International Journal of Mathematics and
Computer Science, **17**(2022), no. 4, 1639–1642

On the Diophantine equation $n^x + 19^y = z^2$, where $n \equiv 2 \pmod{57}$

Nongluk Viriyapong, Chokchai Viriyapong

Mathematics and Applied Mathematics Research Unit
Department of Mathematics
Mahasarakham University
Maha Sarakham, 44150, Thailand

email: nongluk.h@msu.ac.th, chokchai.v@msu.ac.th

(Received June 3, 2022, Accepted July 8, 2022)

Abstract

In this article, we show that the Diophantine equation $n^x + 19^y = z^2$ has only one solution $(n, x, y, z) = (2, 3, 0, 3)$, where n is a positive integer with $n \equiv 2 \pmod{57}$ and x, y, z are non-negative integers.

1 Introduction

Many mathematicians investigated the non-negative solutions (x, y, z) of Diophantine equations of the form $a^x + b^y = z^2$, where a and b are fixed. In 2011, Suvarnamani [1] gave some non-negative solutions of the Diophantine equation $2^x + p^y = z^2$ when p is an odd prime number. In 2013, Sroysang [2] showed that $(x, y, z) = (3, 0, 3)$ is the only non-negative solution of the Diophantine equation $2^x + 19^y = z^2$. In 2021, Tangjai and Chubthaisong [3] showed that $(p, x, y, z) = (2, 0, 3, 3)$ is the only non-negative integer solution of the Diophantine equation $3^x + p^y = z^2$ when p is prime such that $p \equiv 2 \pmod{3}$ and y is not divisible by 4.

In this paper, we find all non-negative solutions of the Diophantine equation $n^x + 19^y = z^2$ when n is a positive integer with $n \equiv 2 \pmod{57}$.

Key words and phrases: Diophantine equation, congruence.
AMS (MOS) Subject Classifications: 11D61.
Chokchai Viriyapong is the Corresponding author.
ISSN 1814-0432, 2022, http://ijmcs.future-in-tech.net

2 Preliminaries

Throughout this paper, $a \equiv b \pmod{m}$ (or simply $a \equiv_m b$) always means a is congruent to b modulo m, where a, b, m are integers such that $m \geqslant 1$. Moreover, we use the notation $a \equiv_m b, c$ to denote $a \equiv_m b$ or $a \equiv_m c$.

In 1844, Catalan raised the following conjecture [4] and which was proved, in 2004, by Mihailescu [5]:

Theorem 2.1 (Catalan's conjecture). $(a, b, x, y) = (3, 2, 2, 3)$ *is the unique solution of the Diophantine equation* $a^x - b^y = 1$, *where* a, b, x *and* y *are integers with* $\min\{a, b, x, y\} > 1$.

Next, we shall give two lemmas that follow from the Catalan's conjecture.

Lemma 2.2. [2] *The Diophantine equation* $1 + 19^y = z^2$ *has no non-negative integer solution where* y *and* z *are non-negative integers.*

Lemma 2.3. [6] $(n, x, z) = (2, 3, 3)$ *is the unique solution for the Diophantine equation* $n^x + 1 = z^2$, *where* n *is a positive integer such that* $n + 1$ *is not a square,* x *and* z *are non-negative integers.*

3 Main Results

Now, we shall prove three lemmas which will be useful in our work.

Lemma 3.1. *For all positive odd integer* x, $2^x \equiv_{19} 2, 3, 8, 10, 12, 13, 14, 15, 18$.

Proof. We will prove by induction that $2^{2n-1} \equiv_{19} 2, 3, 8, 10, 12, 13, 14, 15, 18$, for all $n \in \mathbb{N}$. If $n = 1$, then $2^1 \equiv_{19} 2$. Thus the statement is true for $n = 1$. Assume that it is true for $n = k$. Then

$$2^{2k-1} \equiv_{19} 2, 3, 8, 10, 12, 13, 14, 15, 18.$$

Thus

$$2^{2k+1} \equiv_{19} 8, 12, 13, 2, 10, 14, 18, 3, 15.$$

Hence, the statement is true for $n = k + 1$. □

Lemma 3.2. *For any integer* z, $z^2 \equiv_{19} 0, 1, 4, 5, 6, 7, 9, 11, 16, 17$.

Proof. Let z be an integer. Then

$$z \equiv_{19} 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18.$$

Thus

$$z^2 \equiv_{19} 0, 1, 4, 9, 16, 6, 17, 11, 7, 5, 5, 7, 11, 17, 6, 16, 9, 4, 1.$$

This implies that $z^2 \equiv_{19} 0, 1, 4, 5, 6, 7, 9, 11, 16, 17$. □

Lemma 3.3. *For all positive integers* n *with* $n \equiv_{57} 2$*,* $n+1$ *is not a square.*

Proof. Let n be a positive integer such that $n \equiv_{57} 2$. Suppose that $n+1$ is a square. Then $n+1 = z^2$, for some integer z. Since $n \equiv_{57} 2$, $n+1 \equiv_{19} 3$. Thus $z^2 \equiv_{19} 3$ which contradicts Lemma 3.2. Hence $n+1$ is a not square. □

Next, we shall give our main result.

Theorem 3.4. *The Diophantine equation* $n^x + 19^y = z^2$ *has exactly one solution* $(n, x, y, z) = (2, 3, 0, 3)$ *where* n *is a positive integer with* $n \equiv_{57} 2$ *and* x*,* y*,* z *are non-negative integers.*

Proof. Assume that there exist a positive integer n and non-negative integers x, y, z such that $n \equiv_{57} 2$ and $n^x + 19^y = z^2$. Since $n \equiv_{57} 2$, by Lemma 3.3, $n+1$ is a not square. If $y = 0$, then by Lemma 2.3, $(n, x, y, z) = (2, 3, 0, 3)$ is a solution for the equation $n^x + 19^y = z^2$. Now, we assume that $y \geqslant 1$. By Lemma 2.2, we have $x \geqslant 1$. If x is even, then $n^x \equiv_3 1$ because $n \equiv_3 2$. Since $19^y \equiv_3 1$, $z^2 \equiv_3 2$ which is a contradiction. As a result, x is odd. Since $n \equiv_{19} 2$, by Lemma 3.1, we have

$$n^x \equiv_{19} 2, 3, 8, 10, 12, 13, 14, 15, 18.$$

Since $19^y \equiv 0 \pmod{19}$, we obtain

$$z^2 \equiv_{19} 2, 3, 8, 10, 12, 13, 14, 15, 18.$$

which contradicts Lemma 3.2.

Therefore, $(2, 3, 0, 3)$ is a unique solution (n, x, y, z) of the Diophantine equation $n^x + 19^y = z^2$, where n is a positive integer such that $n \equiv 2$ (mod 57) and x, y, z are non-negative integers. □

Finally, we shall apply Theorem 3.4 when $n = 59, 116$.

Corollary 3.5. *The Diophantine equation* $59^x + 19^y = z^2$ *has no non-negative integer solution, where* x*,* y *and* z *are non-negative integers.*

Proof. Since $59 \equiv_{57} 2$ and $59 \neq 2$, by Theorem 3.4, the Diophantine equation $59^x + 19^y = z^2$ has no non-negative integer solution. □

Corollary 3.6. *The Diophantine equation* $116^x + 19^y = z^2$ *has no non-negative integer solution, where* x*,* y *and* z *are non-negative integers.*

Proof. Since $116 \equiv_{57} 2$ and $116 \neq 2$, by Theorem 3.4, the Diophantine equation $116^x + 19^y = z^2$ has no non-negative integer solution. □

4 Conclusion

In this article, we proved that the Diophantine equation $n^x + 19^y = z^2$ has the unique non-negative solution $(n, x, y, z) = (2, 3, 0, 3)$ where $n \equiv_{57} 2$. The above result can apply to the case where n is prime or composite as can be easily seen from Corollaries 3.5 and 3.6.

Acknowledgment. This research project was financially supported by Mahasarakham University.

References

[1] A. Suvarnamani, Solutions of the Diophantine equations $2^x + p^y = z^2$, Int. J. Math. Sci. Appl., **1**, no. 3, (2011), 1415–1419.

[2] B. Sroysang, More on the Diophantine equation $2^x + 19^y = z^2$, Int. J. Pure Appl. Math., **88**, no.1, (2013), 157–160.

[3] W. Tangjai, C. Chubthaisong, On the Diophantine equation $3^x + p^y = z^2$ where $p \equiv 2 (\mathrm{mod}\, 3)$, WSEAS Transactions on Mathematics, **20**, (2021), 283–287.

[4] E. Catalan, Note extraite dune lettre adressee a lediteur, J. Reine Angew. Math., **27**, (1844), 192.

[5] S. Mihailescu, Primary cyclotomic units and a proof of Catalan's conjecture, J. Reine Angew. Math., **572**, (2004), 167-195.

[6] N. Viriyapong, C. Viriyapong, On the Diophantine Equation $n^x + 13^y = z^2$ where $n \equiv 2 \pmod{57}$ and $n + 1$ is not a square number, *WSEAS Transactions on Mathematics*, **20**, (2021), 442–445.