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Unified products and matched pairs analysis of dual mock-Lie algebras

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Abstract

This paper focuses on finite-dimensional dual mock-Lie algebras. Let $\mathcal{H}$ be a dual mock-Lie algebra and $\mathcal{V}$ a vector space containing $\mathcal{H}$ as a subspace. All dual mock-Lie algebra structures on $\mathcal{V}$ containing $\mathcal{H}$ as a subalgebra are explicitly described and classified by non-abelian cohomological type objects: $\mathcal{C}^2_{\mathcal{H}}(\mathcal{U},\mathcal{H})$ provides the classification up to an isomorphism that stabilizes $\mathcal{H}$ and will classify all such structures from the viewpoint of the extension problem. Here $\mathcal{U}$ is a complement of $\mathcal{H}$ in $\mathcal{V}$. A general product, called the unified product, is introduced as a tool for our approach. The crossed (resp. bicrossed) products between two dual mock-Lie algebras are introduced as special cases of the unified product: crossed product is responsible for the extension problem while the bicrossed product is responsible for the factorization problem. The description and the classification of all complements of a given extension of dual mock-Lie algebras are given as a converse of the factorization problem.

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1 Introduction

Dual mock-Lie algebras are as noted in [20] the intersection of anti-commutative and anti-associative algebras. They play a very significant role in the theory of non-associative algebras. Their importance is motivated by the fact that an algebraically coherent variety of anti-commutative algebras is either a variety of Lie algebras or a variety of anti-associative algebras (c.f. [6]).

A dual mock-Lie algebra consists of a vector space $\mathcal{H}$ with a bilinear map $(.) : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ such that $p.q = -q.p$ and $p.(q.r) = -(p.q).r$, for all $p, q, r \in \mathcal{H}$. The structure theory of dual mock-Lie algebras is exciting and rich but needs further development. There are recent interesting works on the classification of such algebras. In [18], the authors provide a classification of all indecomposable 7-dimensional 2-step nilpotent dual mock-Lie algebras. Next, Kaygorodov et al. [17] gave the classification of all 6-dimensional nilpotent anti-commutative algebras. More recently, Camacho et al. [21] used these two last results to classify algebraically and geometrically low dimensional dual mock-Lie algebras. The reader can find more information about this structure for instance in [19, 9, 5, **?**, 1].

The outline of this paper is as follows: As a starting point, in the first section, after defining notations and conventions that will be used throughout the article and recalling some basic concepts related to dual mock-Lie algebras, we examine the extending structure's problem (E-S problem): Let $\mathcal{H}$ be a dual mock-Lie algebra and $\mathcal{V}$ a vector space containing $\mathcal{H}$ as a subspace. Describe and classify up to an isomorphism of dual mock-Lie algebras that stabilizes $\mathcal{H}$ the set of all dual mock-Lie algebra structures (.) that can be defined on $\mathcal{V}$ such that $\mathcal{H}$ is a dual mock-Lie subalgebra of $(\mathcal{V}, .)$.

We propose the following strategy for the study of the E-S problem: First, we establish in Theorem 2.1 the unified product $\mathcal{H}\natural\mathcal{U}$ that is associated with a dual mock-Lie algebra $\mathcal{H}$ and a space $\mathcal{U}$ that are related by two actions and a cocycle. Further, dual mock-Lie algebra structure $(\mathcal{V}, ._{\mathcal{V}})$ on $\mathcal{V}$ contains $\mathcal{H}$ as a subalgebra if and only if there exists an isomorphism of dual mock-Lie algebras $(\mathcal{V}, ._{\mathcal{V}}) \cong \mathcal{H}\natural\mathcal{U}$ as shown in Theorem 2.2. Furthermore, a theoretical explanation of the E-S problem can be found in Theorem 2.5: a non-abelian cohomological type object $\mathcal{C}^2_{\mathcal{H}}(\mathcal{U}, \mathcal{H})$ is constructed; it parameterizes and classifies all dual mock-Lie algebras which stabilize $\mathcal{H}$ as a subalgebra with codimension equal to the dimension of $\mathcal{U}$. The unified product is a general structure that includes special cases such as crossed products, bi-crossed products, semi-direct products, and skew-crossed products derived from dual mock-Lie algebras. Section 3 discusses all of these special cases in detail, emphasizing the role of problems arising related to each one. We define matched pairs of dual mock-Lie algebras and the related bi-crossed product in Definition 3.3. The Galois group of the extension $\mathcal{H} \subseteq \mathcal{H} \bowtie \mathcal{U}$ is uniquely computed in Corollary 3.6 as a subgroup of the semidirect product of groups $\mathrm{GL}_{\mathbb{F}}(\mathcal{U}) \rtimes \mathrm{Hom}_{\mathbb{F}}(\mathcal{U}, \mathcal{H})$ if $\mathcal{H} \bowtie \mathcal{U}$ is the bi-crossed product related with a matched pair $(\mathcal{H}, \mathcal{U}, \lhd, \rhd)$ of dual mock-Lie algebras. In Theorem 3.10, we discuss an application of crossed products as the main characters in theory for sorting finite dimensional supersolvable dual mock-Lie algebras.

Throughout this paper, $\mathcal{H}, \mathcal{U}$ are two vector spaces on a field $\mathbb{F}$ of characteristic, not 2 nor 3. A bilinear map $h : \mathcal{H} \times \mathcal{H} \to \mathcal{U}$ is said to be skew-symmetric if $h(x_1, x_2) = -h(x_2, x_1)$, for all $x_1, x_2 \in \mathcal{H}$.

Definition 1.1. *a dual mock-Lie algebra consists of vector space $\mathcal{H}$ with a bilinear map $(.) : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ such that $p.q = -q.p$ and $p.(q.r) = -(p.q).r$, for all $p, q, r \in \mathcal{H}$.*

Example 1.2. *Let $\mathcal{H}$ be a 7-dimentional vector space and $(u_i)_{1 \leq i \leq 7}$ a basis of $\mathcal{H}$. The product $(\cdot)$ given on $\mathcal{H}$ by*

$$u_1 \cdot u_2 = u_4,\ u_1 \cdot u_3 = u_5,\ u_2 \cdot u_3 = u_6,\ u_1 \cdot u_6 = -u_2 u_5 = u_3 u_4 = u_7,$$

defines a dual mock-Lie structure on $\mathcal{H}$. This dual mock-Lie algebra is denoted by $\mathcal{F}_{\overline{AA}}(3)$ in page 13 of [5].

A left $\mathcal{H}$-module is a space $\mathcal{U}$ endowed with a bilinear mapping $\rhd: \mathcal{H} \times \mathcal{U} \to \mathcal{U}$, called action, so that

$$(p.q) \rhd x = -p \rhd (q \rhd x_1) \tag{1.1}$$

for any $p, q \in \mathcal{H}$ and $x_1 \in \mathcal{U}$. Moreover, we denote the entity of all (left) $\mathcal{H}$-modules with action-preserving linear maps as morphisms by ${}_{\mathcal{H}}\mathcal{W}$. A right $\mathcal{H}$-module is a space $\mathcal{U}$ defined with a bilinear map $\lhd: \mathcal{U} \times \mathcal{H} \to \mathcal{U}$ so that

$$x_1 \lhd (p.q) = -(x_1 \lhd p) \lhd q \tag{1.2}$$

for any $p, q \in \mathcal{H}$ and $x_1 \in \mathcal{U}$.

Now, we discuss the ES problem for dual mock-Lie algebras. We will begin by introducing the following:

Definition 1.3. *Consider the dual mock-Lie algebra $\mathcal{H}$, and the space $\mathcal{V}$ containing $\mathcal{H}$. Two dual mock-Lie algebra structures$(.)$ and $(.')$ on $\mathcal{V}$ which contain $\mathcal{H}$ as a subalgebra, are called equivalent, and we abbreviate this by*

$(\mathcal{V}, .) \equiv (\mathcal{V}, .')$, if there exists a dual mock-Lie algebra isomorphism $\psi : (\mathcal{V}, .) \to (\mathcal{V}, .')$ which stabilizes $\mathcal{H}$; i.e., $\psi(p) = p$, for all $p \in \mathcal{H}$. The notation $\mathrm{Extd}(\mathcal{V}, \mathcal{H})$ represents the collection of all equivalence classes of all dual mock-Lie algebras structures on $\mathcal{V}$ that contain $\mathcal{H}$ as a subalgebra with respect to the equivalence relation $\equiv$.

$\mathrm{Extd}(\mathcal{V}, \mathcal{H})$ *is the sorting object of the E-S problem. In this section, we show that the extended $(\mathcal{V}, \mathcal{H})$ is parameterized by a cohomological type object which is denoted by $\mathcal{C}^2_{\mathcal{H}}(\mathcal{U}, \mathcal{H})$, where $\mathcal{U}$ is a complement of $\mathcal{H}$ in $\mathcal{V}$, that is $\mathcal{V} = \mathcal{H} + \mathcal{U}$ and $\mathcal{H} \cap \mathcal{U} = 0$.*

Definition 1.4. *Suppose $\mathcal{H}$ is a dual mock-Lie algebra and $\mathcal{U}$ a space. An extending datum of $\mathcal{H}$ through $\mathcal{U}$ is a system $\mho(\mathcal{H},\mathcal{U}) = (\triangleleft, \triangleright, h, \{-,-\})$ consisting of four bilinear maps*

$$\triangleleft : \mathcal{U} \times \mathcal{H} \to \mathcal{U}, \quad \triangleright : \mathcal{U} \times \mathcal{H} \to \mathcal{H}, \quad h : \mathcal{U} \times \mathcal{U} \to \mathcal{H}, \quad \{-,-\} : \mathcal{U} \times \mathcal{U} \to \mathcal{U}.$$

Let $\mho(\mathcal{H},\mathcal{U}) = (\triangleleft, \triangleright, h, \{-,-\})$ be an extending datum.

The notation $\mathcal{H}\natural_{\mho(\mathcal{H},\mathcal{U})}\mathcal{U} = \mathcal{H}\natural\mathcal{U}$ is the vector space $\mathcal{H} \times \mathcal{U}$ together with the bilinear mapping $\star : (\mathcal{H} \times \mathcal{U}) \times (\mathcal{H} \times \mathcal{U}) \to \mathcal{H} \times \mathcal{U}$ defined for all $p, q \in \mathcal{H}$ and $x_1, x_2 \in \mathcal{U}$ by:

$$(p, x_1)\star(q, x_2) := (p.q + x_1 \triangleright q - x_2 \triangleright p + h(x_1, x_2), \quad \{x_1, x_2\} + x_1 \triangleleft q - x_2 \triangleleft p) \tag{1.3}$$

With the multiplication given by above equation the object $\mathcal{H}\natural\mathcal{U}$ is called the unified product of $\mathcal{H}$ and $\mathcal{U}$ if it is a dual mock-Lie algebra. In this case the extending datum $\mho(\mathcal{H},\mathcal{U}) = (\triangleleft, \triangleright, h, \{-,-\})$ is known as a dual mock-Lie E-S of $\mathcal{H}$ through space $\mathcal{U}$. The actions of $\mho(\mathcal{H},\mathcal{U})$ are the maps $\triangleleft$ and $\triangleright$ and the cocycle of $\mho(\mathcal{H},\mathcal{U})$ is h.

Suppose $\mho(\mathcal{H},\mathcal{U})$ is an extending datum of $\mathcal{H}$ through space $\mathcal{U}$. Then, the very useful computations that hold in $\mathcal{H}\natural\mathcal{U}$ follow the given relations for all $p, q \in \mathcal{H}$ and $x_1, x_2 \in \mathcal{U}$:

$$(p, 0) \star (q, x_2) = (p.q - x_2 \triangleright p, -x_2 \triangleleft p) \tag{1.4}$$

$$(0, x_1) \star (q, x_2) = (x_1 \triangleright q + h(x_1, x_2), x_1 \triangleleft q + \{x_1, x_2\}) \tag{1.5}$$

2 Main results: Describing extensions of dual mock-Lie algebras and applications

Theorem 2.1. *Let $\mho(\mathcal{H},\mathcal{U}) = (\triangleleft, \triangleright, h, \{-,-\})$ be an extending datum of a dual mock-Lie algebra $\mathcal{H}$ through $\mathcal{U}$. The following assertions are equivalent:*

(1) *$\mathcal{H}\natural\mathcal{U}$ is a unified product;*

(2) *The following compatibilities hold for any $p, q \in \mathcal{H}, x_1, x_2, x_3 \in \mathcal{U}$:*

(V1) $h : \mathcal{U} \times \mathcal{U} \to \mathcal{H}$ *and* $\{-,-\} : \mathcal{U} \times \mathcal{U} \to \mathcal{U}$ *are skew-symmetric maps;*

(V2) $(\mathcal{U}, \triangleleft)$ *is a right $\mathcal{H}$-module;*

(V3) $x_1 \triangleright (p.q) = (x_1 \triangleright q).p + (x_1 \triangleleft q) \triangleright p$;

(V4) $\{x_1, x_2\} \triangleleft p = -\{x_1, x_2 \triangleleft p\} - x_1 \triangleleft (x_2 \triangleright p)$;

(V5) $\{x_1, x_2\} \triangleright p = -x_1 \triangleright (x_2 \triangleright p) + p.h(x_1, x_2) - h(x_1, x_2 \triangleleft p)$;

(V6) $h(x_1, \{x_2, x_3\})h(\{x_1, x_2\}, x_3) + x_1 \triangleright h(x_2, x_3) - x_3 \triangleright h(x_1, x_2) = 0$

(V7) $\{x_1, \{x_2, x_3\}\} + \{\{x_1, x_2\}, x_3\} + x_1 \triangleleft h(x_2, x_3) - x_3 \triangleleft h(x_1, x_2) = 0$

Proof. Despite the lengthy and laborious computation, the proof is straightforward. We show only the main steps. First, it is easy to show that multiplication (1.3) is anti-commutative iff both $h : \mathcal{U} \times \mathcal{U} \to \mathcal{H}$ and $\{-,-\} : \mathcal{U} \times \mathcal{U} \to \mathcal{U}$ are skew-symmetric maps, that is (V1) holds. The statement (V1) will now be considered true. Thus $\mathcal{H}\natural\mathcal{U}$ is a dual mock-Lie algebra iff the anti-associativity property is satisfied; that is, for all $p, q, r \in \mathcal{H}$ and $x_1, x_1, x_3 \in \mathcal{U}$:

$$(p, x_1) \star \Big((q, x_2) \star (r, x_3)\Big) = -\Big((p, x_1) \star (q, x_2)\Big) \star (r, x_3) \qquad (2.6)$$

We have $(p, x_1) = (p, 0) + (0, x_1)$ since $\mathcal{H}\natural\mathcal{U}$. Consequently, (2.6) holds iff it is true for all generators of $\mathcal{H}\natural\mathcal{U}$, that is for the set $\{(p, 0) \mid p \in \mathcal{H}\} \cup \{(0, x_1) \mid x_1 \in \mathcal{U}\}$. As under anti-associativity property (2.6) is invariant, there are only three cases to consider. By using equation (1.4), it is easy to see that for the triple $(p, 0), (q, 0), (r, 0)$ equation (2.6) holds. Now, we can prove that for $(p, 0)$, $(q, 0)$, $(0, x_1)$ equation (2.6) holds , iff (V2) and (V3) also hold. Furthermore, (V4) and (V5) can be proved to be true iff $(p, 0)$, $(0, x_1)$, and $(0, x_2)$ are true. In conclusion, (2.6) holds for $(0, x_1), (0, x_2), (0, x_3)$ iff (V6) and (V7) hold. This completes the proof. □

We denote the collection of all dual mock-Lie E-S of $\mathcal{H}$ through space $\mathcal{U}$ by $\mathcal{AA}(\mathcal{H}, \mathcal{U})$. That is all systems $\mho(\mathcal{H}, \mathcal{U}) = (\triangleleft, \triangleright, h, \{-,-\})$ fulfilling the compatibility conditions (V1)-(V7) of Theorem 2.1. Observe that $\mathcal{AA}(\mathcal{H}, \mathcal{U})$ is nonempty because it includes the E-S $\mho(\mathcal{H}, \mathcal{U}) = (\triangleleft, \triangleright, h, \{-,-\})$ in which all bilinear mappings are trivial. For this case, the associated unified product $\mathcal{H}\natural\mathcal{U} = \mathcal{H}\times\mathcal{U}$, the direct product between $\mathcal{H}$ and the abelian dual mock-Lie algebra $\mathcal{U}$.

Consider $\mho(\mathcal{H},\mathcal{U}) = (\triangleleft, \triangleright, h, \{-,-\}) \in \mathcal{AA}(\mathcal{H},\mathcal{U})$, a dual mock-Lie algebra E-S and $\mathcal{H}\natural\mathcal{U}$ the associated unified product. Then the canonical inclusion

$$i_{\mathcal{H}} : \mathcal{H} \to \mathcal{H}\natural\mathcal{U}, \quad i_{\mathcal{H}}(p) = (p, 0) \tag{2.7}$$

is an injective dual mock-Lie algebra mapping. Therefore, we can see $\mathcal{H}$ as a dual mock-Lie subalgebra of $\mathcal{H}\natural\mathcal{U}$ through the identification $\mathcal{H} \cong i_{\mathcal{H}}(\mathcal{H}) \cong \mathcal{H} \times \{0\}$. On the other hand, we will demonstrate that any dual mock-Lie algebra structure on $\mathcal{U}$ containing $\mathcal{H}$ as a subalgebra is isomorphic to a unified product.

Theorem 2.2. *Suppose $\mathcal{H}$ is a dual mock-Lie algebra, $\mathcal{V}$ a vector space that contain $\mathcal{H}$ as a subspace and $(.)$ a dual mock-Lie structure on space $\mathcal{V}$ so that $\mathcal{H}$ is a subalgebra in $(\mathcal{V}, .)$. Then there exists a dual mock-Lie Extending structure $\mho(\mathcal{H},\mathcal{U}) = (\triangleleft, \triangleright, h, \{-,-\})$ of $\mathcal{H}$ via a vector subspace $\mathcal{U}$ of $\mathcal{V}$ and an isomorphism of dual mock-Lie algebras $\mathcal{V} \cong \mathcal{H}\natural\mathcal{U}$ that stabilizes $\mathcal{H}$.*

Proof. As, we are working over a field $\mathbb{F}$, there exists $\varphi : \mathcal{V} \to \mathcal{H}$ linear map such that for all $p \in \mathcal{H}$, $\varphi(p) = p$. Then $\mathcal{U} := \ker(\varphi)$ is a subspace of $\mathcal{U}$ and also complement of $\mathcal{H}$ in $\mathcal{U}$. Now, We can define the extending datum of $\mathcal{H}$ via space $\mathcal{U}$ for any $p \in \mathcal{H}$ and $x_1, x_2 \in \mathcal{U}$ as below:

$$\begin{aligned} \triangleright = \triangleright_{\varphi} &: \mathcal{U} \times \mathcal{H} \to \mathcal{H}, & x \triangleright p &:= \varphi(x_1.p) \\ \triangleleft = \triangleleft_{\varphi} &: \mathcal{U} \times \mathcal{H} \to \mathcal{V}, & x_1 \triangleleft \varphi &:= x_1.p - \varphi(x_1.p) \\ h = h_{\varphi} &: \mathcal{U} \times \mathcal{U} \to \mathcal{H}, & h(x_1, x_2) &:= \varphi(x_1.x_2) \\ \{,\} = \{,\}_{\varphi} &: \mathcal{U} \times \mathcal{U} \to \mathcal{U}, & \{x_1, x_2\} &:= x_1.x_2 - \varphi(x_1.x_2) \end{aligned}$$

First, it is obvious that the aforementioned mappings are well defined bilinear maps: $x_1 \triangleleft p \in \mathcal{U}$ and $\{x_1, x_2\} \in \mathcal{U}$, for all $p \in \mathcal{H}$ and $x_1, x_2 \in \mathcal{U}$. We will prove that $\mho(\mathcal{H},\mathcal{U}) = (\triangleleft, \triangleright, h, \{-,-\})$ is a dual mock-Lie E-S of $\mathcal{H}$ through space $\mathcal{U}$ and $\psi : \mathcal{H}\natural\mathcal{U} \to \mathcal{V}$, $\psi(p, x_1) := p+x_1$ is an isomorphism of dual mock-Lie algebras that stabilizes $\mathcal{H}$. On the basis of Theorem 2.1, the process we use is the following: $\psi : \mathcal{H} \times \mathcal{U} \to \mathcal{V}$ defined as $\psi(p, x_1) := p + x_1$ is a linear isomorphism between the dual mock-Lie algebra $\mathcal{U}$ and the direct product of $\mathcal{H} \times \mathcal{U}$ with the inverse defined by $\psi^{-1}(x_2) := (\varphi(x_2), x_2 - \varphi(x_2))$, for all $x_2 \in \mathcal{V}$. Therefore, there is a unique dual mock-Lie algebra structure, $(\star)$, on vector spaces $\mathcal{H} \times \mathcal{U}$ such that ψ is an isomorphism of dual mock-Lie algebras and this exclusive multiplication on $\mathcal{H} \times \mathcal{U}$ for any x_1, $x_2 \in \mathcal{U}$ and $p, q \in \mathcal{H}$ is given by:

$$(p, x_1) \star (q, x_2) := \psi^{-1}(\psi(p, x_1).\psi(q, x_2))$$

Now, the objective is to show that the multiplication coincides with the one associated to $(\triangleleft_\varphi, \triangleright_\varphi, h_\varphi, \{-,-\}_\varphi)$ as defined by (1.3). In fact, for any x_1, $x_2 \in \mathcal{U}$ and $p, q \in \mathcal{H}$ we have:

$$\begin{aligned}(p, x_1) \star (q, x_2) =& \psi^{-1}(\psi(p, x_1).\psi(q, x_2)) = \psi^{-1}(p.q + p.x_2 + x_1.q + x_1.x_2)\\ =& (\varphi(p.q), p.q - \varphi(p.q)) + (\varphi(p.x_2), p.x_2 - \varphi(p.x_2))\\ &+ (\varphi(x_1.q), x_1.q - \varphi(x_1.q)) + (\varphi(x_1.x_2), x_1.x_2 - \varphi(x_1.x_2))\\ =& (\varphi(p.q) + \varphi(p.x_2) + \varphi(x_1.q) + \varphi(x_1.x_2), p.q + p \star x_2\\ &+ x_1.q + x_1.x_2 - \varphi(p.q) - \varphi(p.x_2) - \varphi(x_1.q) - \varphi(x_1.x_2))\\ =& (p.q + \varphi(p.x_2) + \varphi(x_1.q) + \varphi(x_1.x_2), p.x_2 - \varphi(p.x_2) +\\ &+ x_1.q - \varphi(x_1.q) + x_1.x_2 - \varphi(x_1.x_2))\\ =& (p.q - x_2 \triangleright p + x_1 \triangleright q + h(x_1, x_2), \{x_1, x_2\} + x_1 \triangleleft b - x_2 \triangleleft p)\end{aligned}$$

as required. Note that in the above computation the anti-commutativity of $\star$ was intensively used. Furthermore, the following figure is clearly anti-commutative which explains that ψ stabilizes $\mathcal{H}$ and this completes the proof. □

The classification of all dual mock-Lie algebra structures on $\mathcal{V}$ that contain $\mathcal{H}$ as a subalgebra can be reduced to the classification of all unified products $\mathcal{H}\natural\mathcal{U}$, associated to all dual mock-Lie E-S $\mho(\mathcal{H},\mathcal{U}) = (\triangleleft, \triangleright, h, \{-,-\})$, for a given complement $\mathcal{U}$ of $\mathcal{H}$ in $\mathcal{V}$ by using Theorem 2.2.

To construct a cohomological type object (c.f. [3]) $\mathcal{C}^2_{\mathcal{H}}(\mathcal{U},\mathcal{H})$ parameterized by the classifying sets $\text{Extd}(\mathcal{V},\mathcal{H})$ defined in Definition 1.3, we would like to introduce the following:

Lemma 2.3. *Suppose that $\mho(\mathcal{H},\mathcal{U}) = (\triangleleft, \triangleright, h, \{-,-\})$ and $\mho'(\mathcal{H},\mathcal{U}) = (\triangleleft', \triangleright', h', \{-,-\}')$ are two dual mock-Lie algebra E-S of $\mathcal{H}$ through space $\mathcal{U}$ and $\mathcal{H}\natural\mathcal{U}$, respectively $\mathcal{H}\natural'\mathcal{U}$, the associated unified products. Then there exists a bijection between the set of all morphisms of dual mock-Lie algebras $\eta : \mathcal{H}\natural\mathcal{U} \to \mathcal{H}\natural'\mathcal{U}$ which stabilize $\mathcal{H}$ and the pairs (m, n), where $m : \mathcal{U} \to \mathcal{H}, n : \mathcal{U} \to \mathcal{U}$ are two linear mappings that satisfy the following of conditions of compatibility for any $x_1, x_2 \in \mathcal{U}, p \in \mathcal{H}$:*

(M1) $n(x_1) \triangleleft' p = n(x_1 \triangleleft p)$, that is $\mathcal{U}$ is a morphism of right $\mathcal{H}$-modules;

(M2) $n(x_1) \triangleright' p = m(x_1 \triangleleft p) + x_1 \triangleright p + p.m(x_1)$;

(M3) $n(\{x_1, x_2\}) = \{n(x_1), n(x_2)\}' + n(x_1) \triangleleft' m(x_2) - n(x_2) \triangleleft' m(x_1)$;

(M4) $m(\{x_1, x_2\}) = m(x_1).m(x_2) + n(x_1) \rhd' m(x_2) - n(x_2) \rhd' m(x_1) + h'(n(x_1), n(x_2)) - h(x_1, x_2)$.

Under the above bijection the morphism of dual mock-Lie algebras $\eta = \eta_{(m,n)} : \mathcal{H}\natural\mathcal{U} \to \mathcal{H}\natural'\mathcal{U}$ *corresponding to* (m, n) *is given for any* $p \in \mathcal{H}$ *and* $x_1 \in \mathcal{U}$ *by:*

$$\eta(p, x_1) = (p + m(x_1), n(x_1))$$

Furthermore, $\eta = \eta_{(m,n)}$ *is an isomorphism iff* $n : \mathcal{U} \to \mathcal{U}$ *is bijective.*

Proof. The linear map η making the diagram below commutative

$$\begin{array}{ccc} \mathcal{H} & \xrightarrow{i_{\mathcal{H}}} & \mathcal{H}\natural\mathcal{U} \\ {\scriptstyle id_{\mathcal{H}}}\downarrow & & \downarrow{\scriptstyle \eta} \\ \mathcal{H} & \xrightarrow{id_{\mathcal{H}}} & \mathcal{H}\natural'\mathcal{U} \end{array}$$

is uniquely obtained by two linear maps $m : \mathcal{U} \to \mathcal{H}, n : \mathcal{U} \to \mathcal{U}$ such that $\eta : (p, x_1) = (p + m(x_1), n(x_1))$, for all $x_1 \in \mathcal{U}$ and $p \in \mathcal{H}$. Indeed, if we denote $\eta(0, x_1) = (m(x_1), n(x_1)) \in \mathcal{H} \times \mathcal{U}$ for all $x_1 \in \mathcal{U}$, we obtain:

$$\begin{aligned} \eta(p, x_1) &= \eta((p, 0) + \eta(0, x_1)) = \eta(p, 0) + \eta(0, x_1) \\ &= (p, 0) + (m(x_1), n(x_1)) = (p + m(x_1), n(x_1)) \end{aligned}$$

Consider $\eta = \eta_{(m,n)}$, such a linear map; that is, $\eta(p, x_1) = (p + m(x_1), n(x_1))$, for some linear mappings $m : \mathcal{U} \to \mathcal{H}, n : \mathcal{U} \to \mathcal{U}$. We show that η is a morphism of dual mock-Lie algebras iff the conditions of compatibility (M1)-(M4) hold. For this, it is enough to show that the compatibility

$$\eta((p, x_1) \star (q, x_2)) = \eta(p, x_1) \star' \eta(q, x_2)] \tag{2.8}$$

satisfy for all generators $\mathcal{H}\natural\mathcal{U}$. Likewise, we will skip the lengthy calculations and only show the important steps. First, easy to see that for all $p, q \in \mathcal{H}$, the pair $(p, 0)$, $(b, 0)$ hold for (2.8). Secondly, we can show that for the pair $(p, 0), (0, x_1)$ (2.8) holds iff (M1) and (M2) are true. In the end, (2.8) holds for the pair $(0, x_1), (0, x_2)$ iff (M3) and (M4) are satisfied. The last assertion follows just after noting that if $n : \mathcal{U} \to \mathcal{V}$ is bijective, then $\psi_{(m,n)}$ is an isomorphism of dual mock-Lie algebras with an inverse given for any $q \in \mathcal{H}$ and $y \in \mathcal{U}$ according to:

$$\eta^{-1}_{(m,n)}(q, x_2) = \left(q - m\left(n^{-1}(x_2)\right), n^{-1}(x_2)\right)$$

The proof is now complete. □

For the sake of classification we introduce the subsequent:

Definition 2.4. *Suppose $\mathcal{H}$ is a dual mock-Lie algebra and $\mathcal{U}$ a vector space. Two dual mock-Lie algebra extending systems of $\mathcal{H}$ through $\mathcal{U}$, $\mho(\mathcal{H},\mathcal{U}) = (\lhd, \rhd, h, \{-,-\})$ and $\mho'(\mathcal{H},\mathcal{U}) = (\lhd', \rhd', h', \{-,-\}')$ are referred to as equivalent, and we denote this by means of $\mho(\mathcal{H},\mathcal{U}) \equiv \mho'(\mathcal{H},\mathcal{U})$, if there exists a pair of linear mappings (m,n), where $m : \mathcal{U} \to \mathcal{H}$ and $v \in \mathrm{Aut}_{\mathbb{F}}(\mathcal{U})$ such that $(\lhd', \rhd', h', \{-,-\}')$ is described through $(\lhd, \rhd, h, \{-,-\})$ using (m,n) for all $p \in \mathcal{H}, x_1, x_2 \in \mathcal{U}$ as below:*

$$
\begin{aligned}
x_1 \lhd' p =& n\left(n^{-1}(x_1) \lhd p\right)\\
x_1 \rhd' p =& -m\left(n^{-1}(x_1) \lhd p\right) - n^{-1}(x_1) \rhd p - x_1 \rhd p\\
h'(x_1,x_2) =& h\left(n^{-1}(x_1), n^{-1}(x_2)\right) + m\left(\left\{n^{-1}(x_1), n^{-1}(x_2)\right\}\right)\\
&+ m\left(n^{-1}(x_1)\right).m\left(n^{-1}(x_2)\right)\\
&- m\left(n^{-1}(x_1) \lhd m\left(n^{-1}(x_2)\right)\right) - n^{-1}(x_1) \rhd m\left(n^{-1}(x_2)\right)\\
&+ m\left(n^{-1}(x_2) \lhd m\left(n^{-1}(x_1)\right)\right) + n^{-1}(x_2) \rhd m\left(n^{-1}(x_1)\right)\\
\{x_1,x_2\}' =& n\left(\left\{n^{-1}(x_1), n^{-1}(x_2)\right\}\right) - n\left(n^{-1}(x_1) \lhd m\left(n^{-1}(x_2)\right)\right)\\
&- n\left(n^{-1}(x_2) \lhd m\left(n^{-1}(x_1)\right)\right)
\end{aligned}
$$

Based on the results of this section, we can solve the dual mock-Lie algebra E-S problem as below:

Theorem 2.5. *Consider $\mathcal{H}$ as a dual mock-Lie algebra, $\mathcal{V}$ as a space containing $\mathcal{H}$ as subspace, and $\mathcal{U}$ as $\mathcal{H}$ complement in $\mathcal{V}$. Then:*

(I) $\equiv$ is an equivalence relation on $\mathcal{AA}(\mathcal{H},\mathcal{U})$ of all dual mock-Lie algebra E-S of $\mathcal{H}$ via $\mathcal{U}$. We denote the quotient set by $\mathcal{C}^2_{\mathcal{H}}(\mathcal{U},\mathcal{H}) := \mathcal{AA}(\mathcal{H},\mathcal{U})/\equiv$.

(II) The map

$$
\mathcal{C}^2_{\mathcal{H}}(\mathcal{U},\mathcal{H}) \to \mathrm{Extd}(\mathcal{V},\mathcal{H}), \quad \overline{(\lhd, \rhd, h, \{-,-\})} \to (\mathcal{H}\natural\mathcal{U}, \star)
$$

is bijective, here $\overline{(\lhd, \rhd, h, \{-,-\})}$ is an equivalence class of $(\lhd, \rhd, h, \{-,-\})$ through $\equiv$.

Proof. From Theorem 2.1, Theorem 2.2 and Lemma 2.3 we see that $\mho(\mathcal{H},\mathcal{U}) \equiv \mho'(\mathcal{H},\mathcal{U})$ according to the Definition 2.4 iff there exists an isomorphism of dual mock-Lie algebras $\psi : \mathcal{H}\natural\mathcal{U} \to \mathcal{H}\natural'\mathcal{U}$ which stabilizes $\mathcal{H}$. Hence, $\equiv$ is an equivalence relation on $\mathcal{AA}(\mathcal{H},\mathcal{U})$ of all dual mock-Lie algebra E-S $\mho(\mathcal{H},\mathcal{U})$ and from Theorem 2.2 and Lemma 2.3 conclusion follows. □

3 Applications of unified products

This section describes the prominent special cases of unified products, with the names semidirect/crossed/bi-crossed/skew crossed products, as well as their applications. We assume the following rule:

if one of $\triangleleft, \triangleright, f$ or $\{-,-\}$ mappings of $\mho(\mathcal{H},\mathcal{U}) = (\triangleleft, \triangleright, h, \{-,-\})$ is trivial, it will be left out of the quadruple $(\triangleleft, \triangleright, h, \{-,-\})$.

3.1 Matched pairs

Consider $\mho(\mathcal{H},\mathcal{U}) = (\triangleleft, \triangleright, h, \{-,-\})$ an extending datum of $\mathcal{H}$ via $\mathcal{U}$, so that h is the trivial map, that is $h(x_1, x_2) = 0$ for all $x_1, x_2 \in \mathcal{U}$. Then, by using Theorem 2.1, we get that $\mho(\mathcal{H},\mathcal{U}) = (\triangleleft, \triangleright, \{-,-\})$ is a dual mock-Lie E-S of $\mathcal{H}$ via the space $\mathcal{U}$ iff $(\mathcal{V}, \{-,-\})$ is a dual mock-Lie algebra and for all $p, q \in \mathcal{H}, x_1, x_2 \in \mathcal{U}$, the following compatibilities are satisfied:

(1) $(\mathcal{U}, \triangleleft)$ is a right $\mathcal{H}$-module.

(2) $x_1 \triangleright (p.q) = (x_1 \triangleright q).p + (x_1 \triangleleft q) \triangleright p$;

(3) $\{x_1, x_2\} \triangleleft p = -\{x_1, x_2 \triangleleft p\} - x_1 \triangleleft (x_2 \triangleright p)$;

(4) $\{x_1, x_2\} \triangleright p = -x_1 \triangleright (x_2 \triangleright p)$; (i.e $(\mathcal{H}, \triangleright)$ is a left $\mathcal{U}$-module).

Definition 3.1. *A (δ, γ)-derivation is a linear map $D : \mathcal{H} \to \mathcal{H}$ which satisfies*

$$D(xy) = \delta D(x)y + \gamma x D(y)),$$

where δ, γ are some fixed elements of the ground field. The space of (δ, γ)-derivations is denoted by $Der_{(\delta,\gamma)}(\mathcal{H})$.

Remark 3.2. *The space containing all dual mock-Lie algebra's (δ, γ)-derivations does not possess canonical dual mock-Lie algebra structure like Lie algebras.*

By Following the ([13], Theorem 4.1) we define the following definition:

Definition 3.3. *Suppose $\mathcal{H} = (\mathcal{H}, .)$ and $\mathcal{U} = (\mathcal{U}, \{-,-\})$ are two dual mock-Lie algebras.*

Then $(\mathcal{H}, \mathcal{U}, \triangleleft, \triangleright)$ is known as matched pair of dual mock-Lie algebras if $(\mathcal{U}, \triangleleft)$ is a right $\mathcal{H}$, $(\mathcal{H}, \triangleright)$ is a left $\mathcal{U}$-module, and for all $p, q \in \mathcal{H}, x_1, x_2 \in \mathcal{U}$ the given compatibilities satisfy:

(MP1): $\quad x_1 \triangleright (p.q) = (x_1 \triangleright q).p + (x_1 \triangleleft q) \triangleright p;$

(MP2): $\{x_1, x_2\} \lhd p = -\{x_1, x_2 \lhd p\} - x_1 \lhd (x_2 \rhd p);$

For $(\mathcal{H}, \mathcal{U}, \lhd, \rhd)$ *the notation* $\mathcal{H} \bowtie \mathcal{U}$ *represent the unified product* $\mathcal{H}\natural_{\mho(\mathcal{H},\mathcal{U})}\mathcal{V}$ *and will be known as the bi-crossed product of* $(\mathcal{H}, \mathcal{U}, \lhd, \rhd)$*. Therefore,* $\mathcal{H} \bowtie \mathcal{U} = \mathcal{H} \times \mathcal{U}$ *as a vector space with multiplication for all* $p, q \in \mathcal{H}$ *and* $x_1, x_2 \in \mathcal{U}$ *follows as:*

$$(p, x_1) \star (q, x_2) := (p.q + x_1 \rhd q - x_2 \rhd p, \{x_1, x_2\} + x_1 \lhd q - x_2 \lhd p)$$

Example 3.4. *Suppose the matched pair* $(\mathcal{H}, \mathcal{U}, \lhd, \rhd)$ *so that* $\lhd$ *is the trivial mapping. Then* $\mathcal{H} \bowtie \mathcal{U}$ *the associated bi-crossed product was first defined like in [12] with the name of semidirect product. The associated bi-crossed product will be denoted by* $\mathcal{H} \bowtie \mathcal{U}$ *and uniquely, the semidirect product* $\mathcal{H} \bowtie \mathcal{U}$ *is associated to a left* $\mathcal{U}$*-module structure* $(\mathcal{H}, \rhd)$ *so that for any* $\mathcal{H}$*,* $p, q \in \mathcal{H}$ *and* $x_1 \in \mathcal{U}$ *:*

$$x_1 \rhd (p.q) = -p.(x_1 \rhd q);$$

or equivalently for all $x_1 \in \mathcal{U}$ *the map* $x_1 \rhd - : A \to \mathcal{H}$ *is an* $(0, -1)$*-derivation of* $\mathcal{H}$*.*

The bi-crossed product of two dual mock-Lie algebras is the key to solving a factorization problem: Suppose $\mathcal{H}$ and $\mathcal{U}$ are two given dual mock-Lie algebras. Analyze and classify all dual mock-Lie algebras $\mathcal{U}$ that factorize via $\mathcal{H}$ and $\mathcal{U}$, that is $\mathcal{U}$ contains $\mathcal{H}$ and $\mathcal{U}$ as dual mock-Lie subalgebras such that $\mathcal{V} = \mathcal{H} + \mathcal{U}$ and $\mathcal{H} \cap \mathcal{U} = \{0\}$.

In fact, Theorem 2.2 allows us to prove the dual mock-Lie algebra version of ([8] Theorem 3.9):

Proposition 3.5. *A dual mock-Lie algebra* $\mathcal{U}$ *factorizes via two given dual mock-Lie algebras* $\mathcal{H}$ *and* $\mathcal{U}$ *iff there exists a matched pair* $(\mathcal{H}, \mathcal{U}, \lhd, \rhd)$ *such that* $\mathcal{V} \cong \mathcal{H} \bowtie \mathcal{U}$*.*

Proof. First, note that $\mathcal{U} \cong \{0\} \times \mathcal{U}$ and $\mathcal{H} \cong \mathcal{H} \times \{0\}$ are dual mock-Lie subalgebras of $\mathcal{H} \bowtie \mathcal{U}$ and no doubt $\mathcal{H} \bowtie \mathcal{U}$ factorizes via $\{0\} \times \mathcal{V}$ and $\mathcal{H} \times \{0\}$. Conversely, consider that a dual mock-Lie algebra $\mathcal{U}$ factorizes via two dual mock-Lie subalgebras $\mathcal{H}$ and $\mathcal{U}$. Since $\mathcal{U}$ is a subalgebra of $\mathcal{V}$, the cocycle $h = h_\varphi : \mathcal{U} \times \mathcal{U} \to \mathcal{H}$ established in the proof of Theorem 2.2 is just the trivial map that is for all $x_1, x_2 \in \mathcal{U}$, $h_\varphi(x_1, x_2) = 0$. Therefore, the unified product $M_{\mho(\mathcal{H},\mathcal{U})}\mathcal{V} = \mathcal{H} \bowtie \mathcal{U}$ coincides with the bi-crossed product of the dual mock-Lie algebras $\mathcal{H}$ and $\mathcal{U} := \mathrm{Ker}(\varphi)$. □

The factorization problem can be restated based on Corollary 3.5 as below: Suppose $\mathcal{H}$ and $\mathcal{U}$ are two given dual mock-Lie algebras. The objective is to identify all the matched pairs $(\mathcal{H},\mathcal{U},\triangleleft,\triangleright)$ and classify up to an isomorphism all bi-crossed products $\mathcal{H}\bowtie\mathcal{U}$.

The problem will be discussed separately in a forthcoming paper because of its significant applications to the theory of dual mock-Lie algebras.

Here, we compute the Galois group of the dual mock-Lie algebra extension $\mathcal{H}\subseteq\mathcal{H}\bowtie\mathcal{U}$. For $(\mathcal{H},\mathcal{U},\triangleleft,\triangleright)$, we establish the Galois group $\mathrm{Gl}(\mathcal{H}\bowtie\mathcal{U}/\mathcal{H})$ of the extension $\mathcal{H}\subseteq\mathcal{H}\bowtie\mathcal{U}$, which is the subgroup of $\mathrm{Aut}_{\text{ dualmock-Lie}}(\mathcal{H}\bowtie\mathcal{U})$ of all dual mock-Lie algebra automorphisms of $\mathcal{H}\bowtie\mathcal{U}$ that stabilize $\mathcal{H}$:

$$\mathrm{Gl}(\mathcal{H}\bowtie\mathcal{U}/\mathcal{H}):=\{\eta\in\mathrm{Aut}_{dualmock-Lie}(\mathcal{H}\bowtie\mathcal{U})\mid\eta(p)=p,\forall p\in\mathcal{H}\}$$

Based on the result of Lemma 2.3, we get a bijective map between the collection of all elements $\eta\in\mathrm{Gl}(\mathcal{H}\bowtie\mathcal{U}/\mathcal{H})$ and the collection all pairs $(n,m)\in\mathrm{GL}_{\mathbb{F}}(\mathcal{U})\times\mathrm{Hom}_{\mathbb{F}}(\mathcal{U},\mathcal{H})$, with the following compatibility assertions for any $p\in\mathcal{H}$ and $x_1,x_2\in\mathcal{U}$:

(G1) $n(x_1)\triangleleft p=n(x_1\triangleleft p)$;
(G2) $n(x_1)\triangleright p=m(x_1\triangleleft p)+x_1\triangleright p-p;m(x_1)$;
(G3) $n(\{x_1,x_2\})=\{n(x_1),n(x_2)\}+n(x_1)\triangleleft m(x_2)-n(x_2)\triangleleft m(x_1)$;
(G4) $m(\{x_1,x_2\})=m(x_1).m(x_2)+n(x_1)\triangleright m(x_2)-n(x_2)\triangleright m(x_1)$.

The bijection is such that $\eta=\eta_{(n,m)}\in\mathrm{Gl}(\mathcal{H}\bowtie\mathcal{U}/\mathcal{H})$ related to $(n,m)\in\mathrm{GL}_{\mathbb{F}}(\mathcal{U})\times\mathrm{Hom}_{\mathbb{F}}(\mathcal{U},\mathcal{H})$ is defined by $\eta(p,x_1):=(p+m(x_1),n(x_1))$, for all $p\in\mathcal{H}$ and $x_1\in\mathcal{U}$. We highlight that $\eta_{(n,m)}$ is in fact an element of $\mathrm{Gl}(\mathcal{H}\bowtie\mathcal{U}/\mathcal{H})$ with the inverse defined by $\eta^{-1}_{(n,m)}(p,x_1)=(p-m\left(n^{-1}(x_1)\right),n^{-1}(x_1))$, for all $p\in\mathcal{H}$ and $x_1\in\mathcal{U}$.

The entity of all pairs $(n,m)\in\mathrm{GL}_{\mathbb{F}}(\mathcal{U})\times\mathrm{Hom}_{\mathbb{F}}(\mathcal{U},\mathcal{H})$ fulfilling the compatibility conditions (G1)-(G4) is denoted by $\mathbb{G}^{\mathcal{U}}_{\mathcal{H}}(\triangleleft,\triangleright)$. It is easy to see that $\mathbb{G}^{\mathcal{U}}_{\mathcal{H}}(\triangleleft,\triangleright)$ is a subgroup of the semidirect product of groups $\mathrm{GL}_{\mathbb{F}}(\mathcal{U})\rtimes\mathrm{Hom}_{\mathbb{F}}(\mathcal{U},\mathcal{H})$ with the group structure defined for all $n,n'\in\mathrm{GL}_{\mathbb{F}}(\mathcal{U})$ and $m,m'\in\mathrm{Hom}_{\mathbb{F}}(\mathcal{U},\mathcal{H})$ as

$$(n,m)\odot(n',m'):=(n\circ n',m\circ n'+m')$$

Now, for (n,m) and $(n',m')\in\mathbb{G}^{\mathcal{U}}_{\mathcal{H}}(\triangleleft,\triangleright),p\in\mathcal{H}$ and $x_1\in\mathcal{U}$ we have:

$$\eta_{(n,m)}\circ\eta_{(n',m')}(p,x_1)=\Big(p+m'(x_1)+m\left(n'(x_1)\right),n(n'(x_1)\Big)=\eta_{(n\circ n',m\circ n'+m')}(p,x_1).$$

That is $\eta_{(n,m)}\circ\eta_{(n',m')}=\eta_{(n\circ n',m\circ n'+m')}$. To conclude, we have proved the following result:

Corollary 3.6. *Consider $(\mathcal{H},\mathcal{U},\triangleleft,\triangleright)$, a matched pair of dual mock-Lie algebras. Then there exists an isomorphism of groups defined for all $(n,m)\in\mathbb{G}^{\mathcal{U}}_{\mathcal{H}}(\triangleleft,\triangleright)$, $p\in\mathcal{H}$ and $x_1\in\mathcal{U}$ as*

$$\mho:\mathbb{G}^{\mathcal{U}}_{\mathcal{H}}(\triangleleft,\triangleright)\to \mathrm{Gl}(\mathcal{H}\bowtie\mathcal{U}/\mathcal{H}),\quad \mho(n,m)((p,x_1)):=(p+m(x_1),n(x_1))$$

For instance, there exists an embedding $\mathrm{Gl}(\mathcal{H}\bowtie\mathcal{U}/\mathcal{H})\hookrightarrow \mathrm{GL}_{\mathbb{F}}(\mathcal{U})\rtimes\mathrm{Hom}_{\mathbb{F}}(\mathcal{U},\mathcal{H})$.

3.2 Supersolvable dual mock-Lie algebras

Definition 3.7. *An n-dimensional dual mock-Lie algebra $\mathcal{U}$ is known as supersolvable if there exists a finite chain of ideals of $\mathcal{V}$*

$$0=I_0\subset I_1\subset\cdots\subset I_n=\mathcal{V}$$

such that for all $j=0,\dots,n-1$, I_j has codimension 1 in I_{j+1}.

Consider $\mho(\mathcal{H},\mathcal{U})=(\triangleleft,\triangleright,h,\{-,-\})$, such that $\triangleleft$ is trivial. That is $x_1\triangleleft p=0$, for all $x_1\in\mathcal{U}$ and $p\in\mathcal{H}$. So, $\mho(\mathcal{H},\mathcal{U})=(\triangleright,h,\{-,-\})$ is a dual mock-Lie E-S of $\mathcal{H}$ through $\mathcal{U}$ iff for all $p,q\in\mathcal{H}$ and $x_1,x_2,x_3\in\mathcal{U}$, the given compatibilities are true:

(CP1) $h:\mathcal{U}\times\mathcal{U}\to\mathcal{H}$ is a symmetric map;

(CP2) $x_1\triangleright(p.q)=-p.(x_1\triangleright q)$;

(CP3) $\{x_1,x_2\}\triangleright p=-x_1\triangleright(x_2\triangleright p)$;

(CP4) $h(x_1,\{x_2,x_3\})-h(\{x_1,x_2\},x_3)+x_1\triangleright h(x_2,x_3)-x_3\triangleright h(x_1,x_2)=0$;

(CP5) $(\mathcal{U},\{-,-\})$ is a dual mock-Lie algebra.

Definition 3.8. *A system $(\mathcal{H},\mathcal{U},\triangleright,h)$ consisting of two dual mock-Lie algebras $\mathcal{H}$, $\mathcal{U}$ and two bilinear maps $\triangleright:\mathcal{U}\times\mathcal{H}\to\mathcal{H}$, $h:\mathcal{U}\times\mathcal{U}\to\mathcal{H}$ satisfying the above mentioned four compatibility assertions is called as crossed system of $\mathcal{H}$ and $\mathcal{U}$.*

The associated unified product $\mathcal{H}\natural_{\mho(\mathcal{H},\mathcal{U})}\mathcal{U}=\mathcal{H}\#^h_{\triangleright}\mathcal{U}$ is the crossed product of the dual mock-Lie algebras $\mathcal{H}$ and space $\mathcal{U}$ and is defined as: $\mathcal{H}\natural_{\mho(\mathcal{H},\mathcal{U})}\mathcal{U}=\mathcal{H}\#^h_{\triangleright}\mathcal{U}$ with the multiplication given for any $p,q\in\mathcal{H}$ and x_1, $x_2\in\mathcal{U}$ by:

$$(p,x_1)\star(q,x_2):=(p.q+x_1\triangleright q-x_2\triangleright p+h(x_1,x_2),\{x_1,x_2\})$$

For a crossed system $(\mathcal{H},\mathcal{U},\triangleright,h)$, $\mathcal{H}\cong\mathcal{H}\times\{0\}$ is an ideal in $\mathcal{H}\#^h_\triangleright\mathcal{U}$ as

$$(p,0)\star(q,x_2):=(p.q+x_2\triangleright p,0).$$

Conversely, crossed products explain all dual mock-Lie algebra structures on $\mathcal{V}$ in such a way that $\mathcal{H}$ becomes an ideal of $\mathcal{U}$.

Proposition 3.9. *Suppose $\mathcal{H}$ is a dual mock-Lie algebra, $\mathcal{V}$ a space containing $\mathcal{H}$ as a subspace. Then any dual mock-Lie algebra structure on space $\mathcal{V}$ which contains $\mathcal{H}$ as an ideal is isomorphic to a crossed product of dual mock-Lie algebras $\mathcal{H}\#^h_\triangleright\mathcal{U}$.*

Proof. Assume that $\star$ is a dual mock-Lie algebra structure on $\mathcal{V}$ such that $\mathcal{H}$ is an ideal in $\mathcal{U}$. Particularly, $\mathcal{H}$ is a subalgebra of $\mathcal{V}$ and therefore we can impose Theorem 2.2. For this case the action $\triangleleft=\triangleleft_\varphi$ of the dual mock-Lie E-S $\mho(\mathcal{H},\mathcal{U})=(\triangleleft_\varphi,\triangleright_\varphi,h_\varphi,\{-,-\}_\varphi)$ established in the proof of Theorem 2.2 is trivial. As for $x_1\in\mathcal{U}$ and $p\in\mathcal{H}$, $x_1\star p\in\mathcal{H}$, therefore $\varphi(x_1\star p)=x_1\star p$. Thus, $x_1\triangleleft_\varphi p=0$, i.e. the unified product $\mathcal{H}\natural_{\mho(\mathcal{H},\mathcal{U})}\mathcal{U}=\mathcal{H}\#^h_\triangleright\mathcal{U}$ is the crossed product of the dual mock-Lie algebras $\mathcal{U}:=\mathrm{Ker}(\varphi)$ and $\mathcal{H}$. □

There was a detailed study of the crossed product of dual mock-Lie algebras in [11] in relation to Hilbert's extension problem. Our focus is on the use of Proposition 3.9 in a new application: we prove that crossed products play a important role in the classification of finite-dimensional supersolvable dual mock-Lie algebras of finite dimensions.

On the basisd of Corollary 3.9 it is possible to classify all finite-dimensional supersolvable dual mock-Lie algebras by a recursive method. One important step is the description of all crossed products $\mathcal{H}\#^h_\triangleright\mathcal{V}$, for a given dual mock-Lie algebra $\mathcal{H}$ and a 1-dimensional vector space $\mathcal{U}$.

Theorem 3.10. *Suppose $\mathbb{F}$ is a field with characteristic not equal to 3, a dual mock-Lie algebra $\mathcal{H}$ and one-dimensional vector space $\mathcal{U}$ with basis $\{x_1\}$. Then there exists a bijective map between the collection of all crossed systems of $\mathcal{H}$ and $\mathcal{U}$ and the set $\mathcal{S}(\mathcal{H})$ of all $(0,-1)$-derivations of $\mathcal{H}$ satisfying $D^2=0$. Via the above bijective map, the crossed system $(\triangleright,\{-,-\})$ related to $D\in\mathcal{S}(\mathcal{H})$ can be defined for all $p\in\mathcal{H}$ as follows:*

$$x_1\triangleright p=D(p). \tag{3.9}$$

Proof. As we know that $\mathbb{F}$ is a field with characteristic not equal to 3, the only dual mock-Lie algebra structure on $\mathcal{U}:=\mathbb{F}x_1$ is the abelian one, that is $\{x_1,x_1\}=0$. Furthermore, as $\mathcal{U}$ has dimension one the collection of all bilinear mappings $\triangleright$: $\mathcal{U} \times \mathcal{H} \to \mathcal{H}$, is in bijection with the entity of all $D \in \mathrm{End}_{\mathbb{F}}(\mathcal{H})$ and the bijection is given in a way that (3.9) satisfy. Now we will show that the (CP2)-(CP4) compatibilities are equivalent to $D \in \mathcal{S}(\mathcal{H})$. In fact, (CP2) is equivalent to the fact that D is an $(0,-1)-$derivation of $\mathcal{H}$, (CP3) is equivalent to the fact that $D^2 = 0$ and (CP4) is evidently true.

Suppose $D \in \mathcal{S}(\mathcal{H})$. We denote $\mathcal{H}_D := \mathcal{H} \times \mathbb{F}x_1$ the crossed product $\mathcal{H}\#_{\triangleright}\mathbb{F}x_1$ associated to the crossed system (3.9) with the multiplication for all $p, q \in \mathcal{H}$ such that:

$$(p, x_1) \star (q, x_1) = (p.q + D(q) - D(p), 0)$$

□

By applying Corollary 3.9 and Theorem 3.10 we get:

Corollary 3.11. *Suppose* $\mathbb{F}$ *is a field with characteristic not equal to* 3 *and* $\mathcal{H}$ *is a dual mock-Lie algebra. Then a dual mock-Lie algebra* $\mathcal{U}$ *exists that contains* $\mathcal{H}$ *as an ideal of codimension 1 iff there exists a pair* $D \in \mathcal{S}(\mathcal{H})$ *such that* $\mathcal{V} \cong \mathcal{H}_D$*.*

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