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On the Diophantine equation $255^x + 323^y = z^2$

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Abstract

In this article, we prove that $(1, 0, 16)$ and $(0, 1, 18)$ are the only two solutions (x, y, z) for the Diophantine equation $255^x + 323^y = z^2$, where x, y and z are non-negative integers.

1 Introduction

Many mathematicians have been studying the Diophantine equations of the type $a^x + b^y = z^2$, where a and b are fixed. In 2014, Sroysang [1] showed that $(1, 0, 18)$ is the unique non-negative integer solution (x, y, z) of the Diophantine equation $323^x + 325^y = z^2$. In 2022, N. Viriyapong and C. Viriyapong [2] proved that the Diophantine equation $n^x + 19^y = z^2$ has exactly one non-negative solution $(n, x, y, z) = (2, 3, 0, 3)$, where $n \equiv_{57} 2$.

In this paper, we solve the Diophantine equation $255^x + 323^y = z^2$, where x, y and z are non-negative integers.

2 Preliminaries

Throughout this paper, $a \equiv_m b$ always means a is congruent to b modulo m, where a, b, and m are integers such that $m \geqslant 1$. Moreover, we write $a \equiv_m b, c$

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to mean that $a \equiv_m b$ or $a \equiv_m c$.

We now recall the Catalan's conjecture [3] from 1844 which was proved by Mihailescu [4] in 2004.

Theorem 2.1 (Catalan's conjecture). *The Diophantine equation* $a^x - b^y = 1$ *has the unique solution* $(a, b, x, y) = (3, 2, 2, 3)$*, where* a*,* b*,* x *and* y *are integers with* $\min\{a, b, x, y\} > 1$.

Next, we give a lemma that is a consequence of the Catalan's conjecture.

Lemma 2.2. $(1, 16)$ *is the unique non-negative integer solution* (x, z) *for the Diophantine equation* $255^x + 1 = z^2$.

Proof. Assume that there exist non-negative integers x and z such that $255^x + 1 = z^2$. If $x = 0$, then $z^2 = 2$, which is a contradiction. Now, we have $x \geqslant 1$. By Theorem 2.1, $x = 1$. This implies that $z = 16$. The proof is complete. $\square$

Next, we recall the following two lemmas:

Lemma 2.3. [1] *The Diophantine equation* $1 + 323^y = z^2$ *has the unique non-negative integer solution* $(y, z) = (1, 18)$.

Lemma 2.4. [2] *If* z *is an integer, then* $z^2 \equiv_{19} 0, 1, 4, 5, 6, 7, 9, 11, 16, 17$.

3 Main Results

In this section, we begin with a lemma which will be useful in proving our main theorem.

Lemma 3.1. *If* x *is a positive odd integer, then* $8^x \equiv_{19} 8, 12, 18$.

Proof. We prove by induction that $8^{2n-1} \equiv_{19} 8, 12, 18$ for all $n \in \mathbb{N}$. If $n = 1$, then $8^1 \equiv_{19} 8$ and so the statement is true for $n = 1$. Assume that it is true for $n = k$. Then $8^{2k-1} \equiv_{19} 8, 12, 18$ and so $8^{2k+1} \equiv_{19} 18, 8, 12$. Hence, the statement is true for $n = k + 1$ which proves the result. $\square$

Next, we shall give our main result.

Theorem 3.2. *The Diophantine equation* $255^x + 323^y = z^2$ *has exactly the two non-negative integer solutions* $(x, y, z) = (1, 0, 16), (0, 1, 18)$.

Proof. Clearly $z = 0$ cannot happen.
If $y = 0$, then by Lemma 2.2 $(1, 0, 16)$ is the only solution in this case.
If $x = 0$, then by Lemma 2.3 $(0, 1, 18)$ is the only solution in this case.
Now, we consider $x \geqslant 1$ and $y \geqslant 1$. If y is odd, then $z^2 = 255^x + 323^y \equiv_3 2$, which contradicts the fact that $z^2 \equiv_3 0, 1$. Then y is even. If x is even, then $z^2 = 255^x + 323^y \equiv_4 2$, which contradicts the fact that $z^2 \equiv_4 0, 1$. Then x is odd. Since $255 \equiv_{19} 8$, by Lemma 3.1, we have $255^x \equiv_{19} 8, 12, 18$. Since $323^y \equiv_{19} 0$, $z^2 \equiv_{19} 8, 12, 18$, which contradicts Lemma 2.4. Consequently, $(1, 0, 16)$ and $(0, 1, 18)$ are the only two solutions (x, y, z) of the equation. This completes the proof. □

The proof of the following corollary is immediate.

Corollary 3.3. $(x, y, z) = (1, 0, 4)$ *is the unique non-negative integer solution of the Diophantine equation* $255^x + 323^y = z^4$.

4 Conclusion

In this paper, we proved that there are exactly two solutions $(1, 0, 16)$ and $(1, 0, 18)$ for the Diophantine equation $255^x + 323^y = z^2$, where x, y and z are non-negative integers.

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