International Journal of Mathematics and
Computer Science, **17**(2022), no. 4, 1643–1647

Elementary Solution to a Damped and Forced Cubic-Quintic Duffing Equation

Alvaro H. Salas

FIZMAKO Research Group
Department of Mathematics and Statistics
Universidad Nacional de Colombia
sede Manizales, Colombia

email: ahsalass@unal.edu.co

(Received May 24, 2022, Accepted July 9, 2022)

Abstract

The aim of the present work is to solve the cubic-quintic Duffing Equation by using elementary tools. The solution obtained is in terms of trigonometric functions and it is compared with the solution obtained numerically.

1 Introduction

The chaotic Duffing oscillator was designed by the German electrical engineer Georg Duffing (1864-1944) at the beginning of the 20th century, in order to study the buckling motion of a beam. It is a classic model that arises in many branches of physics and engineering and has been used to study everything from the oscillations of a physical pendulum of great amplitude, to image processing and many more. It has provided us with a useful paradigm to study non-linear oscillations and chaotic systems, it has led to the development of new approximate analytical methods based on ideas such as perturbative methods, and to the development of new numerical methods

Keywords and phrases: Duffing equation, Nonlinear electric circuit, damping, forcing.
AMS (MOS) Subject Classifications: 37Cxx, 33E30, 34C15.
ISSN 1814-0432, 2022, http://ijmcs.future-in-tech.net

for the quantitative analysis of chaotic systems. We are interested in solving the following Duffing oscillator equation :

$$\frac{d^2x}{dt^2} + 2\varepsilon\frac{dx}{dt} + \alpha x + \beta x^3 + \gamma x^5 = F\cos\Omega t, \quad x(0) = x_0 \text{ and } x'(0) = \dot{x}_0. \tag{1.1}$$

There are a variety of works in which different variants of equation (1.1) has been analyzed.

2 Solution Method.

We first will consider the unforced case. Let

$$\frac{d^2x}{dt^2} + 2\varepsilon\frac{dx}{dt} + \alpha x + \beta x^3 + \gamma x^5 = 0, \quad x(0) = x_0 \text{ and } x'(0) = \dot{x}_0. \tag{2.2}$$

Assume the equation $x(t) = \exp(-\varepsilon t)\left[x_0\cos\omega(t) + C\sin\omega(t)\right]$, where the function $\omega = \omega(t)$, $\omega(0 = 0$ and the constant C are to be determined. Define the residual function as $R(t) = x''(t) + 2\varepsilon x'(t) + \alpha x + \beta x^3 + \gamma x^5$. We have:

$$R(t) = \left[-\tfrac{1}{8}e^{-5\varepsilon t}\left(\begin{array}{c} x_0\left(-6\beta\left(C^2+x_0^2\right)e^{2\varepsilon t} - 5\gamma\left(C^2+x_0^2\right)^2 + 8\left(\varepsilon^2-\alpha\right)e^{4\varepsilon t}\right) - \\ 8Ce^{4\varepsilon t}\omega''(t) + 8x_0e^{4\varepsilon t}\omega'(t)^2 \end{array}\right)\right]\cos(\omega(t)) + \left[\begin{array}{c} \tfrac{1}{8}Ce^{-5\varepsilon t}\left(6\beta\left(C^2+x_0^2\right)e^{2\varepsilon t} + 5\gamma\left(C^2+x_0^2\right)^2 - 8\left(\varepsilon^2-\alpha\right)e^{4\varepsilon t}\right) - \\ Ce^{-\varepsilon t}\omega'(t)^2 - x_0e^{-\varepsilon t}\omega''(t) \end{array}\right]\sin(\omega(t)) + \text{ h.o.t} \tag{2.3}$$

Equating to zero the coefficients of $\cos(\omega(t))$ and $\sin(\omega(t))$ in the last expression and eliminating $\omega''(t)$ from the resulting system gives the ode $8e^{4t\varepsilon}\left(-\alpha+\varepsilon^2\right) - 6e^{2t\varepsilon}\beta\left(C^2+x_0^2\right) - 5\gamma\left(C^2+x_0^2\right)^2 + 8e^{4t\varepsilon}\omega\prime(t)^2 = 0$. Solving this ordinary differential equation, we get

$$\omega(t) = \int_0^t \sqrt{\alpha - \varepsilon^2 + \tfrac{3}{4}\beta\left(C^2+x_0^2\right)e^{-2\varepsilon\tau} + \tfrac{5}{8}\gamma\left(C^2+x_0^2\right)^2e^{-4\varepsilon\tau}}\,d\tau = F(t) - F(0), \tag{2.4}$$

where

$$F(t) = \frac{\sqrt{\varphi_0}}{10\sqrt{\gamma\varphi_1}\varepsilon} \left(e^{2\varepsilon t} \left(\begin{array}{c} 10\sqrt{2}\sqrt{\gamma}\sqrt{\alpha-\varepsilon^2}\tan^{-1}\left(\frac{8(\varepsilon^2-\alpha)e^{2\varepsilon t}-3\beta(C^2+x_0{}^2)}{2\sqrt{2}\sqrt{\varphi_1}\sqrt{\alpha-\varepsilon^2}}\right) + \\ 3\sqrt{5}\beta\tan^{-1}\left(\frac{5\gamma(C^2+x_0{}^2)+3\beta e^{2\varepsilon t}}{\sqrt{5}\sqrt{\gamma}\sqrt{\varphi_1}}\right) \end{array} \right) - 5\sqrt{\gamma\varphi_1} \right).$$

$$\varphi_0 = \alpha + \tfrac{1}{8}\left(C^2 + x_0{}^2\right) e^{-4\varepsilon t}\left(5\gamma\left(C^2 + x_0{}^2\right) + 6\beta e^{2\varepsilon t}\right) - \varepsilon^2.$$
$$\varphi_1 = 8\left(\varepsilon^2 - \alpha\right) e^{4\varepsilon t} - 6\beta\left(C^2 + x_0{}^2\right) e^{2\varepsilon t} - 5\gamma\left(C^2 + x_0{}^2\right)^2. \tag{2.5}$$

The constant C is determined from the initial condition $x'(0) = \dot{x}_0$ and it is found from the following cubic :

$$\frac{5\gamma}{8}z^3 + \frac{1}{4}\left(3\beta + 5\gamma x_0^2\right) z^2 + \frac{1}{8}\left(8\alpha - 8\varepsilon^2 + 6\beta x_0^2 + 5\gamma x_0^4\right) z - \left(\varepsilon x_0 + \dot{x}_0\right)^2 = 0,\ z = C^2. \tag{2.6}$$

Now, let us consider the damped and/or forced case given by (1.1). Consider the equation $x(t) = y(t) + c_1 \cos\Omega t + c_2 \sin\Omega t$, where $y''(t) + 2\varepsilon y'(t) + \alpha y(t) + \beta y^3(t) + \gamma y^5(t) = 0$ given that $y(0) = x_0 - c_1$ and $y'(0) = \dot{x}_0 - c_1\Omega$.

Define the residual function $\hat{R}(t) = x''(t) + 2\varepsilon x'(t) + \alpha x(t) + \beta x^3(t) + \gamma x^5(t) - F\cos\Omega t$. We have:

$$\hat{R}(t) = \left[\begin{array}{c} \frac{1}{8}\left(-8F + 5\gamma c_1^5 + 16\varepsilon\Omega c_2 + 2c_1^3\left(3\beta + 5\gamma c_2^2\right) + c_1\left(8\alpha - 8\Omega^2 + 6\beta c_2^2 + 5\gamma c_2^4\right)\right) + \\ \frac{3}{2}c_1\left(2\beta + 5\gamma\left(c_1^2 + c_2^2\right)\right) y(t)^2 + 5\gamma c_1 y(t)^4 \end{array} \right] \cos\Omega t + \left[\begin{array}{c} \frac{1}{8}\left(-16\varepsilon\Omega c_1 + \left(8\alpha - 8\Omega^2 + 6\beta c_1^2 + 5\gamma c_1^4\right) c_2 + 2\left(3\beta + 5\gamma c_1^2\right) c_2^3 + 5\gamma c_2^5\right) + \\ \frac{3}{2}c_2\left(2\beta + 5\gamma\left(c_1^2 + c_2^2\right)\right) y(t)^2 + 5\gamma c_2 y(t)^4 \end{array} \right] \sin\Omega t + \text{h.o.t.} \tag{2.7}$$

Equating to zero the coefficients of $\cos\Omega t$ and $\sin\Omega t$ and neglecting the terms containig $y^2(t)$ and $y^4(t)$ gives an algebraic system. The values of c_1 and c_2 will be the least in magnitude real roots to that system.

3 Analysis and Discussion

Let us analyze the accuracy of the obtained results.

Example 1. Let us consider the initial value problem $\ddot{x} + 0.1\dot{x} + x + 2x^3 + 3x^5 = 0$ given that $x(0) = 0$ and $x'(0) = 0.2$. See Figure 1 for a comparison with the numerical solution.

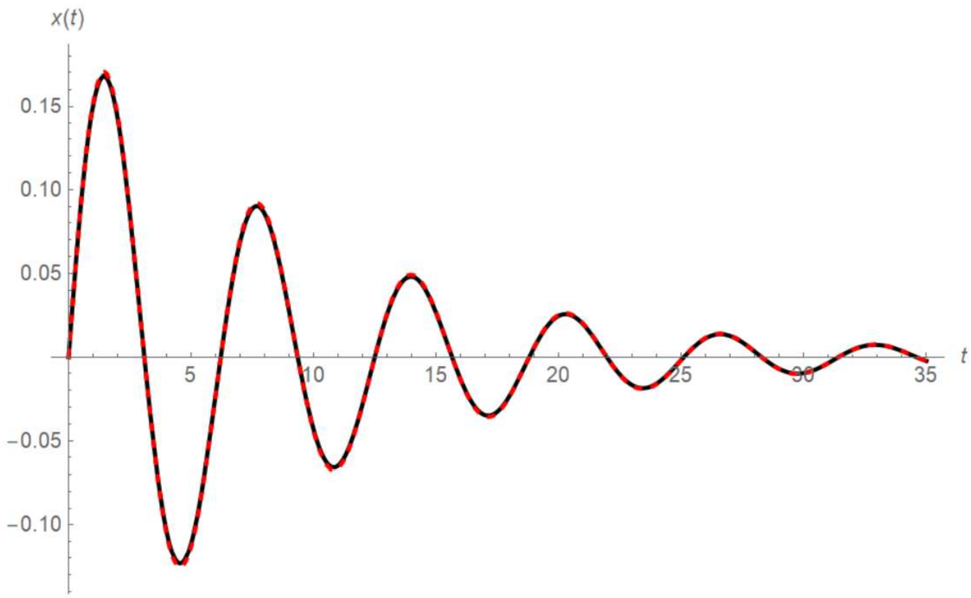

Figure 1. Example 1. $C = 0.195192$. Error $= 0.00216483$

Example 2. Let us consider the i.v.p. $\ddot{x}+0.1\dot{x}+x+2x^3+3x^5 = \cos 2\pi t$ given that $x(0) = 0$ and $x'(0) = 0$. The respective values are : $C = 7.53 \times 10^{-6}$, $c_1 = -0.0259616$ and $c_2 = 0.000847882$. See Figure 2.

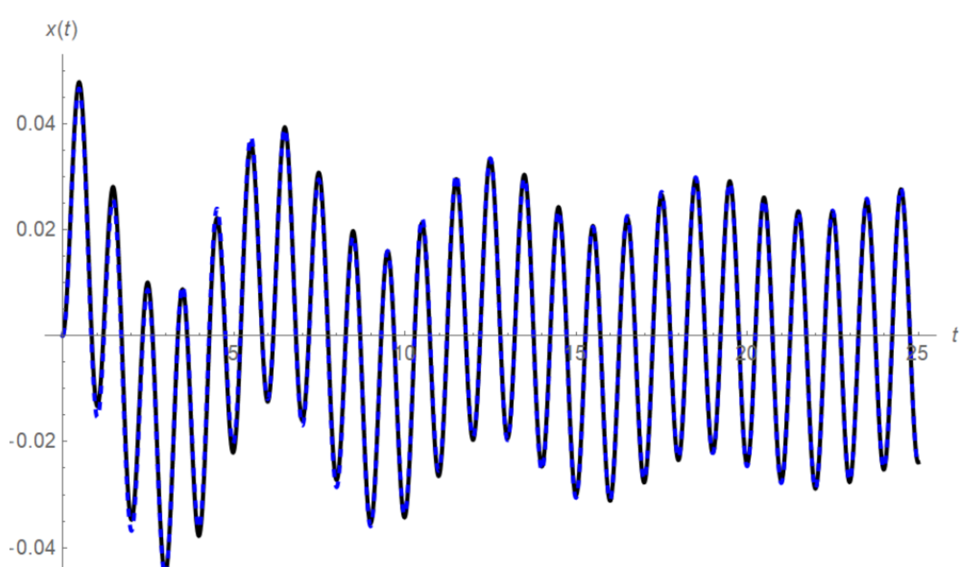

Figure 2. Error $= 0.00238677$

References

[1] N. Nayfeth, D. T. Mook, Non-linear oscillations, John Wiley, New York.

[2] Alvaro H. Salas, Simeon Casanova Trujillo, A New Approach for Solving the Complex Cubic-Quintic Duffing Oscillator Equation for Given Arbitrary Initial Conditions, Math Prob. Engg., Hindawi, https://doi.org/10.1155/2020/3985975

[3] Alvaro H. Salas, S. A. Tantawy, Jairo E. Casillo, Analytical Solution to the Damped Cubic-Quintic Duffing Equation. International Journal of Mathematics and Computer Science, **17,** no. 1, (2022), 425–430.

[4] Alvaro H. Salas, An Elementary Solution to a Duffing Equation. Hindawi, The Scientific World Journal https://doi.org/10.1155/2022/2357258.