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Degree Subtraction Energy of Commuting and Non-Commuting Graphs for Dihedral Groups

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Abstract

Let $\bar{\Gamma}_G$ and Γ_G be the commuting and non-commuting graphs on a finite group G, respectively, having $G\backslash Z(G)$ as the vertex set, where $Z(G)$ is the center of G. The order of $\bar{\Gamma}_G$ and Γ_G is $|G\backslash Z(G)|$, denoted by m. For Γ_G, the edge joining two distinct vertices $v_p, v_q \in G\backslash Z(G)$ if and only if $v_p v_q \neq v_q v_p$, on the other hand, whenever they commute in G, v_p and v_q are adjacent in $\bar{\Gamma}_G$. The degree subtraction matrix (DSt) of Γ_G is denoted by $DSt(\Gamma_G)$, so that its $(p,q)-$entry is equal to $d_{v_p} - d_{v_q}$, if $v_p \neq v_q$, and zero if $v_p = v_q$, where d_{v_p} is the degree of v_p. For $i = 1, 2, \ldots, m$, the maximum of $|\lambda_i|$ as the $DSt-$spectral radius of Γ_G and the sum of $|\lambda_i|$ as $DSt-$energy of Γ_G, where λ_i are the eigenvalues of $DSt(\Gamma_G)$. These notations can be applied analogously to the degree subtraction matrix of the commuting graph, $DSt(\bar{\Gamma}_G)$. Throughout

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this paper, we provide $DSt-$spectral radius and $DSt-$energy of Γ_G and $\bar{\Gamma}_G$ for dihedral groups of order $2n$, where $n \geq 3$. We then present the correlation of the energies and their spectral radius.

1 Introduction

There are many types of graphs whose vertices are elements of a finite group G and two vertices will be linked by an edge subject to the type of graph constructed. In this paper we are concerned with the commuting and non-commuting graphs, having $G \backslash Z(G)$ as its vertices, where $Z(G)$ is the center of G. The non-commuting graph, denoted by Γ_G, with the edge joining two distinct vertices $v_p, v_q \in G \backslash Z(G)$ if and only if $v_p v_q \neq v_q v_p$ [1]. On the other hand, the commuting graph, $\bar{\Gamma}_G$, is the complement of Γ_G with v_p and v_q are joined by an edge whenever $v_p v_q = v_q v_p$ [2]. Here Γ_G and $\bar{\Gamma}_G$ are considered finite, simple, and undirected and their order is $|G \backslash Z(G)|$, denoted by m.

Research on commuting and non-commuting graphs have developed in algebraic graph theory through the years. Several works on the commuting and non-commuting graphs especially for dihedral groups can be seen in [3, 4, 5, 6], which discusses the spectral and energy problem using the spectrum of various matrices associated with Γ_G and $\bar{\Gamma}_G$. The analogous concept of commuting graph for finite non-abelian groups, the spectrum associated with the adjacency matrix is given in [7]. Also, the ordinary spectrum and energy of Γ_G for finite groups inclusive of dihedral groups can be found in [8].

The energy of graph concept was introduced by Gutman in 1978 [9] whose definition relates to the ordinary graph spectrum of the adjacency matrix. This motivates the researchers to study the various graph energies involving different matrices, such as the degree subtraction energy of a graph. Ramane et al. [10] introduced this definition in 2018, the $m \times m$ degree subtraction matrix (DSt) of Γ_G, defined as $DSt(\Gamma_G) = [dst_{pq}]$, where

$$dst_{pq} = \begin{cases} d_{v_p} - d_{v_q}, & \text{if } v_p \neq v_q \\ 0, & \text{if } v_p = v_q, \end{cases}$$

and d_{v_i} be the degree of a vertex $v_i \in G \backslash Z(G)$, for $i = 1, 2, \ldots, m$.

The $DSt-$eigenvalues of $DSt(\Gamma_G)$ denoted by $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_m$ are the roots of the characteristic polynomial of $DSt(\Gamma_G)$, $P_{DSt(\Gamma_G)}(\lambda) = det(\lambda I_m - DSt(\Gamma_G))$, where I_m is an $m \times m$ identity matrix. For $i = 1, 2, \ldots, m$, the maximum of $|\lambda_i|$ is the $DSt-$spectral radius of Γ_G, denoted by $\rho_{DSt}(\Gamma_G)$. The $DSt-$spectrum of Γ_G is denoted by $Spec(\Gamma_G) = \left\{\lambda_1^{k_1}, \lambda_2^{k_2}, \ldots \lambda_m^{k_m}\right\}$ where k_i

are the respective multiplicities of λ_i [11]. Now, $DSt-$energy of Γ_G is defined as $E_{DSt}(\Gamma_G) = \Sigma_{i=1}^{m} |\lambda_i|$. Moreover, the above notations can be applied analogously to the degree subtraction matrix of the commuting graph, $DSt(\bar{\Gamma}_G)$.

Throughout this note, we focus on dihedral groups of order $2n$ for $n \geq 3$, written as

$$D_{2n} = \left\langle a, b : a^n = b^2 = e, bab = a^{-1} \right\rangle,$$

and $Z(D_{2n})$ is the center of D_{2n} defined as $\{e\}$ if n is odd and $\left\{e, a^{\frac{n}{2}}\right\}$ for the even n. The centralizer of the element a^i in D_{2n} is $C_{D_{2n}}(a^i) = \{a^j : 1 \leq j \leq n\}$ and for the element a^ib is either $C_{D_{2n}}(a^ib) = \{e, a^ib\}$, if n is odd, or $C_{D_{2n}}(a^ib) = \left\{e, a^{\frac{n}{2}}, a^ib, a^{\frac{n}{2}+i}b\right\}$, if n is even.

2 Preliminaries

We need some properties for constructing the degree subtraction matrix Γ_G and $\bar{\Gamma}_G$ for $G = G_1 \cup G_2$, where $G_1 = \{a^i : 1 \leq i \leq n\} \backslash Z(D_{2n})$ and $G_2 = \{a^ib : 1 \leq i \leq n\}$. Several results on the vertex degree of Γ_G and $\bar{\Gamma}_G$ are given in Theorem 2.1 and 2.3. The isomorphism of Γ_G and $\bar{\Gamma}_G$ with some types of common graphs are presented in Theorem 2.2 and 2.4.

Theorem 2.1. *[12] Let* Γ_G *be the non-commuting graph on* $G = G_1 \cup G_2$. *Then*

1. *the degree of* a^i *in* Γ_G *is* $d_{a^i} = n$, *and*
2. *the degree of* a^ib *in* Γ_G *is* $d_{a^ib} = \begin{cases} 2(n-1), & \text{if } n \text{ is odd} \\ 2(n-2), & \text{if } n \text{ is even.} \end{cases}$

Theorem 2.2. *[12] Let* Γ_G *be the non-commuting graph on* $G = G_1 \cup G_2$.

1. *If* $G = G_1$, *then* $\Gamma_G \cong \bar{K}_m$, *where* $m = |G_1|$.
2. *If* $G = G_2$, *then* $\Gamma_G \cong \begin{cases} K_n, & \text{if } n \text{ is odd} \\ K_n - \frac{n}{2}K_2, & \text{if } n \text{ is even,} \end{cases}$

for a complete graph K_n *on* n *vertices with* $\bar{K}_n$ *is the complement of* K_n *where* $\frac{n}{2}K_2$ *denotes* $\frac{n}{2}$ *copies of* K_2.

Theorem 2.3. *[5] Let* $\bar{\Gamma}_G$ *be the commuting graph on* $G = G_1 \cup G_2$. *Then*

1. *the degree of* a^i *in* $\bar{\Gamma}_G$ *is* $d_{a^i} = \begin{cases} n-2, & \text{if } n \text{ is odd} \\ n-3, & \text{if } n \text{ is even,} \end{cases}$ *and*

2. *the degree of* a^ib *in* $\bar{\Gamma}_G$ *is* $d_{a^ib} = \begin{cases} 0, & \text{if } n \text{ is odd} \\ 1, & \text{if } n \text{ is even.} \end{cases}$

Theorem 2.4. *[5] Let* $\bar{\Gamma}_G$ *be the commuting graph for* G.

1. *If* $G = G_1$, *then* $\bar{\Gamma}_G \cong K_m$, *where* $m = |G_1|$.
2. *If* $G = G_2$, *then* $\bar{\Gamma}_G \cong \begin{cases} \bar{K}_n, & \text{if } n \text{ is odd} \\ 1-\text{regular graph}, & \text{if } n \text{ is even.} \end{cases}$

The following lemma is used to assist in the determination of the characteristic polynomial of Γ_G and $\bar{\Gamma}_G$ for $G = G_1 \cup G_2$.

Lemma 2.5. *[13] If* a, b, c *and* d *are real numbers, and* J_n *is an* $n \times n$ *matrix whose all entries are equal to one, then the determinant of the* $(n_1 + n_2) \times (n_1 + n_2)$ *matrix of the form*

$$\begin{vmatrix} (\lambda + a)I_{n_1} - aJ_{n_1} & -cJ_{n_1 \times n_2} \\ -dJ_{n_2 \times n_1} & (\lambda + b)I_{n_2} - bJ_{n_2} \end{vmatrix}$$

can be simplified in an expression as

$$(\lambda + a)^{n_1 - 1}(\lambda + b)^{n_2 - 1}\left((\lambda - (n_1 - 1)\, a)\, (\lambda - (n_2 - 1)\, b) - n_1 n_2 cd\right),$$

where $1 \leq n_1,\ n_2 \leq n$ *and* $n_1 + n_2 = n$.

3 Main Results

In this section, we start with finding the degree subtraction energy of commuting and non-commuting graphs, $\bar{\Gamma}_G$ and Γ_G for $G = G_1$ and $G = G_2$.

Theorem 3.1. *Let* $\bar{\Gamma}_G$ *and* Γ_G *be the commuting and non-commuting graphs on* G, *respectively. For* $G = G_1$ *or* G_2, *then*

$$E_{DSt}(\Gamma_G) = E_{DSt}(\bar{\Gamma}_G) = 0.$$

Proof. 1. Let $G = G_1$ and m being the number of elements in G_1. Hence, $m = n - 1$ for odd n, and $m = n - 2$ for even n. Consequently, from Theorem 2.2 (1), the non-commuting graph $\Gamma_G \cong \bar{K}_m$ implies every vertex of Γ_G has degree zero. On the contrary, every vertex of the commuting graph $\bar{\Gamma}_G \cong K_m$ has a degree of either $n - 2$ for odd n, or $n - 3$ for even n. However, by the definition of the degree subtraction matrix, not only the diagonal entries of $DSt(\bar{\Gamma}_G)$ are zero, but all non-diagonal entries are also zero, since $(n-2)-(n-2)=0=(n-3)-(n-3)$. Then, evidently $DSt(\Gamma_G)=DSt(\bar{\Gamma}_G)=[0]$. It follows that zero is the only eigenvalue of $DSt(\Gamma_G)$ and $DSt(\bar{\Gamma}_G)$. Therefore, $E_{DSt}(\Gamma_G)=E_{DSt}(\bar{\Gamma}_G)=0$.

2. When $G=G_2$ and n is odd, Theorem 2.2 (2) gives $\Gamma_G \cong K_n$, which means the degree of each vertex is $n-1$. Consequently, the $(p,q)-$th entry of $DSt(\Gamma_G)$ is $(n-1)-(n-1)=0$, for $v_p \neq v_q$, and it is zero for $v_p = v_q$. Moreover, due to the fact that $\bar{\Gamma}_G \cong \bar{K}_m$ by Theorem 2.4 (2), all entries of $DSt(\bar{\Gamma}_G)$ are also zero. Hence, both $DSt(\Gamma_G)$ and $DSt(\bar{\Gamma}_G)$ are zero matrices. Thus, $E_{DSt}(\Gamma_G)=E_{DSt}(\bar{\Gamma}_G)=0$. Now for the even n case, as it is known from Theorem 2.2 (2), $\Gamma_G \cong K_n - \frac{n}{2}K_2$, which implies d_{a^ib} is $n-2$. Following the definition of the degree subtraction matrix of Γ_G, we know that the non-diagonal entries of $DSt(\Gamma_G)$ are $(n-2)-(n-2)=0$ and zero for the diagonal entries. Similarly, all of the entries of $DSt(\bar{\Gamma}_G)$ are also zero, because the commuting graph $\bar{\Gamma}_G$ is a regular graph with degree one and so $1-1=0$, for $v_p \neq v_q$, and it is zero for $v_p = v_q$. Then, in the same manner, as in the odd n case, we obtain $E_{DSt}(\Gamma_G)=E_{DSt}(\bar{\Gamma}_G)=0$.

□

In the next two theorems, we formulate the characteristic polynomial of $DSt(\bar{\Gamma}_G)$ and $DSt(\Gamma_G)$ for $G=G_1 \cup G_2$.

Theorem 3.2. *Let Γ_G be non-commuting graphs on G, where $G=G_1 \cup G_2 \subset D_{2n}$, then the characteristic polynomial of the degree subtraction matrix of Γ_G is*

1. $P_{DSt(\Gamma_G)}(\lambda)=\lambda^{2n-3}\,(\lambda^2+n(n-1)(n-2)^2)$, *for odd n, and*

2. $P_{DSt(\Gamma_G)}(\lambda)=\lambda^{2(n-2)}\,(\lambda^2+n(n-2)(n-4)^2)$, *for even n.*

Proof. 1. The first proof for the odd n, we know that $Z(D_{2n})=\{e\}$. So Γ_G has $2n-1$ vertices where $G=G_1 \cup G_2$. We write the set G_1 as $\{a,a^2,\ldots,a^{n-1}\}$ and G_2 as $\{b,ab,a^2b,\ldots,a^{n-1}b\}$. Considering Theorem 2.1 we get that $d_{a^i}=n$ and $d_{a^ib}=2(n-1)$, for all $i=1,2,\ldots,n$. Now the degree subtraction matrix of Γ_G is the $(2n-1)\times$

$(2n-1)$ matrix,

$$DSt(\Gamma_G) = \begin{array}{c} \\ a \\ \vdots \\ a^{n-1} \\ b \\ \vdots \\ a^{n-1}b \end{array} \begin{array}{c} \begin{array}{cccccc} a & \ldots & a^{n-1} & b & \ldots & a^{n-1}b \end{array} \\ \begin{pmatrix} 0 & \ldots & 0 & -(n-2) & \ldots & -(n-2) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \ldots & 0 & -(n-2) & \ldots & -(n-2) \\ n-2 & \ldots & n-2 & 0 & \ldots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ n-2 & \ldots & n-2 & 0 & \ldots & 0 \end{pmatrix} \end{array}.$$

Here, the degree subtraction matrix of Γ_G can be obtained as the block matrices

$$DSt(\Gamma_G) = \begin{pmatrix} 0_{n-1} & -(n-2)J_{(n-1)\times n} \\ (n-2)J_{n\times(n-1)} & 0_n \end{pmatrix},$$

and the determinant below is the characteristic polynomial of $DSt(\Gamma_G)$,

$$P_{DSt(\Gamma_G)}(\lambda) = |\lambda I_{2n-1} - DSt(\Gamma_G)| = \begin{vmatrix} \lambda I_{n-1} & (n-2)J_{(n-1)\times n} \\ -(n-2)J_{n\times(n-1)} & \lambda I_n \end{vmatrix}.$$

By Lemma 2.5, with $a = b = 0$, $c = -(n-2)$, $d = n-2$, $n_1 = n-1$ and $n_2 = n$, we get the required result.

2. As it is known for n is even, $Z(D_{2n}) = \{e, a^{\frac{n}{2}}\}$ implies that there are $2n-2$ vertices for Γ_G, where $G = G_1 \cup G_2$, with $n-2$ vertices a^i, $1 \leq i < \frac{n}{2}$, $\frac{n}{2} < i < n$ and n vertices $a^i b$, for $1 \leq i \leq n$. We label the set G_1 as $\{a, a^2, \ldots, a^{\frac{n}{2}-1}, a^{\frac{n}{2}+1}, \ldots, a^{n-1}\}$ and G_2 as $\{b, ab, a^2b, \ldots, a^{n-1}b\}$. A similar argument as given in Theorem 2.1 is $d_{a^i} = n$ and $d_{a^ib} = 2(n-2)$, consequently the degree subtraction matrix of Γ_G is $DSt(\Gamma_G)$ of the size $(2n-2)\times(2n-2)$,

$$\begin{array}{c} \\ a \\ \vdots \\ a^{\frac{n}{2}-1} \\ a^{\frac{n}{2}+1} \\ \vdots \\ a^{n-1} \\ b \\ \vdots \\ a^{n-1}b \end{array} \begin{array}{c} \begin{array}{ccccccccc} a & \ldots & a^{\frac{n}{2}-1} & a^{\frac{n}{2}+1} & \ldots & a^{n-1} & b & \ldots & a^{n-1}b \end{array} \\ \begin{pmatrix} 0 & \ldots & 0 & 0 & \ldots & 0 & -(n-4) & \ldots & -(n-4) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \ldots & 0 & 0 & \ldots & 0 & -(n-4) & \ldots & -(n-4) \\ 0 & \ldots & 0 & 0 & \ldots & 0 & -(n-4) & \ldots & -(n-4) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \ldots & 0 & 0 & \ldots & 0 & -(n-4) & \ldots & -(n-4) \\ n-4 & \ldots & n-4 & n-4 & \ldots & n-4 & 0 & \ldots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ n-4 & \ldots & n-4 & n-4 & \ldots & n-4 & 0 & \ldots & 0 \end{pmatrix} \end{array}.$$

Here $DSt(\Gamma_G)$ can be partitioned as the block matrices

$$DSt(\Gamma_G) = \begin{pmatrix} 0_{n-2} & -(n-4)J_{(n-2)\times n} \\ (n-4)J_{n\times(n-2)} & 0_n \end{pmatrix},$$

and the characteristic polynomial of $DSt(\Gamma_G)$ as follows

$$P_{DSt(\Gamma_G)}(\lambda) = |\lambda I_{2n-2} - DSt(\Gamma_G)| = \begin{vmatrix} \lambda I_{n-2} & (n-4)J_{(n-2)\times n} \\ -(n-4)J_{n\times(n-2)} & \lambda I_n \end{vmatrix}.$$

Lemma 2.5 is further applied with $a = b = 0$, $c = -(n-4)$, $d = n-4$, $n_1 = n-2$ and $n_2 = n$, which leads to

$$P_{DSt(\Gamma_G)}(\lambda) = \lambda^{2(n-2)}\left(\lambda^2 + n(n-2)(n-4)^2\right).$$

□

Theorem 3.3. *Let* $\bar{\Gamma}_G$ *be the commuting graph on* G*, where* $G = G_1 \cup G_2 \subset D_{2n}$*, where* $n \geq 3$*. Then the characteristic polynomial of the degree subtraction matrix of* $\bar{\Gamma}_G$ *is*

1. $P_{DSt(\bar{\Gamma}_G)}(\lambda) = \lambda^{2n-3}(\lambda^2 + n(n-1)(n-2)^2)$*, for odd* n*, and*
2. $P_{DSt(\bar{\Gamma}_G)}(\lambda) = \lambda^{2(n-2)}(\lambda^2 + n(n-2)(n-4)^2)$*, for even* n*.*

Proof. 1. When n is odd and $G = G_1 \cup G_2 \subset D_{2n}$, considering the properties from Theorem 2.3 that $d_{a^i} = n-2$ and $d_{a^ib} = 0$, for all $1 \leq i \leq n$ together with the definition of the degree subtraction matrix, then $DSt(\bar{\Gamma}_G)$ is an $(2n-1)\times(2n-1)$ matrix as follows:

$$DSt(\bar{\Gamma}_G) = \begin{array}{c} \\ a \\ \vdots \\ a^{n-1} \\ b \\ \vdots \\ a^{n-1}b \end{array} \begin{array}{c} \begin{array}{cccccc} a & \dots & a^{n-1} & b & \dots & a^{n-1}b \end{array} \\ \begin{pmatrix} 0 & \dots & 0 & n-2 & \dots & n-2 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & n-2 & \dots & n-2 \\ -(n-2) & \dots & -(n-2) & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ -(n-2) & \dots & -(n-2) & 0 & \dots & 0 \end{pmatrix} \end{array}.$$

In other words, $DSt(\bar{\Gamma}_G)$ can be partitioned into four blocks,

$$DSt(\bar{\Gamma}_G) = \begin{pmatrix} 0_{n-1} & (n-2)J_{(n-1)\times n} \\ -(n-2)J_{n\times(n-1)} & 0_n \end{pmatrix}.$$

Here, the characteristic polynomial of $DSt(\bar{\Gamma}_G)$ is

$$P_{DSt(\bar{\Gamma}_G)}(\lambda) = \begin{vmatrix} \lambda I_{n-1} & -(n-2)J_{(n-1)\times n} \\ (n-2)J_{n\times(n-1)} & \lambda I_n \end{vmatrix}.$$

Using Lemma 2.5 with $a = b = 0$, $c = n-2$, $d = -(n-2)$, $n_1 = n-1$ and $n_2 = n$, it is clear that

$$P_{DSt(\bar{\Gamma}_G)}(\lambda) = \lambda^{2n-3}\left(\lambda^2 + n(n-1)(n-2)^2\right).$$

2. By Theorem 2.3 for the even n, we know that $d_{a^i} = n-3$ and $d_{a^ib} = 1$. For $G = G_1 \cup G_2 \subset D_{2n}$, in the same way for labeling G_1 and G_2 with the proof of Theorem 3.2 (2), we then obtain the degree subtraction matrix of $\bar{\Gamma}_G$, $DSt(\bar{\Gamma}_G)$ is an $(2n-2)\times(2n-2)$ matrix,

$$\begin{array}{c} \\ a \\ \vdots \\ a^{\frac{n}{2}-1} \\ a^{\frac{n}{2}+1} \\ \vdots \\ a^{n-1} \\ b \\ \vdots \\ a^{n-1}b \end{array}
\begin{array}{c} \begin{array}{cccccccccc} a & \dots & a^{\frac{n}{2}-1} & a^{\frac{n}{2}+1} & \dots & a^{n-1} & b & \dots & a^{n-1}b \end{array} \\
\left(\begin{array}{ccccccccc}
0 & \dots & 0 & 0 & \dots & 0 & n-4 & \dots & n-4 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & \dots & 0 & 0 & \dots & 0 & n-4 & \dots & n-4 \\
0 & \dots & 0 & 0 & \dots & 0 & n-4 & \dots & n-4 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & \dots & 0 & 0 & \dots & 0 & n-4 & \dots & n-4 \\
-(n-4) & \dots & -(n-4) & -(n-4) & \dots & -(n-4) & 0 & \dots & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
-(n-4) & \dots & -(n-4) & -(n-4) & \dots & -(n-4) & 0 & \dots & 0
\end{array}\right) \end{array}.$$

We then provide the block matrices of $DSt(\bar{\Gamma}_G)$,

$$DSt(\bar{\Gamma}_G) = \begin{pmatrix} 0_{n-2} & (n-4)J_{(n-2)\times n} \\ -(n-4)J_{n\times(n-2)} & 0_n \end{pmatrix}.$$

Here, the characteristic polynomial of $DSt(\bar{\Gamma}_G)$ is

$$P_{DSt(\bar{\Gamma}_G)}(\lambda) = \begin{vmatrix} \lambda I_{n-2} & -(n-4)J_{(n-2)\times n} \\ (n-4)J_{n\times(n-2)} & \lambda I_n \end{vmatrix}.$$

Again by Lemma 2.5 with $a = b = 0$, $c = n-4$, $d = -(n-4)$, $n_1 = n-2$ and $n_2 = n$, we get

$$P_{DSt(\bar{\Gamma}_G)}(\lambda) = \lambda^{2(n-2)}\left(\lambda^2 + n(n-2)(n-4)^2\right).$$

□

Now as a result of two preceding theorems, we can relate the characteristic polynomial of Γ_G and $\bar{\Gamma}_G$ as shown in the following corollary:

Corollary 3.4. *Let* Γ_G *and* $\bar{\Gamma}_G$ *be the non-commuting and commuting graphs on* G*, respectively, where* $G = G_1 \cup G_2$*, then* $P_{DSt(\Gamma_G)}(\lambda) = P_{DSt(\bar{\Gamma}_G)}(\lambda)$.

In the next discussion, we focus on the relationship between the degree subtraction energy of commuting and non-commuting graphs for $G = G_1 \cup G_2$. First, we need to find the spectrum and the spectral radius of Γ_G and $\bar{\Gamma}_G$ as presented below.

Theorem 3.5. *Let* Γ_G *be the non-commuting graph and* $\bar{\Gamma}_G$ *be the commuting graph on* G*, where* $G = G_1 \cup G_2$*, then* $DSt-$*spectral radius are*

1. $\rho_{DSt}(\Gamma_G) = \rho_{DSt}(\bar{\Gamma}_G) = (n-2)\sqrt{n(n-1)}$, *for odd* n, *and*
2. $\rho_{DSt}(\Gamma_G) = \rho_{DSt}(\bar{\Gamma}_G) = (n-4)\sqrt{n(n-2)}$, *for even* n.

Proof.

1. The result according to Corollary 3.4 is that the spectrum of Γ_G and $\bar{\Gamma}_G$ are the same. Theorem 3.2 (1) and Theorem 3.3 (1) give one real eigenvalue and two complex eigenvalues obtained from $P_{DSt(\Gamma_G)}(\lambda)$ and $P_{DSt(\bar{\Gamma}_G)}(\lambda)$, for odd n. They are $\lambda_1 = 0$ of multiplicity $2n-3$, $\lambda_2 = i(n-2)\sqrt{n(n-1)}$ of multiplicity 1 and a single $\lambda_2 = -i(n-2)\sqrt{n(n-1)}$. Hence, the spectrum of Γ_G and $\bar{\Gamma}_G$ are as follows:

$$Spec(\Gamma_G) = Spec(\bar{\Gamma}_G) = \left\{ \left(i(n-2)\sqrt{n(n-1)}\right)^1, (0)^{2n-3}, \left(-i(n-2)\sqrt{n(n-1)}\right)^1 \right\}.$$

Evidently, the $DSt-$spectral radius of Γ_G and $\bar{\Gamma}_G$ is

$$\rho_{DSt}(\Gamma_G) = \rho_{DSt}(\bar{\Gamma}_G) = (n-2)\sqrt{n(n-1)}.$$

2. The eigenvalues of Γ_G and $\bar{\Gamma}_G$ for even n are given by the roots of $P_{DSt(\Gamma_G)}(\lambda) = P_{DSt(\bar{\Gamma}_G)}(\lambda) = 0$ which is obtained from Theorem 3.2 (2) and Theorem 3.3 (2). The first eigenvalue is $\lambda_1 = 0$ with the multiplicity $2(n-2)$, the other two eigenvalues are $\lambda_2 = i(n-4)\sqrt{n(n-2)}$ and $\lambda_3 = -i(n-4)\sqrt{n(n-2)}$ of multiplicity 1, respectively. So that the spectrum of Γ_G and $\bar{\Gamma}_G$ are

$$Spec(\Gamma_G) = Spec(\bar{\Gamma}_G) = \left\{ \left(i(n-4)\sqrt{n(n-2)}\right)^1, (0)^{2(n-2)}, \left(-i(n-4)\sqrt{n(n-2)}\right)^1 \right\}.$$

Taking the maximum modulus eigenvalues, then we get the $DSt-$spectral radius of Γ_G and $\bar{\Gamma}_G$ as follows

$$\rho_{DSt}(\Gamma_G) = \rho_{DSt}(\bar{\Gamma}_G) = (n-4)\sqrt{n(n-2)}.$$

□

Theorem 3.6. *Let $\bar{\Gamma}_G$ and Γ_G be the commuting and non-commuting graph on G, respectively, where $G = G_1 \cup G_2$, then the degree subtraction energy for Γ_G and $\bar{\Gamma}_G$ are*

1. $E_{DSt}(\Gamma_G) = E_{DSt}(\bar{\Gamma}_G) = 2(n-2)\sqrt{n(n-1)}$, *for odd n, and*
2. $E_{DSt}(\Gamma_G) = E_{DSt}(\bar{\Gamma}_G) = 2(n-4)\sqrt{n(n-2)}$, *for even n.*

Proof. 1. Calculating the eigenvalues from $Spec(\Gamma_G)$ and $Spec(\bar{\Gamma}_G)$ in Theorem 3.5 (1), the degree subtraction energy of Γ_G and $\bar{\Gamma}_G$ are then given by

$$\begin{aligned} E_{DSt}(\Gamma_G) =& E_{DSt}(\bar{\Gamma}_G) \\ =&(2n-3)|0| + \left|i(n-2)\sqrt{n(n-1)}\right| + \left|-i(n-2)\sqrt{n(n-1)}\right| \\ =&2(n-2)\sqrt{n(n-1)}. \end{aligned}$$

2. Using $Spec(\Gamma_G)$ and $Spec(\bar{\Gamma}_G)$ given in Theorem 3.5 (2) for the even n, we get the degree subtraction energy of Γ_G and $\bar{\Gamma}_G$,

$$\begin{aligned} E_{DSt}(\Gamma_G) =& E_{DSt}(\bar{\Gamma}_G) \\ =&2(n-2)|0| + \left|i(n-4)\sqrt{n(n-2)}\right| + \left|-i(n-4)\sqrt{n(n-2)}\right| \\ =&2(n-4)\sqrt{n(n-2)}. \end{aligned}$$

□

By observing Theorem 3.5 and Theorem 3.6, we find the following relation.

Corollary 3.7. *Let Γ_G and $\bar{\Gamma}_G$ be the non-commuting and commuting graphs on G, respectively, where $G = G_1 \cup G_2$, then $E_{DSt}(\Gamma_G) = E_{DSt}(\bar{\Gamma}_G) = 2\rho_{DSt}(\Gamma_G) = 2\rho_{DSt}(\bar{\Gamma}_G)$.*

4 Conclusion

In this paper, we present the formula of $DSt-$spectrum, $DSt-$spectral radius, and $DSt-$energy of Γ_G and $\bar{\Gamma}_G$ for $G = D_{2n}\backslash Z(D_{2n})$. $DSt-$energy is similar for both Γ_G and $\bar{\Gamma}_G$, which is either $2(n-2)\sqrt{n(n-1)}$, for odd n, or $2(n-4)\sqrt{n(n-2)}$, for even n, and also equal to twice their $DSt-$spectral radius.

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References

[1] A. Abdollahi, S. Akbari, H. R. Maimani, Non-Commuting Graph of a Group, J. Algebra, **298,** no. 2, (2006), 468–492.

[2] R. Brauer, K. A. Fowler, On Groups of Even Order, Ann. Math., **62,** no. 3, (1955), 565–583.

[3] R. Mahmoud, N. H. Sarmin, A. Erfanian, On the Energy of Non-Commuting Graph of Dihedral Groups, AIP Conf. Proc., **1830,** (2017), 1–5.

[4] M. U. Romdhini, A. Nawawi, C. Y. Chen, Degree Exponent Sum Energy of Commuting Graph for Dihedral Groups, Malays. J. Sci., **41,** no. 1, (2022a), 40–46.

[5] M. U. Romdhini, A. Nawawi, Maximum and Minimum Degree Energy of Commuting Graph for Dihedral Groups, Sains Malays., **51,** no. 12, (2022b), 4145–4151.

[6] M. U. Romdhini, A. Nawawi, Degree Sum Energy of Non-Commuting Graph for Dihedral Groups, Malays. J. Sci., **41,** no. 1, (2022c), 34–39.

[7] J. Dutta, R. K. Nath, Spectrum of Commuting Graphs of Some Classes of Finite Groups, Matematika, **33,** no. 1, (2017), 87–95.

[8] W. N. T. Fasfous, R. K. Nath, Spectrum and Energy of Non-Commuting Graphs of Finite Groups, retrieved from ArXiv:2002.10146v1, (2020), 1–22.

[9] I. Gutman, The Energy of Graph, Ber. Math.-Stat. Sekt. Forschungsz. Graz, **103,** (1978), 1–22.

[10] H. S. Ramane, K. C. Nandeesh, G. A. Gudodagi, B. Zhou, Degree Subtraction Eigenvalues and Energy of Graphs, Comput. Sci. J. Mold., **2,** no. 77, (2018), 146–162.

[11] R. A. Horn, C. A. Johnson, Matrix Analysis, 1985.

[12] S. M. S. Khasraw, I. D. Ali, R. R. Haji, On the Non-Commuting Graph of Dihedral Group, Electron. J. Graph Theory Appl., **8,** no. 2, (2020), 233–239.

[13] H. S. Ramane, S. S. Shinde, Degree Exponent Polynomial of Graphs Obtained by Some Graph Operations, Electron. Notes Discrete Math., **63,** (2017), 161–168.