{2, 3\}$. We may assume that $i = 2$. We define $c_1 : V(M(G)) \to \mathbb{N}_{x_2+1}$ using the following algorithm:

$c_1(v_1) \leftarrow 2$; $c_1(v_1v_2) \leftarrow 1$
$c_1(v_1v_{1,1}) \leftarrow 1$; $c_1(v_{1,1}) \leftarrow x_2 + 1$
for $j \in \mathbb{N}_{x_1} \setminus \{1\}$ **do**
    $c(v_1v_{1,j}) \leftarrow j + 1$; $c(v_{1,j}) \leftarrow 1$
end for
$c_1(v_2) \leftarrow 1$; $c_1(v_2v_3) \leftarrow 1$
for $j \in \mathbb{N}_{x_2}$ **do**
    $c(v_2v_{2,j}) \leftarrow 1$; $c(v_{2,j}) \leftarrow j + 1$
end for
$c_1(v_3) \leftarrow 2$
if $x_3 > 0$ **then**
    $c_1(v_3v_4) \leftarrow 1$; $c_1(v_3v_{3,1}) \leftarrow x_2$; $c_1(v_{3,1}) \leftarrow 1$
    Set $\mathbb{N}_{x_2+1} \setminus \{1, 2, x_2\} = \{\alpha_1, \alpha_2, ..., \alpha_{x_2-2}\}$.
    for $j \in \mathbb{N}_{x_3} \setminus \{1\}$ **do**
        $c_1(v_3, v_{3,j}) \leftarrow 1$; $c_1(v_{3,j}) \leftarrow \alpha_{j-1}$
    end for
else
    $c_1(v_3v_4) \leftarrow x_2 + 1$
end if
$c_1(v_4) \leftarrow x_2 + 1$; $c_1(v_4v_{4,1}) \leftarrow x_2 - 1$; $c_1(v_{4,1}) \leftarrow 1$
Set $\mathbb{N}_{x_2+1} \setminus \{2, x_2 \pm 1\} = \{\beta_1, \beta_2, ..., \beta_{x_2-2}\}$.
if $x_4 \geq 2$ **then**
    for $j \in \mathbb{N}_{x_4} \setminus \{1\}$ **do**
        $c_1(v_4v_{4,j}) \leftarrow 1$; $c_1(v_{4,j}) \leftarrow \beta_{j-1}$
    end for
end if

Case 2. Suppose $i \in \{1, 4\}$. We may assume that $i = 1$. We define $c_2 : V(M(G)) \to \mathbb{N}_{x_1+1}$ using the following algorithm:

$c_2(v_1) \leftarrow 1$; $c_2(v_1v_2) \leftarrow 1$
$c_2(v_1v_{1,1}) \leftarrow 1$; $c_2(v_{1,1}) \leftarrow x_1 + 1$
$c_2(v_1v_{1,2}) \leftarrow 1$; $c_2(v_{1,2}) \leftarrow x_1$
for $j \in \mathbb{N}_{x_1} \setminus \{1, 2\}$ **do**
    $c_2(v_1v_{1,j}) \leftarrow j - 1$; $c_2(v_{1,j}) \leftarrow 1$
end for
$c_2(v_2) \leftarrow x_1$; $c_2(v_2v_3) \leftarrow x_1 + 1$
for $j \in \mathbb{N}_{x_2}$ **do**
    $c_2(v_2v_{2,j}) \leftarrow 1$; $c_2(v_{2,j}) \leftarrow j + 1$
end for
$c_2(v_3) \leftarrow 1$; $c_2(v_3v_4) \leftarrow 1$
for $j \in \mathbb{N}_{x_3}$ **do**
    $c_2(v_3v_{3,j}) \leftarrow 1$; $c_3(v_{3,j}) \leftarrow j + 1$
end for
$c_2(v_4) \leftarrow x_1$
$c_2(v_4v_{4,1}) \leftarrow x_1 - 1$; $c_2(v_{4,1}) \leftarrow 1$
for $j \in \mathbb{N}_{x_4} \setminus \{1\}$ **do**
    $c_2(v_4v_{4,j}) \leftarrow 1$; $c_2(v_{4,j}) \leftarrow j - 1$
end for

□

Figure 5 shows a set coloring generated as in Case 1 under Theorem 3.1.

Table 3: The NC of each vertex of $M(G)$, under the coloring c_1 in Case 1

Vertex	Neighbors	NC
v_1	$v_1v_2; v_1v_{1,j}\ \forall j \in \mathbb{N}_{x_1}$	$\mathbb{N}_{x_1+1} \setminus \{2\}$
v_1v_2	$v_1; v_2; v_1v_{1,j}\ \forall j \in \mathbb{N}_{x_1};$ $v_2v_{2,j}\ \forall j \in \mathbb{N}_{x_2}; v_2v_3$	$\mathbb{N}_{x_1+1}$
$v_1v_{1,j}$	If $j=1$: $v_1; v_{1,1};\ v_1v_{1,h}\ \forall h \neq 1;\ v_1v_2$	$\mathbb{N}_{x_1+1} \cup \{x_2+1\}$
	If $j>1$: $v_1; v_{1,j};\ v_1v_{1,h}\ \forall h \neq j;\ v_1v_2$	$\mathbb{N}_{x_1+1} \setminus \{j+1\}$
$v_{1,j}$	If $j=1$: $v_1v_{1,1}$	$\{1\}$
	If $j>1$: $v_1v_{1,j}$	$\{j+1\}$
v_2	$v_1v_2; v_2v_3;\ v_2v_{2,j}\ \forall j \in \mathbb{N}_{x_2}$	$\{1\}$
v_2v_3	$v_2; v_3; v_1v_2; v_3v_4;$ $v_2v_{2,j}\ \forall j \in \mathbb{N}_{x_2};\ v_2v_{3,h}\ \forall h \in \mathbb{N}_{x_3}$	$x_3>0: \{1,2,x_2\};$ $x_3=0: \{1,2,x_2+1\}$
$v_2v_{2,j}$	$v_2; v_{2,j}; v_1v_2; v_2v_3;\ v_2v_{2,h} \forall h \neq j$	$\{1, j+1\}$
$v_{2,j}$	$v_2v_{2,j}$	$\{1\}$
v_3	$v_2v_3; v_3v_4;$ $v_3v_{3,j}\ \forall j \in \mathbb{N}_{x_3}$	$x_3=0: \{1, x_2+1\};$ $x_3>0: \{1, x_2\}$
v_3v_4	$v_3; v_4; v_2v_3;$ $v_3v_{3,j}\ \forall j \in \mathbb{N}_{x_3};\ v_4v_{4,h}\ \forall h \in \mathbb{N}_{x_4}$	$x_3=0: \{1,2,x_2 \pm 1\};$ $x_3>0: \{1,2,x_2 \pm 1, x_2\}$
$v_3v_{3,j}$	If $j=1$: $v_3;\ v_3v_{3,h}\ \forall h; v_2v_3; v_3v_4$	Needs $x_3 \geq 1: \{1,2\}$
	If $j>1$: $v_3; v_{3,j};$ $v_3v_{3,h}\ \forall h \neq j; v_2v_3; v_3v_4$	Needs $x_3 \geq 2: \{1,2,x_2,\alpha_{j-1}\}$, where $\alpha_{j-1} \notin \{1,2,x_2\}$
$v_{3,j}$	If $j=1$: $v_3v_{3,1}$	Needs $x_3 \geq 1:\ \{x_2\}$
	If $j>1$: $v_3v_{3,j}$	Needs $x_3 \geq 2:\ \{1\}$
v_4	$v_3v_4; v_4v_{4,j}\ \forall j \in \mathbb{N}_{x_4}$	If $x_3=0:\ \{x_2 \pm 1\}$ if $x_4=1$ or $\{1, x_2 \pm 1\}$ if $x_4 \geq 2$
		If $x_3>0:\ \{1, x_2-1\}$
$v_4v_{4,j}$	If $j=1$: $v_4; v_{4,1};\ v_4v_{4,h}\ \forall h \neq 1; v_3v_4$	$\{1, x_2+1\}$
	If $j>1$: $v_4; v_{4,j};$ $v_4v_{4,h}\ \forall h \neq j; v_3v_4$	$\{1, x_2 \pm 1, \beta_{j-1}\}$, where $\beta_{j-1} \in \mathbb{N}_{x_2+1} \setminus \{2, x_2 \pm 1\}$
$v_{4,j}$	If $j=1$: $v_4v_{4,1}$	$\{x_2-1\}$
	If $j>1$: $v_4v_{4,j}$	$\{1\}$

4 Conclusion

As a continuation of previous works, this paper focused on the set coloring problem on the middle graph of different tree families such as brooms, double brooms, and caterpillars. We constructed set colorings of such graphs using algorithms or explicit formulas. By proving the optimality of these set

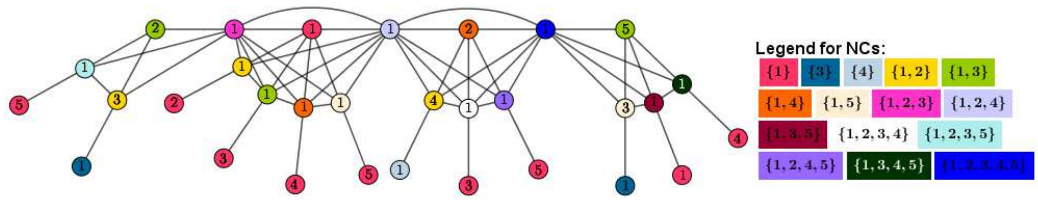

Figure 5: A set 5-coloring of $M(P_4(2, 4, 3, 3))$

colorings, we obtained the set chromatic number for these different graph families.

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