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Dual Spherical Curves According to Bishop Frame in the Dual 3-Space

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Abstract

Using curves in Euclidean 3-space with Bishop vectors, we study the tangent, first and second normals, Darboux lines and their dual spherical indicatrices. In addition, we investigate the development of the ruled surfaces that correspond to Bishop vectors of a dual curve in Dual 3-space D^3. Moreover, we calculate the dual angles and lengths of pitch of the closed ruled surfaces. Furthermore, we obtain some relations between Bishop and Darboux vectors of these surfaces. Finally, we give two computational examples in support of our main results.

1 Introduction

Study [1] and Clifford [2] studied dual spherical motion expressed with the help of dual unit vectors. In analogy with the complex numbers, Clifford

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defined the dual numbers and showed that they form an algebra but not a field. Later, Study introduced the dual angle. Dual numbers were introduced in the 19th century by Clifford as a tool for his geometrical investigations. In addition, Study generalized their applications to rigid body kinematics in their principal of transference. As it is well-known, the analytical tools in the study of 3-dimensional kinematics and differential geometry of ruled surfaces are based on dual vector calculus. Some works on this field in dual 3-space have been introduced by many geometers [3, 4, 5, 6, 7] and obtained some interesting results.

In the Euclidean three-space E^3, lines combined with one of their two directions can be represented by unit dual vectors over the ring of dual numbers. The most important properties of real vector analysis are valid for the dual vectors. The oriented lines in E^3 are in one-to-one correspondence with the points of a dual unit sphere. A dual point on dual unit sphere in D^3 corresponds to a line in E^3 and two different points on D^3 represent two skew-lines in E^3 in general. A differentiable curve on dual unit sphere in D^3 represents a ruled surface in E^3 (for more details see [8, 9]). The fundamental idea of this work depends on replacing points by lines as basic concepts of geometry. In the light of this, the author [10] showed that we can study a ruled surface as a curve on the dual unit sphere by using Blaschke approach. In [11], it can be seen that a dual curve can be defined as the set of dual points.

Dual spherical geometry, expressed with the help of dual unit vectors is closely analogous to real spherical geometry, expressed with help of real unit vectors. Therefore, the properties of elementary real spherical geometry can also be carried over by analogy into the geometry of lines in E^3.

In this study, we consider spherical motion for a dual curve and present the analysis of this motion with Bishop frame in three-dimensional dual space D^3. The dual angle and lengths of pitch of the closed ruled surfaces are computed. In addition to, some important results are given as special cases which the line is being the tangent, the first and second unit normals of the base curve under study.

2 Fundamental concepts

In what follows, we briefly give the mathematical formulations and basic concepts on classical differential geometry that we need in our study.

2.1 Dual space

Let p and p^* be two real numbers and $\varepsilon \neq 0, \varepsilon^2 = 0$. A dual number $\hat{p}$ is an ordered pair of the form $(p,\ p^*)$ for all $p,\ p^* \in R$. Let the set $R \times R$ be denoted as D, where

$$D = \{\hat{p} = p + \varepsilon p^* : p, p^* \epsilon \mathbb{R}\}. \tag{2.1}$$

Dual numbers form a ring over the real number field. Two inner operations and an equality on D are defined as follows:

1. $\oplus : D \times D \to D$ for $\hat{p} = (p,\ p^*), \hat{q} = (q,\ q^*)$ defined as $\hat{p} \oplus \hat{q} = (p +\ q) + \varepsilon(\ p^* +\ q^*)$, is called the addition in D.
2. $\odot : D \times D \to D$ for $\hat{p} = (p,\ p^*), \hat{q} = (q,\ q^*)$ defined as $\hat{p} \odot \hat{q} = pq + \varepsilon(\ p^*q + pq^*)$, is called the multiplication in D.
3. For the equality of $\hat{p}$ and $\hat{q}$ we have $\hat{p} = \hat{q} \Leftrightarrow p = q$, and $p^* = q^*$.

The dual number $\hat{p} = p + \varepsilon p^*$ divided by the dual number $\hat{q} = q + \varepsilon q^*$ provided $q \neq 0$ can be defined as

$$\frac{\hat{p}}{\hat{q}} = \frac{p + \varepsilon\ p^*}{q + \varepsilon q^*} = \frac{p}{q} + \varepsilon \frac{p^* q - q^* p}{q^2}. \tag{2.2}$$

If the operations of addition, multiplication and equality on $D = R \times R$ with the set of real numbers $\mathbb{R}$ are defined as above, the set D is called the *dual number system* and the element (p, p^*) of D is called a *dual number*. In a dual number $\hat{p} = (p, p^*) \in D$, the real number p is called the *real part* of $\hat{p}$ and the real number p^* is called the *dual part* of $\hat{p}$. The dual number $(1, 0) = 1$ is called a real unit in D. The dual number $(0, 1)$ is denoted by ε and is called *dual unit*. The set of

$$\begin{aligned} D^3 &= D \times D \times D = \{\overrightarrow{\hat{p}} : \overrightarrow{\hat{p}} = \vec{p} + \varepsilon \vec{p}^*, \vec{p}, \vec{p}^* \epsilon R^3\}; \\ \vec{p} &= (p_1, p_2, p_3), \vec{p}^* = (p_1^*, p_2^*, p_3^*) \end{aligned} \tag{2.3}$$

is a module of the ring D. For any $\overrightarrow{\hat{p}}, \overrightarrow{\hat{q}} \in D^3$, the inner product and the vector product are defined by

$$< \overrightarrow{\hat{p}}, \overrightarrow{\hat{q}} > = < \vec{p}, \vec{q} > + \varepsilon(< \vec{p}, \vec{q}^* > + < \vec{p}^*, \vec{q} >), \tag{2.4}$$

$$\overrightarrow{\hat{p}} \times \overrightarrow{\hat{q}} = \vec{p} \times \vec{q} + \varepsilon(\vec{p} \times \vec{q}^* + \vec{p}^* \times \vec{q}), \tag{2.5}$$

respectively. If $p \neq 0$, then the norm is defined by

$$\left\| \overrightarrow{\hat{p}} \right\| = \sqrt{< \overrightarrow{\hat{p}}, \overrightarrow{\hat{p}} >} = \|\vec{p}\| + \varepsilon \frac{< \vec{p}, \vec{p}^* >}{\|\vec{p}\|}. \tag{2.6}$$

A dual vector $\overrightarrow{\hat{p}}$ with norm 1 is called a *dual unit vector*. Let $\overrightarrow{\hat{p}} = \vec{p} + \varepsilon \vec{p}^* \in D^3$, the set

$$S^2 = \{ \overrightarrow{\hat{p}} = \vec{p} + \varepsilon \vec{p}^* : \left\| \overrightarrow{\hat{p}} \right\| = (1, 0);\ \vec{p}, \vec{p}^* \in R^3 \} \tag{2.7}$$

is called *the dual unit sphere* with the center $\hat{O}$ in D^3. Let $p_i(t)$ and $p_i^*(t)$,$1 \leq i \leq 3$ be differentiable real valued functions, the dual space curve

$$\begin{aligned} \hat{\alpha} \quad &: \quad I \subset R \to \ D^3; \\ t \ \longrightarrow \quad & \overrightarrow{\hat{\alpha}}(t) = \vec{\alpha}(t) + \varepsilon \vec{\alpha}^*(t), \end{aligned} \tag{2.8}$$

in D^3 is differentiable. We call the real part $\vec{\alpha}(t)$ the indicatrix of $\overrightarrow{\hat{\alpha}}(t)$. The dual arc length of the curve $\overrightarrow{\hat{\alpha}}(t)$ is defined as

$$\hat{s} = \int_{t_0}^{t} \left\| \overrightarrow{\hat{\alpha}}(t)' \right\| dt = \int_{t_0}^{t} \|\vec{\alpha}(t)'\|\, dt + \varepsilon \int_{t_0}^{t} \left\| \langle \tilde{\mathbf{T}}, \vec{\alpha}^*(t)' \rangle \right\| dt = s + \varepsilon s^*. \tag{2.9}$$

where $\tilde{\mathbf{T}}$ is a unit tangent vector of $\vec{\alpha}$ with arclength parameter s. For the dual space curve $\overrightarrow{\hat{\alpha}}$ with the arclength parameter $\hat{s}$, the dual unit tangent vector ca be expressed as

$$\overrightarrow{\hat{\mathbf{T}}} = \frac{d\overrightarrow{\hat{\alpha}}}{d\hat{s}} = \frac{d\overrightarrow{\hat{\alpha}}}{ds}\frac{ds}{d\hat{s}} \tag{2.10}$$

2.2 Bishop frame

The Frenet frame is constructed for the curve of 3-time continuously differentiable non-degenerate curves. However, curvature may vanish at some points on the curve. That is, the second derivative of the curve may be zero. In this situation, we need an alternative frame in E^3. Therefore, in [10], Bishop defined a new frame for a curve and he called it Bishop frame which is well defined even if the curve has vanishing second derivative in 3-dimensional

Euclidean space. Let $\alpha : I \subset R \to E^3$ be a unit speed curve in the Euclidean 3-space E^3. The Frenet equations along α is defined as [10, 11]

$$\begin{pmatrix} \mathbf{T}' \\ \mathbf{N}' \\ \mathbf{B}' \end{pmatrix} = \begin{pmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{pmatrix} \begin{pmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{pmatrix}, \tag{2.11}$$

where $\{\mathbf{T}(s), \mathbf{N}(s), \mathbf{B}(s)\}$ is the moving Frenet frame along α and the functions $\kappa(s)$ and $\tau(s)$ are the curvature and the torsion, respectively.
The Bishop frame of 1-type is an alternative approach to defining a moving frame that is well defined even when the curve has vanishing second derivative. We can parallel transport an orthonormal frame along $\alpha(s)$ simply by parallel transporting each component of the frame. This frame is denoted by $\{\mathbf{T}(s), \mathbf{N}_1(s), \mathbf{N}_2(s)\}$. Then, its derivatives are expressed as [11]

$$\begin{pmatrix} \mathbf{T}' \\ \mathbf{N}_1' \\ \mathbf{N}_2' \end{pmatrix} = \begin{pmatrix} 0 & k_1(s) & k_2(s) \\ -k_1(s) & 0 & 0 \\ -k_2(s) & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{T} \\ \mathbf{N}_1 \\ \mathbf{N}_2 \end{pmatrix}. \tag{2.12}$$

Here, the relation between Frenet and Bishop frames is given as follows:

$$\begin{pmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \mathbf{T} \\ \mathbf{N}_1 \\ \mathbf{N}_2 \end{pmatrix}, \tag{2.13}$$

with

$$\begin{aligned} \theta(s) &= \arctan(\frac{k_2}{k_1}), k_1 \neq 0, \quad \tau(s) = \frac{d\theta(s)}{ds}, \quad \kappa(s) = \sqrt{k_1^2 + k_2^2}, \\ k_1 &= \kappa\cos\theta, \ k_2 = \kappa\sin\theta, \end{aligned} \tag{2.14}$$

where, k_1 and k_2 are called the first and second Bishop curvatures effectively correspond to a Cartesian coordinate system for the polar coordinates κ, θ.

3 Dual spherical motion and ruled surfaces

In this section, we define a dual curve $\hat{\alpha}$ in D^3 with the assistance of unit speed curve α in E^3. Then, by using an orthonormal Bishop frame along $\hat{\alpha}$ we give a one parameter spherical motion in D^3. Moreover, the Darboux vector of this motion is calculated with the Darboux indicatrix of the dual curve. Here, let $\alpha : I \to E^3$ be a unit speed curve and its Bishop frame of type-1 is $\{\mathbf{T}(s), \mathbf{N}_1(s), \mathbf{N}_2(s)\}$. Accordingly, we can easily give dual spherical curves in D^3 as follows

$$\begin{aligned} \hat{\alpha}(s) &= \int [\mathbf{T}(s) + \varepsilon\alpha(s) \wedge \mathbf{T}(s)\ ]ds, \\ \hat{\gamma}(s) &= \int [\mathbf{N}_1(s) + \varepsilon\alpha(s) \wedge \mathbf{N}_1(s)]ds, \\ \hat{\delta}(s) &= \int [\mathbf{N}_2(s) + \varepsilon\alpha(s) \wedge \mathbf{N}_2(s)]ds. \end{aligned} \quad (3.15)$$

And, let $\hat{\mathbf{T}}$, $\hat{\mathbf{N}}_1$ and $\hat{\mathbf{N}}_2$ be the dual Bishop tangent, first and second unit normals of $\hat{\alpha}$ given by

$$\hat{\mathbf{T}} = \frac{d\hat{\alpha}}{ds} = \mathbf{T} + \varepsilon(\alpha \wedge \mathbf{T}), \hat{\mathbf{N}}_1 = \mathbf{N}_1 + \varepsilon(\alpha \wedge \mathbf{N}_1), \hat{\mathbf{N}}_2 = \mathbf{N}_2 + \varepsilon(\alpha \wedge \mathbf{N}_2). \quad (3.16)$$

At this time, the ruled surfaces in E^3 correspond to the dual curves $\hat{\mathbf{T}}$, $\hat{\mathbf{N}}_1$ and $\hat{\mathbf{N}}_2$ are the ruled surfaces that drawn by the lines $\mathbf{T}$, $\mathbf{N}_1$ and $\mathbf{N}_2$ of α. That is,

$$\Psi_{\hat{\mathbf{T}}}(u,v) = \alpha(u) + v\ \mathbf{T}(u), \Psi_{\hat{\mathbf{N}}_1}(u,v) = \alpha(u) + v\ \mathbf{N}_1(u), \Psi_{\hat{\mathbf{N}}_2}(u,v) = \alpha(u) + v\ \mathbf{N}_2(u). \quad (3.17)$$

Let $\hat{\alpha}$ be a closed spherical dual curve of class C^1on a unit dual sphere S^1 in D^3. Then, $\hat{\alpha}$ can be written as

$$\begin{aligned} \hat{\alpha} &: \quad S^1 \to D^3; \\ s &\to \quad \hat{\alpha}(s) = \alpha(s) + \varepsilon \oint \alpha \wedge \mathbf{T} ds = \oint \left[\mathbf{T} + \varepsilon\left(\alpha \wedge \mathbf{T}\right)\right] ds. \end{aligned} \quad (3.18)$$

In light of this, the Bishop dual matrix corresponding to the dual spherical motion can be expressed as:

$$\begin{pmatrix} \hat{\mathbf{T}}' \\ \hat{\mathbf{N}}_1' \\ \hat{\mathbf{N}}_2' \end{pmatrix} = \begin{pmatrix} 0 & k_1 & k_2 \\ -k_1 & 0 & \varepsilon \\ -k_2 & -\varepsilon & 0 \end{pmatrix} \begin{pmatrix} \hat{\mathbf{T}} \\ \hat{\mathbf{N}}_1 \\ \hat{\mathbf{N}}_2 \end{pmatrix}, \quad (3.19)$$

On the other hand [10] the Darboux vector $\hat{\Omega}$ of the motion is defined as:

$$\hat{\Omega} = \varepsilon\hat{\mathbf{T}} - k_2\hat{\mathbf{N}}_1 + k_1\hat{\mathbf{N}}_2, \quad (3.20)$$

with

$$\begin{aligned} \hat{\Omega} \wedge \hat{\mathbf{T}} &= k_1\hat{\mathbf{N}}_1 + k_2\hat{\mathbf{N}}_2 = \hat{\mathbf{T}}', \\ \hat{\Omega} \wedge \hat{\mathbf{N}}_1 &= -k_1\hat{\mathbf{T}} + \varepsilon\hat{\mathbf{N}}_2 = \hat{\mathbf{N}}_1', \\ \hat{\Omega} \wedge \hat{\mathbf{N}}_2 &= -k_2\hat{\mathbf{T}} - \varepsilon\hat{\mathbf{N}}_1 = \hat{\mathbf{N}}_2'. \end{aligned} \quad (3.21)$$

Theorem 3.1. *The dual spherical curves* $\hat{\mathbf{T}}, \hat{\mathbf{N}}_1$ *and* $\hat{\mathbf{N}}_2$ *on dual unit sphere are not involutes of* $\hat{\varrho} = \frac{\hat{\Omega}}{\|\hat{\Omega}\|}$.

Proof. Since

$$\hat{\Omega} = \varepsilon\hat{\mathbf{T}} - k_2\hat{\mathbf{N}}_1 + k_1\hat{\mathbf{N}}_2, \tag{3.22}$$

we have

$$\hat{\varrho} = \frac{\varepsilon\hat{\mathbf{T}} - k_2\hat{\mathbf{N}}_1 + k_1\hat{\mathbf{N}}_2}{\sqrt{k_1^2 + k_2^2}}. \tag{3.23}$$

Also, we get

$$\frac{d\ \hat{\varrho}}{ds} = \frac{1}{\kappa^3}\left\{\begin{array}{c} -\varepsilon(k_1k_1' + k_2k_2')\hat{\mathbf{T}} + (-k_2'(k_1^2 + k_2^2) + k_2(k_1k_1' + k_2k_2') + \varepsilon k_1(k_1^2 + k_2^2))\hat{\mathbf{N}}_1 + \\ (k_1'(k_1^2 + k_2^2) - k_1(k_1k_1' + k_2k_2') + \varepsilon k_2(k_1^2 + k_2^2))\hat{\mathbf{N}}_2 \end{array}\right\}. \tag{3.24}$$

Using Eqs. (3.5), we get

$$< \frac{d\ \hat{\varrho}}{ds}, \frac{d\ \hat{\mathbf{T}}}{ds} >= \frac{\varepsilon(k_1^2 + k_2^2)^2 + k_2(k_1k_1' + k_2k_2')(k_1 - k_2) + (k_1^2 + k_2^2)(k_2k_1' - k_1k_2')}{\kappa^3} \neq 0, \tag{3.25}$$

$$< \frac{d\ \hat{\varrho}}{ds}, \frac{d\ \hat{\mathbf{N}}_1}{ds} >= \frac{\varepsilon k_1(k_1k_1' + k_2k_2')}{\kappa^3} \neq 0, \tag{3.26}$$

$$< \frac{d\ \hat{\varrho}}{ds}, \frac{d\ \hat{\mathbf{N}}_2}{ds} >= \frac{\varepsilon k_2(k_1k_1' + k_2k_2')}{\kappa^3} \neq 0. \tag{3.27}$$

This completes the proof.

Hereafter, by presenting the ruled surfaces in E^3 that correspond to dual curves $\hat{\mathbf{T}}, \hat{\mathbf{N}}_1$ and $\hat{\mathbf{N}}_2$, we give their special cases: If

$$\Psi_{\hat{\mathbf{T}}}(u, v) = \alpha(u) + v\ \mathbf{T}(u), \tag{3.28}$$

then the distribution parameter for this surface is given by [11]

$$\lambda_{\hat{\mathbf{T}}} = \frac{det(\alpha', \mathbf{T}, \mathbf{T}')}{\langle \mathbf{T}', \mathbf{T}' \rangle} = \frac{det(\mathbf{T}, \mathbf{T}, k_1\mathbf{N}_1 + k_2\mathbf{N}_2)}{\sqrt{k_1^2 + k_2^2}} = 0. \tag{3.29}$$

It is easy to see that the distribution parameters for $\Psi_{\hat{\mathbf{N}}_1}, \Psi_{\hat{\mathbf{N}}_2}$ are also equal zero. Therefore, we can formulate the following lemma:

Lemma 3.2. *Let* $\hat{\mathbf{T}}, \hat{\mathbf{N}}_1$ *and* $\hat{\mathbf{N}}_2$ *be the tangent, the first and second unit normal indicatrices of the dual curve* $\hat{\alpha}$. *Then the corresponding ruled surfaces are* ***developable***.

Now, by taking two dual unit spheres moving with respect to each other, we define different motion on the system. So, let $\hat{S}_1 = \{\hat{O}, \vec{e}_1, \vec{e}_2, \vec{e}_3\}$ and $\hat{S}_2 = \{\hat{O}, \hat{\mathbf{T}}, \hat{\mathbf{N}}_1, \hat{\mathbf{N}}_2\}$be two orthonormal coordinate systems of fixed and moving unit dual spheres with the same origin, respectively. On the base curve $\alpha$of the closed surface $\Psi(s,v)$, we investigate the one parameter dual spherical motion (dual rotation) between $\hat{S}_1$ and $\hat{S}_2$denoted by $\hat{S}_1/\hat{S}_2$. In this case, the *Steiner vector* is expressed as

$$\hat{\Gamma} = \oint \hat{\Omega}\ ds = \oint \left(\varepsilon \hat{\mathbf{T}} - k_2 \hat{\mathbf{N}}_1 + k_1 \hat{\mathbf{N}}_2\right)\ ds. \tag{3.30}$$

Let $\hat{X}$ be the director vector of Ψ given by

$$\hat{X} = \hat{x}_1 \hat{\mathbf{T}} + \hat{x}_2 \hat{\mathbf{N}}_1 + \hat{x}_3 \hat{\mathbf{N}}_2; \left\|\hat{X}\right\| = 1. \tag{3.31}$$

Then *the dual angle* of pitch of the closed ruled surface generated by $\hat{X}$ of closed curve $\hat{\alpha}(s)$ is

$$\hat{\theta} = - < \hat{\Gamma}, \hat{X} > . \tag{3.32}$$

Thus, using (3.16) and (3.17), we get

$$\hat{\theta} = \left(x_2 \oint k_2 ds - x_3 \oint k_1 ds\right) - \varepsilon \left(x_1 \oint ds - x_2^* \oint k_2 ds + x_3^* \oint k_1 ds\right), \tag{3.33}$$

where the real and dual parts of $\hat{\theta}$ are

$$\begin{aligned} \hat{\theta}_{real} &= \left(x_2 \oint k_2 ds - x_3 \oint k_1 ds\right), \\ \hat{\theta}_{dual} &= -\left(x_1 \oint ds - x_2^* \oint k_2 ds + x_3^* \oint k_1 ds\right). \end{aligned} \tag{3.34}$$

From (3.15), we can discuss different special situations:

Case 3.3. *If* $\hat{X} = \hat{\mathbf{T}}$*, then*

$$\hat{\theta}_{\hat{\mathbf{T}}} = -\varepsilon \oint ds. \tag{3.35}$$

Case 3.4. *If* $\hat{X} = \hat{\mathbf{N}}_1$*, then*

$$\hat{\theta}_{\hat{\mathbf{N}}_1} = \oint k_2 ds. \tag{3.36}$$

Case 3.5. *If* $\hat{X} = \hat{\mathbf{N}}_2$*, then*

$$\hat{\theta}_{\hat{\mathbf{N}}_2} = - \oint k_1 ds. \tag{3.37}$$

4 Computational examples

By using Mathematica, some computational examples for the dual spherical curve of the ruled surface and its evolutee are given and plotted.

Example 4.1. *Let $\hat{\alpha}$ be a dual curve in D^3 defined by*

$$\begin{aligned}\hat{\alpha}(\hat{s}) &= \left(\sin\hat{s}, \sin\hat{s}\ \cos\hat{s}, \cos^2\hat{s}\right)\\ &= \left(\sin s, \sin s\ \cos s, \cos^2 s\right) + \varepsilon s^*\left(\cos s, \cos 2s, -\sin 2s\right); \hat{s} = s + \varepsilon s^*.\end{aligned}$$

The corresponding ruled surface is given by

$$\begin{aligned}r_\alpha(s,v) &= \alpha \wedge \alpha^* + v\alpha\\ &= \left(-\cos^2 s + v\ \sin s, \cos^3 s + \sin s \sin 2s + v\cos s\sin s, -\sin^3 s + v\ \cos^2 s\right).\end{aligned}$$

After some calculations, we obtain

$$\hat{\mathbf{T}} = \left(\frac{\sqrt{2}\cos\hat{s}}{\sqrt{3+\cos 2\hat{s}}}, \frac{\sqrt{2}\cos 2\hat{s}}{\sqrt{3+\cos 2\hat{s}}}, -\frac{\sqrt{2}\sin 2\hat{s}}{\sqrt{3+\cos 2\hat{s}}}\right),$$

$$\hat{\mathbf{N}} = \left(-\frac{2\sin\hat{s}}{\sqrt{3+\cos 2\hat{s}}\sqrt{13+3\cos 2\hat{s}}}, -\frac{12\sin 2\hat{s}+\sin 4\hat{s}}{2\sqrt{3+\cos 2\hat{s}}\sqrt{13+3\cos 2\hat{s}}}, -\frac{4\left(\cos^4\hat{s}+\cos 2\hat{s}\right)}{\sqrt{3+\cos 2\hat{s}}\sqrt{13+3\cos 2\hat{s}}}\right),$$

$$\hat{\mathbf{B}} = \left(-\frac{2\sqrt{2}}{\sqrt{13+3\cos 2\hat{s}}}, \frac{2\sqrt{2}\cos^3\hat{s}}{\sqrt{13+3\cos 2\hat{s}}}, \frac{-3\sin\hat{s}-\sin 3\hat{s}}{\sqrt{26+6\cos 2\hat{s}}}\right).$$

Also, we obtain

$$\begin{aligned}\hat{\kappa} &= \frac{2\sqrt{13+3\cos 2\hat{s}}}{(3+\cos 2\hat{s})^{3/2}}, \quad \hat{\tau} = -\frac{12\cos\hat{s}}{13+3\cos 2\hat{s}},\\ \hat{\theta} &= \int \hat{\tau}(\hat{s})d\hat{s} = \frac{1}{2}\sqrt{\frac{3}{2}}\ln\left(\frac{8\sqrt{6}-12\sin\hat{s}}{2\sqrt{6}+3\sin\hat{s}}\right),\end{aligned}$$

$$\hat{k}_1 = \hat{\kappa}\cos\hat{\theta} = \frac{2\sqrt{13+3\cos 2\hat{s}}}{(3+\cos 2\hat{s})^{3/2}}\left(\cos\left[\frac{1}{2}\sqrt{\frac{3}{2}}\ln\left(\frac{8\sqrt{6}-12\sin\hat{s}}{2\sqrt{6}+3\sin\hat{s}}\right)\right]\right),$$

$$\hat{k}_2 = \hat{\kappa}\sin\hat{\theta} = \frac{2\sqrt{13+3\cos 2\hat{s}}}{(3+\cos 2\hat{s})^{3/2}}\left(\sin\left[\frac{1}{2}\sqrt{\frac{3}{2}}\ln\left(\frac{8\sqrt{6}-12\sin\hat{s}}{2\sqrt{6}+3\sin\hat{s}}\right)\right]\right).$$

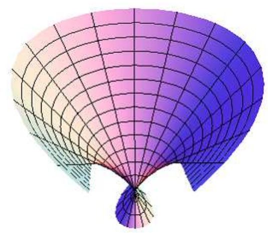

Figure 1: The ruled surface corresponding to the dual unit spherical curve $\hat{\alpha}$.

Example 4.2. *Let $\Psi(s,v)$ be a ruled surface of E^3 defined by [12].*

$$\Psi(s,v) = \gamma(s) + v\delta(s); v \in \mathbb{R},$$

where

$$\gamma(s) = \left(\frac{-2}{1+\cos^2 s}, \frac{2\cos^3 s}{1+\cos^2 s}, \frac{-\sin s - 2\sin s \ \cos^2 s}{1+\cos^2 s} \right)$$

and

$$\delta(s) = \left(\sin s, \sin s \ \cos s, \cos^2 s\right).$$

The ruled surface Ψ is written as the dual vector function $\hat{\gamma}$ given by

$$\hat{\gamma}(s) = \hat{\mathbf{U}}(s) = \delta(s) + \varepsilon \ \gamma(s) \wedge \delta(s),$$

which can be expressed as follows:

$$\hat{\gamma}(\hat{s}) = \hat{\mathbf{U}}(\hat{s}) = \left(\sin \hat{s}, \sin \hat{s} \ \cos \hat{s}, \cos^2 \hat{s}\right).$$

Now, we can write the orthonormal moving frame $\left\{\hat{\mathbf{U}}_1, \hat{\mathbf{U}}_2, \hat{\mathbf{U}}_3\right\}$along this dual curve as follows

$$\hat{\mathbf{U}}_1 = \hat{\mathbf{U}}(\hat{s}), \qquad \hat{\mathbf{U}}_2(\hat{s}) = \frac{\hat{\mathbf{U}}_1'}{\left\| \hat{\mathbf{U}}_1' \right\|}, \qquad \hat{\mathbf{U}}_3(\hat{s}) = \hat{\mathbf{U}}_1 \wedge \hat{\mathbf{U}}_2.$$

$$\begin{aligned}
\hat{\mathbf{U}}_1 &= \left\{\sin \hat{s}, \cos \hat{s} \sin \hat{s}, \cos^2 \hat{s}\right\}, \\
\hat{\mathbf{U}}_2 &= \left\{ \frac{\sqrt{2}\cos \hat{s}}{\sqrt{3+\cos 2\hat{s}}}, \frac{\sqrt{2}\cos 2\hat{s}}{\sqrt{3+\cos 2\hat{s}}}, -\frac{\sqrt{2}\sin 2\hat{s}}{\sqrt{3+\cos 2\hat{s}}} \right\}, \\
\hat{\mathbf{U}}_3 &= \left\{ -\frac{\sqrt{2}\cos^2 \hat{s}}{\sqrt{3+\cos 2\hat{s}}}, -\frac{-5\cos \hat{s} + \cos 3\hat{s}}{2\sqrt{2}\sqrt{3+\cos 2\hat{s}}}, -\frac{\sqrt{2}\sin^3 \hat{s}}{\sqrt{3+\cos 2\hat{s}}} \right\},
\end{aligned}$$

and

$$P = \left\| \hat{\mathbf{U}}_1' \right\| = \frac{\sqrt{3+\cos 2\hat{s}}}{\sqrt{2}},$$

$$Q = \frac{\det(\mathbf{U}_1, \mathbf{U}_1', \mathbf{U}_1'')}{\| \mathbf{U}_1' \|^3} = -\frac{\sqrt{2}(5+\cos 2\hat{s})\sin\hat{s}}{(3+\cos 2\hat{s})^{3/2}},$$

hence, the dual geodesic curvature

$$\hat{\Sigma} := \frac{Q}{P} = -\frac{2(5+\cos 2\hat{s})\sin\hat{s}}{(3+\cos 2\hat{s})^2},$$

the dual curvature $\mathcal{K} = \mathcal{K}(\hat{s})$ and the dual torsion $\mathcal{T} = \mathcal{T}(\hat{s})$ of $\hat{\gamma}$ are calculated as follows

$$\mathcal{K} = \sqrt{1+\hat{\Sigma}^2} = \sqrt{1+\frac{4(5+\cos 2\hat{s})^2\sin^2\hat{s}}{(3+\cos 2\hat{s})^4}},$$

$$\mathcal{T} = \pm\frac{\hat{\Sigma}'}{1+\hat{\Sigma}^2} = \frac{-106\cos\hat{s}+9\cos 3\hat{s}+\cos 5\hat{s}}{2(3+\cos 2\hat{s})^3\left(1+\frac{4(5+\cos 2\hat{s})^2\sin^2\hat{s}}{(3+\cos 2\hat{s})^4}\right)}.$$

We obtain the evolute of the dual unit spherical curve of ruled surface as follows:

$$\mathbf{E}_{\hat{\gamma}}(\hat{s}) = \left(\hat{A}_1(\hat{s}), \hat{A}_2(\hat{s}), \hat{A}_3(\hat{s})\right),$$

where

$$\hat{A}_1(\hat{s}) = \frac{1-\frac{8}{(3+\cos 2\hat{s})^2}-\frac{2}{3+\cos 2\hat{s}}+\frac{\sqrt{2}}{\sqrt{3+\cos 2\hat{s}}}-\frac{\sqrt{3+\cos 2\hat{s}}}{\sqrt{2}}}{\sqrt{1+\frac{4(5+\cos 2\hat{s})^2\sin^2\hat{s}}{(3+\cos 2\hat{s})^4}}},$$

$$\hat{A}_2(\hat{s}) = \frac{\frac{\sqrt{2}\cos^3\hat{s}}{\sqrt{3+\cos 2\hat{s}}}-\frac{2\cos\hat{s}\ (5+\cos 2\hat{s})\sin^2\hat{s}}{(3+\cos 2\hat{s})^2}+\frac{\sqrt{2}\sin\hat{s}\ \sin 2\hat{s}}{\sqrt{3+\cos 2\hat{s}}}}{\sqrt{1+\frac{4(5+\cos 2\hat{s})^2\sin^2\hat{s}}{(3+\cos 2\hat{s})^4}}},$$

$$\hat{A}_3(\hat{s}) = \frac{\left(-1+\frac{4}{(3+\cos 2\hat{s})^2}-\frac{2\sqrt{2}}{\sqrt{3+\cos 2\hat{s}}}+\frac{\sqrt{3+\cos 2\hat{s}}}{\sqrt{2}}\right)\sin\hat{s}}{\sqrt{1+\frac{4(5+\cos 2\hat{s})^2\sin^2\hat{s}}{(3+\cos 2\hat{s})^4}}}.$$

Also, in the case that $\hat{s} = \frac{3\pi}{2}$, we get

$$\hat{\Sigma}'_{\mathbf{E}}(\frac{3\pi}{2}) = 0,\ \hat{\Sigma}''_{\mathbf{E}}(\frac{3\pi}{2}) = -8.$$

Then the evolute of the dual unit spherical curve $\hat{\gamma}$ at $\hat{s} = \frac{3\pi}{2}$ is diffeomorphic to the ordinary cusp.

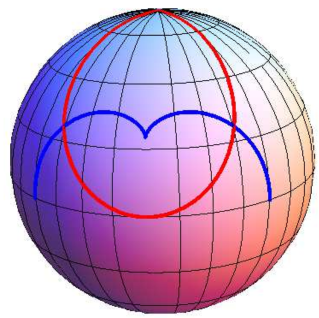

Figure 2: The dual spherical curve $\hat{\gamma}$ (the red color) of the ruled surface Ψ and its evolute (the blue color).

5 Conclusion

In this paper, we studied the spherical motion of dual curves in the dual space D^3. The distribution parameters of ruled surfaces were investigated by considering a unit speed curve in a Euclidean 3-space and a dual curve in D^3 with the same parameter and Bishop frame fields. In addition, we gave special cases for dual angles of pitch of the closed ruled surfaces. Finally, depending on the dual spherical indicatrices of the ruled surfaces, the values of distribution parameter of these ruled surfaces showed a change.

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References

[1] E. Study, Geometry der Dynamen, Lepzig, 1901.

[2] W. K. Clifford, Preliminary Sketch of Biquaternions, Proc. Lond. Soc., **4,** no. 64, (1873), 381–395.

[3] Y. Yusuf, Dual Motions and Real Spatial Motions, Int. Math. Forum, **26,** no. 7, (2012), 1263–1278.

[4] M. Önder, H. H. Uğurlu, Normal and Spherical Curves in Dual Space D^3, Mediterr. J. Math., **10,** (2013), 1527–1537.

[5] H. W. Guggenheimer, Differential geometry, McGraw-Hill, NY, 1963.

[6] S. Őzkaldı , K. İlarslan, Y. Yaylı On Mannheim partner curve in dual space, An. Ştiinţ. Univ. Ovidius Constanta, **17,** no. 2, (2009), 131–142.

[7] S. G. Papageorgiou, N. Aspragathos, Transformation and normal vector calculation of parametrically defined surfaces based on dual vectors and screw theory: Application to Phong's Shading Model, Computer Graphics Forum, **25,** (2006), 183–195.

[8] L. R. Bishop, There is more than one way to frame a curve, Amer. Math. Monthly, **82,** no. 3, 1975, 246–251.

[9] M. P. Do Carmo, Differential geometry of curves and surfaces, Prentice Hall, Englewood Cliffs, NJ, 1976.

[10] Y. Yayli, On the motion of the Frenet vectors and spacelike ruled surfaces in the Minkowski 3-Space, Mathematical and Computational Applications, **5,** no. 1, (2000), 49–55.

[11] T. Shifrin, Differential Geometry: A first Course in Curves and Surfaces (Preliminary Version), University of Georgia, 2010.

[12] Yanlin Li, Donghe Pei, Evolutes of dual spherical curves for ruled surfaces, Math. Meth. Appl. Sci., **39,** (2016), 3005–3015.