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Analytical Solution to a Damped and Forced Oscillator Equation with Power Law Nonlinearity

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Abstract

In this paper we derive approximate analytical solution to a damped and forced oscillator with power law nonlinear restoring force. We illustrate the obtained results with concrete examples.

1 Introduction

In this paper, we derive approximate analytical solution to the following oscillator equation:

$$\ddot{x} + 2\varepsilon\dot{x} + ax + bx\,|x|^{\alpha-1} = \gamma\cos\omega t,\ x(0) = x_0 \text{ and } x'(0) = \dot{x}_0, \alpha > 0. \quad (1.1)$$

Let $\varepsilon = \gamma = 0$. In the case when $\alpha = 1$, we have the linear second order ode $\ddot{x}+ax+bx = 0$. If $\alpha = 3$, then we obtain an undamped and unforced Duffing equation $\ddot{x} + ax + bx^3 = 0$ whose general solution is written in the form $x(t) = c_0\text{cn}\left(\sqrt{a + bc_0^2}t + c_1, \frac{bc_0^2}{2(a+bc_0^2)}\right)$. If $\alpha = 5$, hen we have another Duffing equation having the form $\ddot{x} + ax + bx^5 = 0$ whose general solution may also be expressed in terms of the Jacobian elliptic function cn. In general, for arbitrary parameter values $\varepsilon > 0$, $\alpha > 0$ and $ab\gamma \neq 0$ the oscillator (1.1) does not admit a closed form solution. Our idea is to approximate the odd parity function $f(x) = ax + bx\,|x|^{\alpha-1}$ by means of some cubic-quintic polynomial making use of Chebyshev techniques, say $f(x) = px + qx^3 + rx^5$ for $|x| \leq A$. Then, we reduce the original problem to the following initial value problem

$$\ddot{x} + 2\varepsilon\dot{x} + px + qx^3 + rx^5 = \gamma\cos\omega t,\ x(0) = x_0 \text{ and } x'(0) = \dot{x}_0. \quad (1.2)$$

Oscillator (1.2) is chaotic for some parameter values and its chaotic behavior may be studied using Melnikov's method. We are interested in giving approximate analytical solution to it. We will use the following approximation:

$$ax + bx\,|x|^{\alpha-1} \approx px + qx^3 + rx^5 \text{ for } |x| \leq M, \quad (1.3)$$

where

$$\begin{aligned}
p &= a - \tfrac{1}{3}2^{\frac{1}{2}-\frac{3\alpha}{2}}\left(2^\alpha + \left(5-3\sqrt{3}\right)\left(1+\sqrt{3}\right)^\alpha - \left(5+3\sqrt{3}\right)\left(\sqrt{3}-1\right)^\alpha\right)bM^{\alpha-1}\\
q &= \tfrac{1}{3}2^{-\frac{3}{2}(\alpha-1)}\left(2^{\alpha+3} + \left(7-5\sqrt{3}\right)\left(1+\sqrt{3}\right)^\alpha - \left(7+5\sqrt{3}\right)\left(\sqrt{3}-1\right)^\alpha\right)bM^{\alpha-3}\\
r &= -\tfrac{1}{3}2^{\frac{7}{2}-\frac{3\alpha}{2}}\left(2^{\alpha+1} - \left(1+\sqrt{3}\right)\left(\sqrt{3}-1\right)^\alpha - \left(\sqrt{3}-1\right)\left(1+\sqrt{3}\right)^\alpha\right)bM^{\alpha-5}
\end{aligned} \quad (1.4)$$

2 The Solution Method

We will assume that $p > 0$, say $p = \omega_0^2$. Let us consider the following δ-problem:

$$\ddot{x} + \omega_0^2 x + \delta[2\varepsilon\dot{x} + qx^3 + rx^5 - \gamma\cos\omega t] = 0,\ x(0) = x_0 \text{ and } x'(0) = \dot{x}_0. \quad (2.5)$$

Let $x = x_\delta(t)$ be the solution to our δ-problem (2.5). Then the solution to the original problem (1.2) will be $x = x_1(t)$. Following the ideas by Krýlov, Bogoliúbov and Mitropólsky from Kiev's school, we will assume the solution to the δ-problem in the form

$$x = x_\delta(t) = a\cos(\psi) + \sum_{n=1}^{N} \delta^n u_n(a, \psi) + O(\delta^{N+1}), \quad (2.6)$$

where each u_n is a periodic function of $\psi = \psi(t)$, and $a = a(t)$ and ψ are assumed to vary with time according to

$$\frac{da}{dt} \equiv \dot{a} = \sum_{n=1}^{N} \delta^n A_n(a) + O(\delta^{N+1}), \quad (2.7)$$

$$\frac{d\psi}{dt} \equiv \dot{\psi} = \omega_0 + \sum_{n=1}^{N} \psi_n(a) + O(\delta^{N+1}), \quad (2.8)$$

Define the δ-residual as

$$H_\delta(t) = \ddot{x} + \omega_0^2 x + \delta[2\varepsilon\dot{x} + qx^3 + rx^5 - \gamma\cos\omega t]. \quad (2.9)$$

The next step is to write the residual $H_\delta(t)$ as a power series in δ

$$H_\delta(t) = \ddot{x} + \omega_0^2 x + \delta\Upsilon_1 + \delta^2\Upsilon_2 + \delta^3\Upsilon_3 + \cdots. \quad (2.10)$$

For the determination of the unknown functions u_n, ψ_n, A_n, and a, we equate to zero the coefficients Υ_n in Eq. (2.10) and then we can get a system of odes. To avoid the so-called secularity terms, we choose only the solutions that do not contain $\cos\psi$ nor $\sin\psi$. Letting $N = 1$ we obtain the first order approximation as follows :

$$x_\delta(t) = a(t)\cos(\psi(t)) + \delta\frac{1}{384\omega_0^2}\left(384\gamma\cos(t\omega) + 12qa(t)^3\cos(3\psi(t)) + ra(t)^5(15\cos(3\psi(t)) + \cos(5\psi(t)))\right). \quad (2.11)$$

The ordinary differential equations for determining $a = a(t)$ and $\psi = \psi(t)$ are

$$\dot{a}(t) = -\delta\varepsilon a(t) \text{ and } \dot{\psi}(t) = \omega_0 + \frac{\delta}{16\omega_0}\left(6qa(t)^2 + 5ra(t)^4\right) \tag{2.12}$$

let $\delta = 1$. We have

$$\begin{aligned} &a(t) = c_0 \exp(-\varepsilon t) \\ &\text{and} \\ &\psi(t) = \frac{12c_0^2 q\left(1-e^{-2\varepsilon t}\right)+c_0^4 r\left(5-5e^{-4\varepsilon t}\right)+64\varepsilon t\omega_0^2}{64\varepsilon\omega_0} + c_1 \end{aligned} \tag{2.13}$$

The approximate analytical solution is obtained from (2.11) by letting $\delta = 1$. The constants c_0 and c_1 are determined from the initial conditions.

3 Analysis and Discussion.

Let us examine the accuracy of the obtained results.
Example 1. Let

$$\ddot{x} + 2x + x\sqrt{|x|} = 0,\ x(0) = 1 \text{ and } x'(0) = 0. \tag{3.14}$$

We have the following approximation :

$$2x + x\sqrt{|x|} \approx 2.44037x + 1.05471x^3 - 0.507293x^5 \text{ for } -1 \le x \le 1 \tag{3.15}$$

We replace the original problem (3.14) with the i.v.p.

$$\ddot{x} + 2.44037x + 1.05471x^3 - 0.507293x^5 = 0,\ x(0) = 1 \text{ and } x'(0) = 0. \tag{3.16}$$

Approximate analytical solution is obtained from (2.11) - (2.12) - (2.13) as follows :

$$x_{\text{approx}}(t) = 0.995143\cos(1.71337t) + 0.00538528\cos(5.14012t) - 0.000528324\cos(8.56687t) \tag{3.17}$$

See Figure 1.

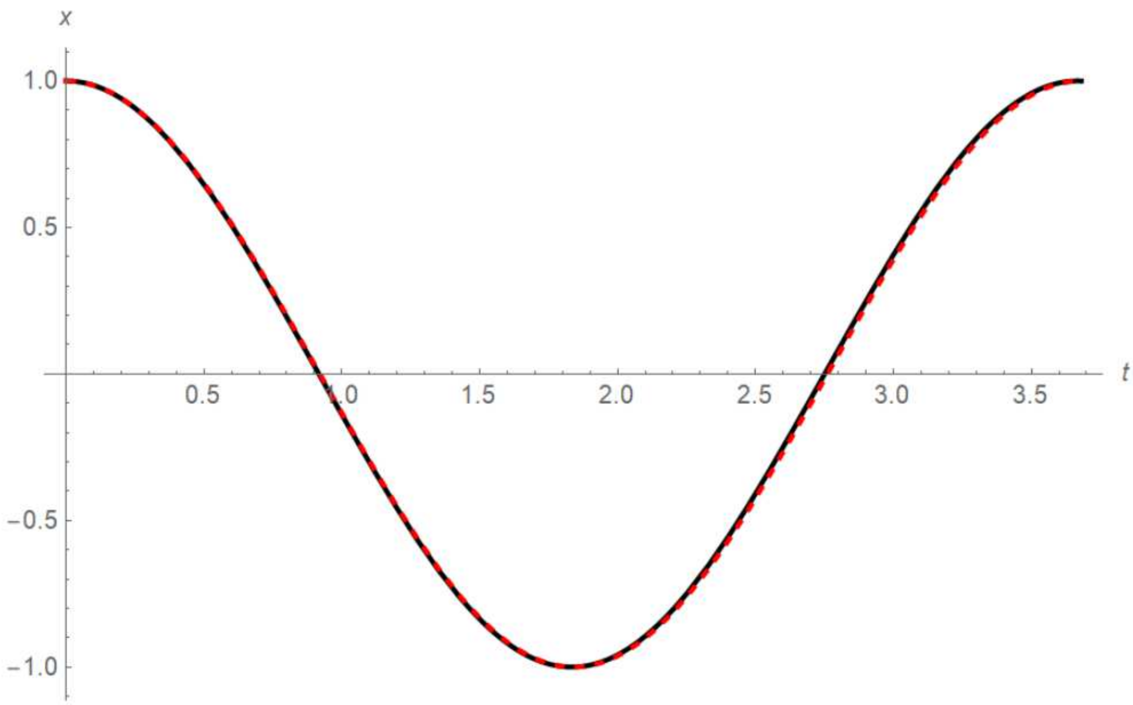

Figure 1. Error = 0.403117

Example 2. Let

$$\ddot{x} + 0.02\dot{x} \ + 5x + 0.1x \ \ |x|^{-2/3} = 0.1\cos t \wedge x(0) = 0.2 \wedge x'(0) = 0.$$

An approximate analytical solution is

$x_{\text{approx}}(t) = 0.183088e^{-0.01t}\cos(\psi(t)) - 0.00102425e^{-0.03t}\cos(3\psi(t)) +$
$0.0000434799e^{-0.05t}(15\cos(3\psi(t)) + \cos(5\psi(t))) + 0.0172426\cos(t),$
where
$\psi(t) = -2.40823t + 1.71573e^{-0.04t} - 8.08347e^{-0.02t} + 6.37212$

See Figure 2.

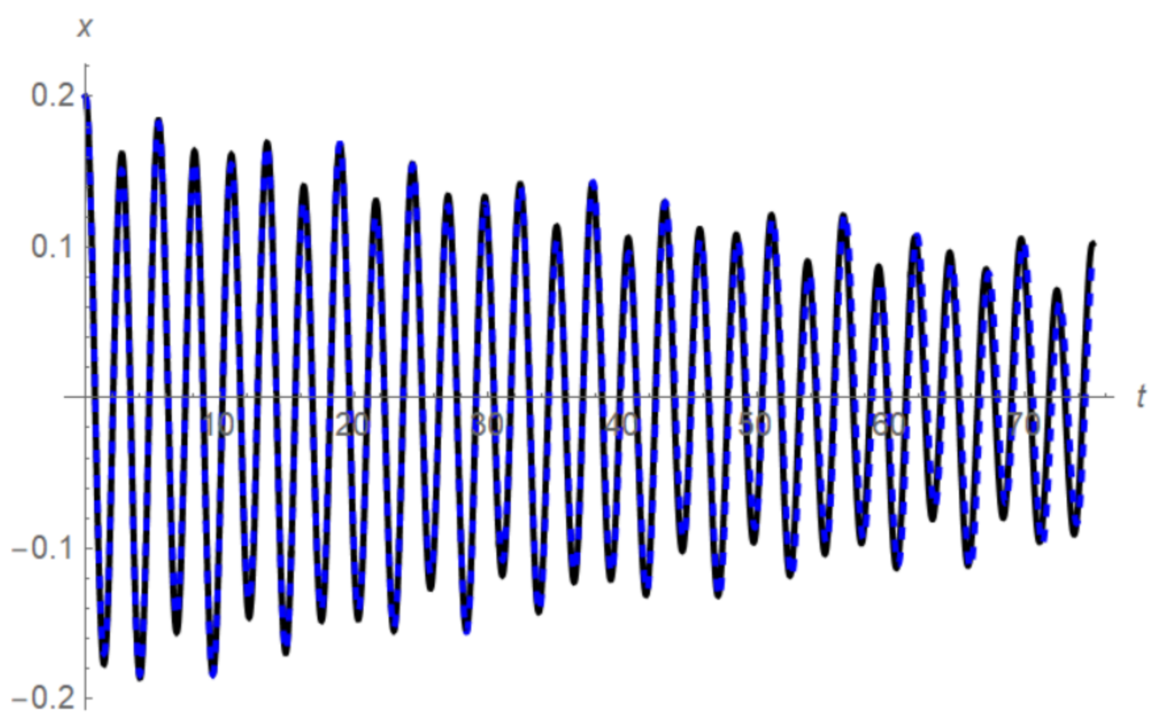

Figure 2. Error = 0.0460566.

References

[1] Alvaro H. Salas, Analytic solution to the pendulum equation for a given initial conditions, Journal of King Saud University - Science, **32,** no. 1, (2020), 974–978.

[2] Alvaro H. Salas, An Elementary Solution to a Duffing Equation, The Scientific World Journal, https://doi.org/10.1155/2022/2357258