International Journal of Mathematics and
Computer Science, **17**(2022), no. 4, 1657–1663

Hosoya, Schultz and modified Schultz polynomials and their topological indices of prime graphs of commutative ring Z_n

Nabeel E. Arif

Department of Mathematics
College of Computer Science and Mathematics
Tikrit University
34001, Tikrit, Iraq

Email: nabarif@tu.edu.iq

(Received April 30, 2022, Revised July 4, 2022, Accepted July 7, 2022)

Abstract

A Hosoya polynomial of any graph is defined by $\sum_{u,v\in V(G)} x^d(u,v)$, while Schultz and modified Schultz polynomials of a graph G are defined as $Sc = \sum_{u,v\in V(G)}(deg(u) + deg(v))x^{d(u,v)}$ and $S^*c = \sum_{u,v\in V(G)}(deg(u)deg(v))x^{d(u,v)}$, respectively. Moreover, Wiener, Harary, Schultz and modified Schultz indices are related. In this paper, we investigate and compute new formulas of these polynomials and their prime graph indices of commutative ring Z_n.

1 Introduction

A *simple graph* $G = (V, E)$ is a finite non-empty set V together with a (maybe empty) set E of unordered pairs of distinct element of V which are called *vertices* and *edges*, respectively. The degree of a vertex u, denoted by $deg(u)$, is the number of vertices adjacent to u in G. The *distance* between two vertices u and v in G, denoted by $d(u, v)$, is the length of the shortest path connecting these two vertices[4, 6].
A bridge exists between graph theory and the algebraic concept of rings.

Key words and phrases: Graph polynomials, topological indices, prime graph, commutative ring.
AMS (MOS) Subject Classifications: 05C31.
ISSN 1814-0432, 2022, http://ijmcs.future-in-tech.net

Bhavanari et al. [7] studied the prime graph (PG) of a ring R with some graph properties. Arif et al. [5] defined a Pseudo Von Neumann regular graph of commutative ring. Walikar et al. [2] studied the Hosoya polynomial (HP) of thorn trees, stars, rings and rods. Ali [1] studied Schultz (ScP) and modified Schultz (MScP) of certain Cog-special graphs. Asir and Rabikka [8] examined the Wiener index of the zero divisor graph of Z_n. Sreeja et al. [3] investigated the graph polynomial, especially the independent polynomial of Fibonacci trees.
These polynomials are interesting. In this work, we study polynomials by connecting graphs and commutative rings. In addition, we investigate the Hosoya (HP), Schultz (ScP) and modified Schultz (MScP) polynomials and their indices, such as Wiener, Harary, Schultz and modified Schultz indices of the prime graph (PG) of commutative ring Z_n.

2 Preliminaries

Definition 2.1. *[7] Let R be a ring. Then $(G(V(G)), E(G))$ is called the prime graph (PG) of R: If $V = R$ and $E(G) = \{\overline{ab}/aRb = 0$ or $bRa = 0$ and $a \neq b\}$ and is denoted by $PG(R)$.*

Theorem 2.2. *[7] Let R be a ring. Assume $dimR = S$ and let $U_i, 1 \leq i \leq S$ be a collection of uniform ideals whose sum is direct and essential in R. The number of triangles in $PG(R) = \sum_{j=1}^{s-1} n_j \Big(\sum_{i=j+1}^{n_i} \Big)$, where $n_i = |U_i| - 1$ for $1 \leq i \leq s$.*

Definition 2.3. *[2] The Hosoya polynomial (HP) of a graph G is defined by*

$$H(G, x) = \sum_{u,v \in V(G)} x^{d(u,v)}.$$

Definition 2.4. *[1] The Schultz polynomial (ScP) of a graph G is defined by*

$$Sc(G, x) = \sum_{u,v \in V(G)} (deg(u) + deg(v)) x^{d(u,v)}.$$

Definition 2.5. *[1] The modified Schultz polynomial (MScP) of a graph G is defined by*

$$S^*c(G, x) = \sum_{u,v \in V(G)} (deg(u)deg(v)) x^{d(u,v)}.$$

3 Main results

Hosoya polynomial (HP) of a PG(Z_n)

Theorem 3.1. *The Hosoya polynomial (HP) of $PG(Z_n)$ is*

$$H(PG(Z_n), \chi) = (n + k - 1)\chi + \sum_{i=2}^{n-1} (n - i)\chi^2 - k\chi^2,$$

where k is the number of triangles in $PG(Z_n)$ such that $k = \sum_{j=1}^{n-1} t_j \left(\sum_{i=j+1}^{n} t_i \right)$ and $t_i = |U_i| - 1$ for $1 \leq i \leq n$, U_i is a collection of uniform ideals whose sum is direct and essential in $R = Z_n$.

Proof. The Hosoya polynomial of $PG\left(Z_n\right), n$ is a prime number, where $(n-1)\chi + \sum_{i=2}^{n-1}(n-i)\chi^2$ because the graph $PG\left(Z_n\right)$ is a star graph. The distance between any two vertices in $PG\left(Z_n\right)$ is either 1 or 2 and so

$$H\left(PG\left(Z_n\right), \chi\right) = (n-1)\chi + k\chi + \sum_{i=2}^{n-1} (n-i)\chi^2 - k\chi^2.$$

That is,

$$H\left(PG\left(Z_n\right), \chi\right) = (n+k-1)\chi + \sum_{i=2}^{n-1} (n-i)\chi^2 - k\chi^2.$$

□

Example 3.2. *The Hosoya polynomial of $PG\left(Z_8\right)$ is $9x + 19x^2$*

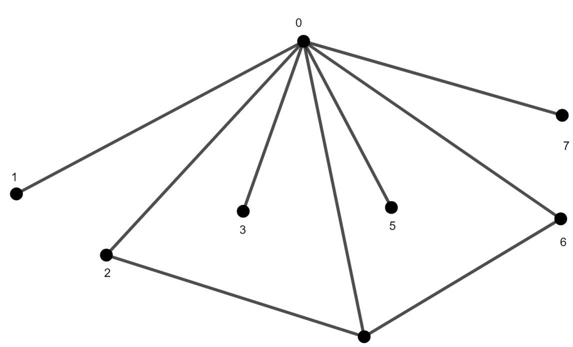

Figure 1: $PG\left(Z_8\right)$

Notably, $n = 8$ and $k = 2$. Then, $H\left(PG\left(Z_n\right), x\right) = (8 + 2 - 1)x + \sum_{i=2}^{7}(8 - i)x^2 - 2x^2 = 9x + 19x^2$.

Theorem 3.3. *The Wiener index (WI) of $PG\left(Z_n\right)$ is*

$$W\left(PG\left(Z_n\right)\right)=(n+k-1)+2.\sum_{i=2}^{n-1}(n-i)$$

Proof. The Wiener index (WI) of $PG\left(Z_n\right)$ is the first derivative of (HP) at $x=1$ [2] as obtained in theorem 3.1; that is, $W(G)=\frac{d}{dx}H(G,x)\big|_{x=1}$ which implies $W\left(PG\left(Z_n\right)\right)=(n+k-1)+\sum_{i=2}^{n-1}(n-i)$.

Example 3.4. $W\left(PG\left(Z_6\right)\right)=(6+2-1)+2\sum_{i=2}^{5}(6-i)=27$ *Since* $n=6$ *and* $k=2$

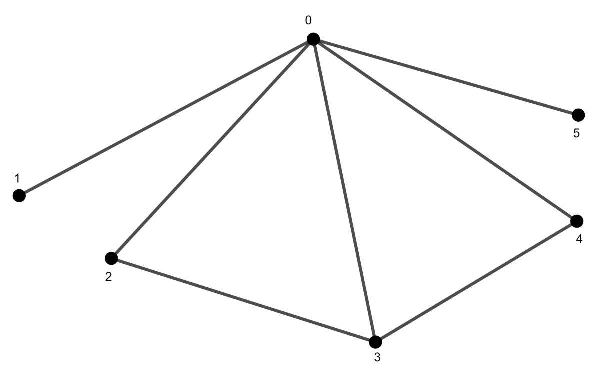

Figure 2: $PG\left(Z_6\right)$

Definition 3.5. *[2] The Harary index (HI) of a graph* G *is*

$$Ha(G)=\int_0^1\frac{H(G,x)}{x}dx.$$

□

Theorem 3.6. *The Harary index (HI) of $PG\left(Z_n\right)$ is*

$$Ha\left(PG\left(Z_n\right)\right)=n+k-1+\frac{1}{2}\sum_{i=2}^{p-1}(p-i)-\frac{1}{2}k.$$

Proof. The integral Hosoya polynomial divides x as follows:

$$\int_0^1\frac{H\left(PG\left(Z_n\right),x\right)}{\chi}d\chi=\int_0^1\frac{(n+k-1)\chi+\sum_{i=2}^{n=1}(n-i)\chi^2-k\chi^2}{\chi}d\chi$$

which implies

$$\mathrm{Ha}\left(PG\left(Z_n\right)\right) = (n+k-1) + \frac{1}{2}\sum_{i=2}^{p-1}(p-i) - \frac{1}{2}k.$$

□

Schultz (ScP) and modified Schultz (MScP) polynomials of the prime graph (PG) of commutative ring Z_n.

Theorem 3.7. *The Schultz polynomial (ScP) of $PG\left(Z_n\right)$ is*

$$Sc\left(PG\left(Z_n\right), x\right) = \left(\frac{n!}{(n-2)!} + k\right)x + (2n-k-2)x^2,$$

where k is the number of triangles in $\underline{PG}\left(Z_n\right)$.

Proof. The Schultz polynomial (ScP) of $PG\left(Z_n\right), n$ is a prime number which is $Sc\left(PG\left(Z_n\right), x\right) = \left(\frac{n!}{(n-2)!}\right)x + 2(n-1)x^2$ because the graph $PG\left(Z_n\right)$ is a star graph. The distance between any two vertices in $PG\left(Z_n\right)$ is either 1 or 2 and so

$$Sc\left(PG\left(Z_n\right), x\right) = \left(\frac{n!}{(n-2)!}\right)x + kx + 2(n-1)x^2 - kx^2;$$

that is,

$$Sc\left(PG\left(Z_n\right), x\right) = \left(\frac{n!}{(n-2)!} + k\right)x + (2n-k-2)x^2$$

□

Theorem 3.8. *The modified Schultz polynomial (MScP) of $PG\left(Z_n\right)$ is*

$$S^*c\left(PG\left(Z_n\right), x\right) = \left((n-1)^2 + k\right)x + \left(\sum_{i=2}^{n-1}(n-i) - k\right)x^2,$$

where k is the number of triangles in $PG\left(Z_n\right)$.

Proof. The modified Schultz polynomial (MScP) of $PG\left(Z_n\right), n$ is a prime number which is defined by

$$S^*c\left(PG\left(Z_n\right), x\right) = (n-1)^2 x + \sum_{i=2}^{n-1}(n-i)x^2$$

because the graph $PG\left(Z_n\right)$ is a star graph. The distance between any two vertices in $PG\left(Z_n\right)$ is either 1 or 2 and so

$$S^*c\left(PG\left(Z_n\right),x\right)=(n-1)^2x+kx+\sum_{i=2}^{n-1}(n-i)x^2-kx^2;$$

that is,

$$S^*c\left(PG\left(Z_n\right),x\right)=\left((n-1)^2+k\right)x+\left(\sum_{i=2}^{n-1}(n-i)-k\right)x^2.$$

□

Results of the topological indices (TI) of Schultz (ScP) and modified Schultz (MScP) polynomials of the prime graph PG of commutative ring Z_n

The Schultz (ScI) and modified Schultz (MScI) indices of a graph G are $Sc(G)=\frac{d}{dx}Sc(G,x)\big|_{x=1}$, $S^*c(G)=\frac{d}{dx}S^*c(G,x)\big|_{x=1}$, respectively.[1]

Theorem 3.9. *The Schultz index (ScI) of $PG\left(Z_n\right)$ is*

$$\left(\frac{n!}{(n-2)!}+k\right)+2(2n-k-2),$$

where k is the number of triangles in $\mathrm{PG}\left(\mathrm{Z_n}\right)$.

Proof. The Schultz index of $PG\left(Z_n\right)$ is the first derivative of Schultz polynomial at x = 1 as obtained in Theorem 3.7. This finding implies the following:

$$Sc(G)=\frac{d}{dx}Sc(G,x)\bigg|_{x=1}=\left(\frac{n!}{(n-2)!}+k\right)+2(2n-k-2)$$

□

Theorem 3.10. *The modified Schultz index (MScI) of $PG\left(Z_n\right)$ is*

$$\left((n-1)^2+k\right)+2\left(\sum_{i=1}^{n-1}(n-i)-k\right),$$

where k is the number of triangles in $PG\left(Z_n\right)$.

Proof. The modified Schultz index of $PG\left(Z_n\right)$ is the first derivative of the modified Schultz polynomial(MScP) at $x=1$ as obtained in Theorem 3.8. This finding implies the following:

$$S^*c(G)=\frac{d}{dx}S^*c(G,x)\bigg|_{x=1}=\left((n-1)^2+k\right)+2\left(\sum_{i=1}^{n-1}(n-i)-k\right).$$

□

4 Conclusion

In this paper, we dealt with the computation of the Hosoya, Schultz and modified Schultz polynomials and topological indices such as Wiener, Harary, Schultz, and modified Schultz polynomials of prime graphs (PG) of commutative ring Z_n. The problem with other graph polynomials and prime graph indices of commutative ring Z_n remains open.

Acknowledgment. The authors wish to thank the College of Computer Science and Mathematics, Tikrit University, for supporting this work and the referee of this paper for his/her valuable comments.

References

[1] A. M. Ali, Schultz and Modified Schultz Polynomials of Some Cog-Special Graphs, Open Access Library Journal, **6,** (2019), e5625.

[2] H. B. Walikar, H. S. Ramane, L. Sindagi, S. S. Shirakola, I. Gutman, Hosoya Polynomial of Thorn Trees, Rods, Rings, and stars, Kragujevac J. Sci., **28,** (2006), 47–56.

[3] K. U. Sreeja, P. B. Vinodkumar, P. B. Ramkumar, A graph polynomial for independent sets of Fibonacci trees, International Journal of Mathematics and Computer Science, **15,** no. 4, (2020), 1129–1133.

[4] M. Rahman, Basic Graph Theory, Springer , 2017.

[5] N. E. Arif, R. Hasani, N. J. Khalel, Pseudo-von Neumann regular graph of commutative ring, Journal of physics, **1879,** (2021), 032012.

[6] S. Bhavanari, S. Kuncham, Discrete Mathematics and Graph Theory, PHL Learning Private Limited, Delhi, 2014.

[7] S. Bhavanari, S. Kuncham, N. Dasari, Prime Graph of a Ring, Journal of Combinatorics, Information and System Sciences, **35,** (2010), 27–42.

[8] T. Asir, V. Rabikka, The Wiener index of the zero-divisor graph of Zn, Discrete Applied Mathematics, (2021), 1–11.