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Percentage Points For Testing Two-Sample Compound Symmetry

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Abstract

The exact null distribution of the likelihood ratio statistic for simultaneous testing the equality of covariance matrices and compound symmetry of two m variate Gaussian models has been obtained and percentage points for $m \leq 7$ have been computed. Inverse Mellin transform and calculus of residues have been used to derive these results.

1 Introduction

Let $\boldsymbol{X}_{g1}, \ldots, \boldsymbol{X}_{gN_g}$ be a random sample from an m-variate normal population with mean vector $\boldsymbol{\mu}_g$ and covariance matrix Σ_g, $g = 1, \ldots, q$. Let $H_{vc(q)}$ denote the hypothesis of multisample compound symmetry, i.e.

$$H_{vc(q)} : \Sigma_1 = \cdots = \Sigma_q = \sigma^2[(1-\rho)I_m + \rho J] \tag{1.1}$$

against general alternatives, where $\sigma^2 > 0$ and ρ $(-1/(m-1) < \rho < 1)$ are unknown scalars, I_m is an identity matrix of order m and J is an $m \times m$

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matrix with each element equals to unity. The modified likelihood ratio statistic $\Lambda^*_{vc(q)}$ for testing $H_{vc(q)}$ is given by (Nagar and Castañeda [4]),

$$\Lambda^*_{vc(q)} = \frac{(m-1)^{n_0(m-1)/2}(mn_0)^{n_0 m/2}}{\prod_{g=1}^{q} n_g^{n_g m/2}} \frac{\prod_{g=1}^{q}\det(A_g)^{n_g/2}}{\{\mathrm{tr}(JA)\}^{n_0/2}[\mathrm{tr}\{(mI_m - J)A\}]^{n_0(m-1)/2}},$$

where $A = \sum_{g=1}^{q} A_g$, $A_g = \sum_{j=1}^{N_g}(\mathbf{X}_{gj} - \mathbf{X}_{g\cdot})(\mathbf{X}_{gj} - \mathbf{X}_{g\cdot})'$, $N_g \mathbf{X}_{g\cdot} = \sum_{j=1}^{N_g}\mathbf{X}_{gj}$, $n_g = N_g - 1$, $g = 1, \ldots, q$, and $n_0 = \sum_{g=1}^{q} n_g$. The h^{th} null moment of $\Lambda^*_{vc(q)}$ is also available as

$$\begin{aligned} E(\Lambda^{*h}_{vc(q)}) = {}& \frac{(m-1)^{n_0(m-1)h/2} n_0^{n_0 mh/2}}{\prod_{g=1}^{q} n_g^{n_g mh/2}} \frac{\Gamma(n_0/2)\Gamma[n_0(m-1)/2]}{\Gamma[n_0(1+h)/2]\Gamma[n_0(m-1)(1+h)/2]} \\ & \times \prod_{g=1}^{q}\prod_{j=1}^{m} \frac{\Gamma[n_g(1+h)/2 - (j-1)/2]}{\Gamma[(n_g - j + 1)/2]}, \end{aligned}$$

where $\mathrm{Re}[n_g(1+h)] > m-1$, $g = 1, \ldots, q$. When $n_1 = \cdots = n_q = n$, the h^{th} null moment of $V = (\Lambda^*_{vc(q)})^{2/n}$ simplifies to

$$\begin{aligned} E(V^h) = {}& (m-1)^{q(m-1)h} q^{qmh} \frac{\Gamma(nq/2)\Gamma[nq(m-1)/2]}{\Gamma[q(n/2+h)]\Gamma[q(m-1)(n/2+h)]} \\ & \times \prod_{j=1}^{m} \frac{\Gamma^q[(n-j+1)/2 + h]}{\Gamma^q[(n-j+1)/2]}. \end{aligned} \tag{1.2}$$

In the case $q = 1$, (1.1) is the usual Wilks' H_{vc} hypothesis for testing compound symmetry (intra-class correlation structure) of the covariance matrix of a multivariate normal model. The problem of testing H_{vc} plays a very useful role in areas like medical research and psychometrics. Such models also arise in the study of familial data (Srivastava [8]). The problems of testing compound symmetry has been studied by many authors. For some early work see Wilks [10] and Votaw [9]. The exact distribution and exact percentage points of V for bivariate and trivariate Gaussian populations are available in Nagar and Castañeda [4].

In this article, we derive the exact distribution of V for $q = 2$ and compute exact percentage points. We use the inverse Mellin transform and residue theorem to derive the distribution of V. Using the exact distribution derived we compute exact percentage points.

2 The Density of V

Substituting $q = 2$ in (1.2) and using Gauss-Legendre multiplication formula for gamma function, the h-th moment of V is simplified as

$$E(V^h) = \frac{\Gamma(n/2+1/2)\prod_{k=1}^{2m-3}\Gamma[n/2+k/(2(m-1))]}{\Gamma(n/2+h+1/2)\prod_{k=1}^{2m-3}\Gamma[n/2+h+k/(2(m-1))]} \times \prod_{j=1}^{m-1}\frac{\Gamma^2[(n-m+j)/2+h]}{\Gamma^2[(n-m+j)/2]}. \tag{2.3}$$

Now, from (2.3), using the inverse Mellin transform and substituting $h+(n-m)/2 = t$, we have the density of V as

$$f(v) = K(n,m)v^{(n-m)/2-1}(2\pi\iota)^{-1}\int_C \eta(t)v^{-t}\,dt,\ 0 < v < 1,$$

where $\iota = \sqrt{-1}$,

$$K(n,m) = \frac{\Gamma(n/2+\frac{1}{2})\prod_{k=1}^{2m-3}\Gamma[n/2+k/(2(m-1))]}{\prod_{j=1}^{m-1}\Gamma^2[(n-m+j)/2]}$$

$$\eta(t) = \frac{\prod_{j=1}^{m-1}\Gamma^2(t+j/2)}{\Gamma(t+m/2+1/2)\prod_{k=1}^{2m-3}\Gamma[t+m/2+k/(2(m-1))]},$$

and C is a suitable contour. Now, canceling common gamma functions for m-even and m-odd separately, we get following simplifications:

(i) m-even

$$\eta_1(t) = \frac{1}{[t+(m-1)/2]^2}\frac{\prod_{j=1}^{m/2-1}\Gamma^2(t+j)\prod_{j=1}^{m/2-1}\Gamma^2(t-1/2+j)}{\prod_{k=1(\neq m-1)}^{2m-3}\Gamma[t+m/2+k/(2(m-1))]}. \tag{2.4}$$

(iii) m-odd

$$\eta_2(t) = \frac{1}{[t+(m-1)/2]^2}\frac{\prod_{j=1}^{(m-3)/2}\Gamma^2(t+j)\prod_{j=1}^{(m-1)/2}\Gamma^2(t-1/2+j)}{\prod_{k=1(\neq m-1)}^{2m-3}\Gamma[t+m/2+k/(2(m-1))]}. \tag{2.5}$$

Now, with the help of (2.4) and (2.5) one can identify poles of the integrand and their order for two different cases. The poles are available by equating to zero each factor of $\prod_{i=1}^{\infty}(t+i)^{a_i}\prod_{i=1}^{\infty}(t-1/2+i)^{b_i}$ where a_i and b_i give the order of the pole at $t=-i$ and $t=-i+1/2$, respectively. The orders a_i and b_i are given by

(i) m-even

$$a_i=\begin{cases} 2i,\ i=1,\ldots,\frac{m}{2}-1 \\ m-2,\ i=\frac{m}{2},\ldots, \end{cases} \qquad b_i=\begin{cases} 2i,\ i=1,\ldots,\frac{m}{2} \\ m-2,\ i=\frac{m}{2}+1,\ldots, \end{cases}, \tag{2.6}$$

(ii) m-odd

$$a_i=\begin{cases} 2i,\ j=1,\ldots,\frac{m-1}{2} \\ m-3,\ i=\frac{m+1}{2},\ldots, \end{cases} \qquad b_i=\begin{cases} 2i,\ i=1,\ldots,\frac{m-1}{2} \\ m-1,\ i=\frac{m+1}{2},\ldots, \end{cases}. \tag{2.7}$$

Now, using the residue theorem we get the density of V as

$$f(v)=K(n,m)v^{(n-m)/2-1}\sum_{i=1}^{\infty}[R_{1i}+R_{2i}],\ 0<v<1, \tag{2.8}$$

where $R_{\ell i}$ is the residue at $t=-i+(\ell-1)/2$, $\ell=1,2$. Further, from the calculus of residues

$$R_{1i}=\frac{1}{(a_i-1)!}\lim_{t\to -i}\frac{\partial^{a_i-1}}{\partial t^{a_i-1}}\left[(t+i)^{a_i}\eta(t)v^{-t}\right]$$

and

$$R_{2i}=\frac{1}{(b_i-1)!}\lim_{t\to -i+1/2}\frac{\partial^{b_i-1}}{\partial t^{b_i-1}}\left[\left(t-\frac{1}{2}+i\right)^{b_i}\eta(t)v^{-t}\right]$$

where $\eta(t)\equiv\eta_1(t)$ and a_i,b_i are given by (2.6) if $m=$ even, and $\eta(t)\equiv\eta_2(t)$ and a_i,b_i are given by (2.7) if $m=$ odd. Clearly, the residues R_{1i} and R_{2i} will be different for the two cases considered above and hence the density will also be different. Considering these cases separately one can derive expressions for residues explicitly. These expressions involve polygamma functions denoted by $\psi^{(s)}(\cdot)$ (for details see Apostol [1], Askey and Roy [2], Gupta, Nagar and Gómez [3], Nagar and Gupta [5, 6], Nagar and Zarrazola [7]). Here we give density for m-odd and m-even as

$$f(v)=K(n,m)v^{(n-m)/2-1}\sum_{i=1}^{\infty}\left[\frac{v^i}{(a_i-1)!}\sum_{u=0}^{a_i-1}\binom{a_i-1}{u}A_{1i0}^{(u)}(-\ln v)^{a_i-1-u}\right.$$
$$\left.+\frac{v^{i-1/2}}{(b_i-1)!}\sum_{u=0}^{b_i-1}\binom{b_i-1}{u}A_{2i0}^{(u)}(-\ln v)^{b_i-1-u}\right],\ 0<v<1, \tag{2.9}$$

with

$$A_{\ell i0}^{(u)} = \sum_{s=0}^{u-1} \binom{u-1}{s} A_{\ell i0}^{(u-1-s)} B_{\ell i0}^{(s)},\ \ell = 1, 2,$$

where, for m even,

$$A_{1i0}^{(0)} = \frac{[(m-1)/2 - i]^{-2}}{\prod_{\ell=1}^{i-1}(\ell - i)^{2\ell}} \frac{\prod_{j=i+1}^{m/2-1} \Gamma^2(j-i) \prod_{j=1}^{m/2-1} \Gamma^2(j-i-1/2)}{\prod_{k=1(\neq m-1)}^{2m-3} \Gamma[m/2 - i + k/(2(m-1))]},$$
$$i = 1, \ldots, \frac{m}{2} - 1,$$

$$A_{1i0}^{(0)} = \frac{[(m-1)/2 - i]^{-2}}{\prod_{\ell=1}^{i-1}(\ell - i)^{a_\ell}} \frac{\prod_{j=1}^{m/2-1} \Gamma^2(j-i-1/2)}{\prod_{k=1(\neq m-1)}^{2m-3} \Gamma[m/2 - i + k/(2(m-1))]},$$
$$i = \frac{m}{2}, \ldots,$$

$$B_{1i0}^{(s)} = 2i\psi^{(s)}(1) + \frac{2(-1)^{s+1}s!}{[(m-1)/2 - i]^{s+1}} + 2\sum_{\ell=1}^{i-1} \frac{(-1)^{s+1}s!\ell}{(\ell - i)^{s+1}} + 2\sum_{j=i+1}^{m/2-1} \psi^{(s)}(j-i)$$
$$+ 2\sum_{j=1}^{m/2-1} \psi^{(s)}\Big(j - i - \frac{1}{2}\Big) - \sum_{\substack{k=1 \\ \neq m-1}}^{2m-3} \psi^{(s)}\Big(\frac{m}{2} - i + \frac{k}{2(m-1)}\Big),\ i = 1, \ldots, \frac{m}{2} - 1,$$

$$B_{1i0}^{(s)} = (m-2)\psi^{(s)}(1) + \frac{2(-1)^{s+1}s!}{[(m-1)/2 - i]^{s+1}} + \sum_{\ell=1}^{i-1} \frac{(-1)^{s+1}s!a_\ell}{(\ell - i)^{s+1}}$$
$$+ 2\sum_{j=1}^{m/2-1} \psi^{(s)}\Big(j - i - \frac{1}{2}\Big) - \sum_{\substack{k=1 \\ \neq m-1}}^{2m-3} \psi^{(s)}\Big(\frac{m}{2} - i + \frac{k}{2(m-1)}\Big),\ i = \frac{m}{2}, \ldots,$$

$$A_{2i0}^{(0)} = \frac{(m/2 - i)^{-2}}{\prod_{\ell=1}^{i-1}(\ell - i)^{2\ell}} \frac{\prod_{j=i+1}^{m/2-1} \Gamma^2(j-i) \prod_{j=1}^{m/2-1} \Gamma^2(j-i+1/2)}{\prod_{k=1(\neq m-1)}^{2m-3} \Gamma[(m+1)/2 - i + k/(2(m-1))]},$$
$$i = 1, \ldots, \frac{m}{2} - 1,$$

$$A_{2i0}^{(0)} = \frac{1}{\prod_{\ell=1}^{m/2-1}(\ell - m/2)^{2\ell}} \frac{\prod_{j=1}^{m/2-1} \Gamma^2(j - m/2 + 1/2)}{\prod_{k=1(\neq m-1)}^{2m-3} \Gamma[1/2 + k/(2(m-1))]},\ i = \frac{m}{2},$$

$$A_{2i0}^{(0)} = \frac{1}{\prod_{\ell=1}^{i-1}(\ell-i)^{b_\ell}} \frac{\prod_{j=1}^{m/2-1}\Gamma^2(j-i+1/2)}{\prod_{k=1(\neq m-1)}^{2m-3}\Gamma[(m+1)/2-i+k/(2(m-1))]},\ i=\frac{m}{2}+1,\dots,$$

$$\begin{aligned} B_{2i0}^{(s)} &= 2i\psi^{(s)}(1) + \frac{2(-1)^{s+1}s!}{(m/2-i)^{s+1}} + 2\sum_{\ell=1}^{i-1}\frac{\ell(-1)^{s+1}s!}{(\ell-i)^{s+1}} + 2\sum_{j=i+1}^{m/2-1}\psi^{(s)}(j-i) \\ &\quad + 2\sum_{j=1}^{m/2-1}\psi^{(s)}\Big(j-i+\frac{1}{2}\Big) - \sum_{\substack{k=1\\ \neq m-1}}^{2m-3}\psi^{(s)}\left(\frac{m+1}{2}-i+\frac{k}{2(m-1)}\right), \\ &\qquad i=1,\dots,\frac{m}{2}-1, \end{aligned}$$

$$\begin{aligned} B_{2i0}^{(s)} &= (m-2)\psi^{(s)}(1) + 2\sum_{\ell=1}^{m/2-1}\frac{\ell(-1)^{s+1}s!}{(\ell-m/2)^{s+1}} \\ &\quad + 2\sum_{j=1}^{m/2-1}\psi^{(s)}\Big(j-\frac{m-1}{2}\Big) - \sum_{\substack{k=1\\ \neq m-1}}^{2m-3}\psi^{(s)}\left(\frac{1}{2}+\frac{k}{2(m-1)}\right),\ i=\frac{m}{2} \end{aligned}$$

$$\begin{aligned} B_{2i0}^{(s)} &= (m-2)\psi^{(s)}(1) + \sum_{\ell=1}^{i-1} b_\ell\frac{(-1)^{s+1}s!}{(\ell-i)^{s+1}} + 2\sum_{j=1}^{m/2-1}\psi^{(s)}\left(j-i+\frac{1}{2}\right) \\ &\quad - \sum_{\substack{k=1\\ \neq m-1}}^{2m-3}\psi^{(s)}\Big(\frac{m+1}{2}-i+\frac{k}{2(m-1)}\Big),\quad i=\frac{m}{2}+1,\dots, \end{aligned}$$

and for m odd,

$$\begin{aligned} A_{1i0}^{(0)} &= \frac{[(m-1)/2-i]^{-2}}{\prod_{\ell=1}^{i-1}(\ell-i)^{2\ell}} \frac{\prod_{j=i+1}^{(m-3)/2}\Gamma^2(j-i)\prod_{j=1}^{(m-1)/2}\Gamma^2(j-i-1/2)}{\prod_{k=1(\neq m-1)}^{2m-3}\Gamma[m/2-i+k/(2(m-1))]}, \\ &\qquad i=1,\dots,\frac{m-3}{2}, \end{aligned}$$

$$A_{1i0}^{(0)} = \frac{1}{\prod_{\ell=1}^{(m-3)/2}[\ell-(m-1)/2]^{2\ell}} \frac{\prod_{j=1}^{(m-1)/2}\Gamma^2(j-m/2)}{\prod_{k=1(\neq m-1)}^{2m-3}\Gamma[1/2+k/(2(m-1))]},\ i=\frac{m-1}{2},$$

$$A_{1i0}^{(0)} = \frac{1}{\prod_{\ell=1}^{i-1}(\ell-i)^{a_\ell}} \frac{\prod_{j=1}^{(m-1)/2}\Gamma^2(j-i-1/2)}{\prod_{k=1(\neq m-1)}^{2m-3}\Gamma[m/2-i+k/(2(m-1))]},\ i=\frac{m+1}{2},\dots,$$

$$B_{1i0}^{(s)} = 2i\psi^{(s)}(1) + \frac{2(-1)^{s+1}s!}{[(m-1)/2-i]^{s+1}} + 2\sum_{\ell=1}^{i-1}\frac{\ell(-1)^{s+1}s!}{(\ell-i)^{s+1}} + 2\sum_{j=i+1}^{(m-3)/2}\psi^{(s)}(j-i)$$
$$+ 2\sum_{j=1}^{(m-1)/2}\psi^{(s)}\Big(j-i-\frac{1}{2}\Big) - \sum_{\substack{k=1\\ \neq m-1}}^{2m-3}\psi^{(s)}\Big(\frac{m}{2}-i+\frac{k}{2(m-1)}\Big),$$
$$i = 1,\ldots,\frac{m-3}{2},$$

$$B_{1i0}^{(s)} = (m-3)\psi^{(s)}(1) + 2\sum_{\ell=1}^{(m-3)/2}\frac{\ell(-1)^{s+1}s!}{[\ell-(m-1)/2]^{s+1}}$$
$$+ 2\sum_{j=1}^{(m-1)/2}\psi^{(s)}\Big(j-\frac{m}{2}\Big) - \sum_{\substack{k=1\\ \neq m-1}}^{2m-3}\psi^{(s)}\Big(\frac{1}{2}+\frac{k}{2(m-1)}\Big),\ i=\frac{m-1}{2},$$

$$B_{1i0}^{(s)} = (m-3)\psi^{(s)}(1) + \sum_{\ell=1}^{i-1}\frac{a_\ell(-1)^{s+1}s!}{(\ell-i)^{s+1}} + 2\sum_{j=1}^{(m-1)/2}\psi^{(s)}\Big(j-i-\frac{1}{2}\Big)$$
$$- \sum_{\substack{k=1\\ \neq m-1}}^{2m-3}\psi^{(s)}\Big(\frac{m}{2}-i+\frac{k}{2(m-1)}\Big),\ i=\frac{m+1}{2},\ldots,$$

$$A_{2i0}^{(0)} = \frac{(m/2-i)^{-2}}{\prod_{\ell=1}^{i-1}(\ell-i)^{2\ell}}\frac{\prod_{j=i+1}^{(m-1)/2}\Gamma^2(j-i)\prod_{j=1}^{(m-3)/2}\Gamma^2(j-i-1/2)}{\prod_{k=1(\neq m-1)}^{2m-3}\Gamma[(m+1)/2-i+k/(2(m-1))]},$$
$$i = 1,\ldots,\frac{m-1}{2},$$

$$A_{2i0}^{(0)} = \frac{(m/2-i)^{-2}}{\prod_{\ell=1}^{i-1}(\ell-i)^{b_\ell}}\frac{\prod_{j=1}^{(m-3)/2}\Gamma^2(j-i+1/2)}{\prod_{k=1(\neq m-1)}^{2m-3}\Gamma[(m+1)/2-i+k/(2(m-1))]},$$
$$i = \frac{m+1}{2},\ldots,$$

$$B_{2i0}^{(s)} = (m-1)\psi^{(s)}(1) + \frac{2(-1)^{s+1}s!}{(m/2-i)^{s+1}} + 2\sum_{\ell=1}^{i-1}\frac{\ell(-1)^{s+1}s!b_\ell}{(\ell-i)^{s+1}}$$
$$+ 2\sum_{j=1}^{(m-3)/2}\psi^{(s)}\Big(j-i+\frac{1}{2}\Big) - \sum_{\substack{k=1\\ \neq m-1}}^{2m-3}\psi^{(s)}\left(\frac{m+1}{2}-i+\frac{k}{2(m-1)}\right),\quad i=\frac{m+1}{2},\ldots$$

3 Computation

The computation of the exact percentage points has been carried out by using $F(v,m) = \int_0^v f(t)\,dt$ where $f(t)$ is given by (2.9). First, $f(t)$ is simplified for $m = 4, 5, 6, 7$ using results on Gamma-, Psi- and polygamma functions. Then, the CDF $F(v,m)$ for $m = 4, 5, 6, 7$ is obtained by integrating term by term these simplified density functions. For each m, $F(v,m)$ is computed for various values of v. It is checked for monotonicity and for conditions $F(v,m) \to 0$ as $v \to 0$ and $F(v,m) \to 1$ as $v \to 1$. Then, v is computed for $m = 4, 5, 6, 7$. These are given in Table 1–4. We have used MATHEMATICA 12.0 to carry out these computations. To compute v for given value of $\alpha = F(v,m)$, we have used `FindRoot` which searches for a numerical solution to the given equation using Newton's method or a variant of the secant method. A six place accuracy has been kept throughout. For higher values of m it is seen that the accuracy is being lost. Hence the tables are given for $m = 4, 5, 6, 7$.

Table 1: percentage points of V for $m = 4$

n	$\alpha = 0.01$	$\alpha = 0.025$	$\alpha = 0.05$	$\alpha = 0.1$
5	0.00000	0.00003	0.00008	0.00022
6	0.00022	0.00049	0.00096	0.00197
7	0.00121	0.00232	0.00393	0.00691
8	0.00373	0.00638	0.00984	0.01571
9	0.00824	0.01303	0.01887	0.02817
10	0.01490	0.02224	0.03070	0.04364
11	0.02361	0.03370	0.04490	0.06137
12	0.03412	0.04700	0.06092	0.08066
13	0.04614	0.06174	0.07817	0.10092
14	0.05935	0.07754	0.09627	0.12170
15	0.07347	0.09405	0.11487	0.14264
16	0.08825	0.11101	0.13368	0.16349
17	0.10348	0.12820	0.15250	0.18406
18	0.11895	0.14544	0.17117	0.20420
19	0.13455	0.16260	0.18956	0.22384
20	0.15015	0.17958	0.20760	0.24291
21	0.16565	0.19629	0.22521	0.26138
22	0.18099	0.21267	0.24236	0.27922
23	0.19611	0.22869	0.25902	0.29643
24	0.21097	0.24432	0.27517	0.31302
25	0.22552	0.25954	0.29082	0.32900
26	0.23977	0.27434	0.30598	0.34437
27	0.25367	0.28871	0.32061	0.35917
28	0.26724	0.30265	0.33476	0.37341
29	0.28046	0.31618	0.34843	0.38710
30	0.29333	0.32929	0.36163	0.40028

Table 2: percentage points of V for $m = 5$

n	$\alpha = 0.01$	$\alpha = 0.025$	$\alpha = 0.05$	$\alpha = 0.1$
6	0.00000	0.00000	0.00000	0.00003
7	0.00003	0.00008	0.00015	0.00033
8	0.00023	0.00045	0.00079	0.00145
9	0.00083	0.00148	0.00237	0.00395
10	0.00213	0.00350	0.00525	0.00817
11	0.00436	0.00674	0.00963	0.01421
12	0.00764	0.01129	0.01553	0.02200
13	0.01205	0.01714	0.02287	0.03135
14	0.01754	0.02422	0.03151	0.04203
15	0.02407	0.03238	0.04127	0.05381
16	0.03153	0.04150	0.05196	0.06645
17	0.03980	0.05142	0.06341	0.07974
18	0.04878	0.06200	0.07544	0.09350
19	0.05834	0.07310	0.08791	0.10758
20	0.06839	0.08461	0.10070	0.12184
21	0.07882	0.09642	0.11370	0.13617
22	0.08955	0.10844	0.12680	0.15050
23	0.10050	0.12058	0.13995	0.16474
24	0.11161	0.13279	0.15307	0.17885
25	0.12281	0.14501	0.16612	0.19278
26	0.13405	0.15718	0.17905	0.20649
27	0.14529	0.16928	0.19182	0.21995
28	0.15651	0.18127	0.20441	0.23316
29	0.16765	0.19312	0.21681	0.24609
30	0.17871	0.20481	0.22898	0.25873

Table 3: percentage points of V for $m = 6$

n	$\alpha = 0.01$	$\alpha = 0.025$	$\alpha = 0.05$	$\alpha = 0.1$
7	0.00000	0.00000	0.00001	0.00003
8	0.00000	0.00001	0.00002	0.00005
9	0.00004	0.00009	0.00015	0.00029
10	0.00018	0.00033	0.00054	0.00094
11	0.00053	0.00089	0.00138	0.00223
12	0.00121	0.00193	0.00283	0.00432
13	0.00234	0.00356	0.00502	0.00735
14	0.00400	0.00586	0.00803	0.01135
15	0.00627	0.00890	0.01187	0.01630
16	0.00917	0.01267	0.01653	0.02217
17	0.01271	0.01717	0.02198	0.02887
18	0.01688	0.02234	0.02815	0.03631
19	0.02164	0.02815	0.03497	0.04441
20	0.02695	0.03454	0.04236	0.05306
21	0.03277	0.04143	0.05025	0.06216
22	0.03906	0.04878	0.05857	0.07166
23	0.04575	0.05652	0.06726	0.08143
24	0.05280	0.06454	0.07629	0.09064
25	0.06016	0.07293	0.08531	0.10112
26	0.06782	0.08151	0.09579	0.11129
27	0.07562	0.09002	0.10327	0.11702
28	0.08366	0.09853	0.11097	0.11702
29	0.09093	0.10729	0.12240	0.14166
30	0.09904	0.11402	0.12236	0.14212

Table 4: percentage points of V for $m = 7$

n	$\alpha = 0.01$	$\alpha = 0.025$	$\alpha = 0.05$	$\alpha = 0.1$
8	0.00000	0.00000	0.00000	0.00000
9	0.00000	0.00000	0.00000	0.00001
10	0.00001	0.00002	0.00003	0.00006
11	0.00004	0.00007	0.00012	0.00021
12	0.00012	0.00022	0.00035	0.00057
13	0.00032	0.00052	0.00079	0.00124
14	0.00068	0.00106	0.00153	0.00231
15	0.00126	0.00189	0.00265	0.00385
16	0.00212	0.00308	0.00421	0.00593
17	0.00330	0.00467	0.00623	0.00853
18	0.00484	0.00669	0.00872	0.01169
19	0.00676	0.00914	0.01178	0.01490
20	0.00910	0.01216	0.01451	0.01857
21	0.01176	0.01510	0.01741	0.01984
22	0.01495	0.01745	0.01866	0.02092
23	0.01819	0.01969	0.02108	0.02732
24	0.01862	0.02012	0.02171	0.02510
25	0.02193	0.02137	0.02395	0.02576
26	0.02205	0.02727	0.02676	0.02837
27	0.02656	0.02712	0.03036	0.02965
28	0.02738	0.02913	0.02996	0.03237
29	0.02742	0.02713	0.03176	0.03253
30	0.03050	0.03597	0.03284	0.03593

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