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A note on superassociative algebra of terms determined by singular mappings on a finite set

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Abstract

In this paper, we study some classes of terms, called trees, which play a key role in both universal algebra and theoretical computer science. We introduce the concept of terms defined by singular transformations and provide some concrete examples. We also prove that the set of such terms together with an operation of type $(n+1)$ for a fixed natural number n defined on that set forms an algebra of type $(n+1)$ satisfying the axiom of superassociativity.

1 Introduction and preliminaries

For a fixed positive integer n, a nonempty set G equipped with one operation of type $(n+1)$ defined on G satisfying the axiom of superassociativity is called a *superassociative algebra* or a *Menger algebra*. Recent trends and different perspectives in superassociative algebras can be found, for example, in [5, 7].

Let $X_n = \{x_1, \ldots, x_n\}$ be a finite set of variables. The *type* is the sequence $\tau = (n_i)_{i \in I}$ of natural numbers which are arities of the function symbols f_i, where I is an index set. By $W_\tau(X_n)$, we denote the set of all

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n-ary terms of type τ. Generally, terms are induced by various kinds of transformations and have been widely studied (for example, full terms in [4], $K^*(n,r)$-full terms in [9], order-decreasing full terms in [10], full terms derived from transformations with restricted range in [8], inductive terms in [6]). For more background and advanced topics in the study of terms, we refer the reader to [3, 11]. From [4], recall that full terms of the type $\tau_n = (n_i)_{i\in I}$, where $n_i = n$ for all $i \in I$ which are classical classes of terms can be inductively defined by a full transformation α in the semigrop T_n of all mappings from $\bar{n} = \{1, \ldots, n\}$ to itself as follows:
(1) If f_i is an n-ary operation symbol of type τ_n, then $f_i(x_{\alpha(1)}, \ldots, x_{\alpha(n)})$ is an n-ary full term of type τ_n.
(2) If f_i is an n-ary operation symbol of type τ_n and $t_1, \ldots, t_n$ are n-ary full terms of type τ_n, then $f_i(t_1, \ldots, t_n)$ is an n-ary full term of type τ_n.
The set of all order-decreasing full terms of type τ_n is denoted by $W^{T_n}_{\tau_n}(X_n)$. Moreover, in [4], the superposition operation R^n on $W^{T_n}_{\tau_n}(X_n)$, $n \in \{1, 2, \ldots\}$, which can be defined as the mapping $R^n : (W^{T_n}_{\tau_n}(X_n))^{n+1} \to W^{T_n}_{\tau_n}(X_n)$ was mentioned and the algebra $\mathcal{M}_{T_n}(\tau_n) := (W^{T_n}_{\tau_n}(X_n), R^n)$ satisfying the superassociative law was constructed.

In this paper, we focus on the concept of a singular transformation which is a mapping in the set $T_n \setminus S_n$, where S_n is the set of all bijective mappings on $\bar{n}$. For more details of sigular transformations [1, 2]. Moreover, We define a particular class of terms by all elements in the set $sing_n$ of all singular transformations on $\bar{n}$ and propose some concrete examples. Furthermore, we study the algebra consisting of the set of our defined terms and an operation of the arity $(n+1)$ satisfying the superassociativity. Finally, we present some future outlook on this subject.

2 Results

We start with the definition of full terms induced by singular transformations on a finite set $\bar{n}$.

Definition 2.1. *Let f_i be an n-ary operation symbol and let β be an arbitrary mapping in $sing_n$. An n-ary singular full term of type τ_n is inductively defined by the following steps:*

(1) $f_i(x_{\beta(1)}, \ldots, x_{\beta(n)})$ is an n-ary singular full term of type τ_n,

(2) if $t_1, \ldots, t_n$ are n-ary singular full terms of type τ_n, then $f_i(t_1, \ldots, t_n)$ is an n-ary singular full term of type τ_n.

The set of all n-ary singular full terms of type τ_n is denoted by $W_{\tau_n}^{sing_n}(X_n)$.

Now we give some concrete examples of singular full terms of some types.

Example 2.2. *Let $\tau_3 = (3,3,3)$ be a type with three ternary operation symbols $f, g,$ and h and let $sing_3$ be the set of all mappings in $T_3 \setminus S_3$. Then $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 3 \end{pmatrix} \in sing_3$. So*

$$f(x_1,x_3,x_1), g(x_2,x_2,x_2), h(x_2,x_3,x_3) \in W_{\tau_3}^{sing_3}(X_3),$$

and $g(h(x_2,x_3,x_3), f(x_1,x_3,x_1), g(x_2,x_2,x_2)) \in W_{\tau_3}^{sing_3}(X_3)$. On the other hand, $f(x_1,x_2,x_3), g(x_3,x_2,x_1), g(x_2,x_3,x_1) \notin W_{\tau_3}^{sing_3}(X_3)$, since $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$, $\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \notin sing_3$.

Observe that both sets $W_{\tau_n}^{sing_n}(X_n)$ and $W_{\tau_n}^{T_n}(X_n)$ are the same if S_n is empty. This means that singular full terms can be regarded as a generalization of full terms of type τ_n.

For every $s, t_1, \ldots, t_n \in W_{\tau_n}^{sing_n}(X_n)$, we define an $(n+1)$-ary operation on the set $W_{\tau_n}^{sing_n}(X_n)$. A mapping $S^n : (W_{\tau_n}^{sing_n}(X_n))^{n+1} \to W_{\tau_n}^{sing_n}(X_n)$, is inductively defined by

(1) $S^n(s,t_1,\ldots,t_n) = f_i\left(t_{\alpha(1)},\ldots,t_{\alpha(n)}\right)$ if $s = f_i(x_{\alpha(1)},\ldots,x_{\alpha(n)})$,

(2) $S^n(s,t_1,\ldots,t_n) = f_i(S^n(s_1,t_1,\ldots,t_n),\ldots,S^n(s_n,t_1,\ldots,t_n))$ if $s = f_i(s_1,\ldots,s_n)$.

Then we can form the algebra $\text{clone}_{sing_n}(\tau_n) := (W_{\tau_n}^{sing_n}(X_n), S^n)$ of type $(n+1)$, where n is a positive integer. Moreover, we prove:

Theorem 2.3. *The algebra $\text{clone}_{sing_n}(\tau_n)$ satisfies the superassociative law, i.e., $S^n(S^n(r,p_1,\ldots,p_n),q_1,\ldots,q_n) = S^n(t,S^n(p_1,q_1,\ldots,q_n),\ldots,S^n(p_n,q_1,\ldots,q_n))$ for all $t, p_j, q_j \in W_{\tau_n}^{sing_n}(X_n), j = 1\ldots,n$.*

Proof. We give a proof by induction on the complexity of a singular full term r. If $r = f_i(x_{\alpha(1)},\ldots,x_{\alpha(n)})$ where $\alpha \in sing_n$, then

$$\begin{aligned}
& S^n(S^n(f_i(x_{\alpha(1)},\ldots,x_{\alpha(n)}),p_1,\ldots,p_n),q_1,\ldots,q_n) \\
= \; & S^n(f_i(p_{\alpha(1)},\ldots,p_{\alpha(n)}),q_1,\ldots,q_n) \\
= \; & f_i(S^n(p_{\alpha(1)},q_1,\ldots,q_n),\ldots,S^n(p_{\alpha(n)},q_1,\ldots,q_n)) \\
= \; & S^n(f_i(x_{\alpha(1)},\ldots,x_{\alpha(n)}),S^n(p_1,q_1,\ldots,q_n),\ldots,S^n(p_n,q_1,\ldots,q_n)).
\end{aligned}$$

In the case $r = f_i(r_1, \ldots, r_n)$ where $r_1, \ldots, r_n \in W_{\tau_n}^{sing_n}(X_n)$, we assume that the theorem is satisfied for all $1 \leq k \leq n$,. Then

$$\begin{aligned}
& S^n(S^n(f_i(r_1, \ldots, r_n), p_1, \ldots, p_n), q_1, \ldots, q_n) \\
= \; & S^n(f_i(S^n(r_1, p_1, \ldots, p_n), \ldots, S^n(r_n, p_1, \ldots, p_n)), q_1, \ldots, q_n) \\
= \; & f_i(S^n(S^n(r_1, p_1, \ldots, p_n), q_1, \ldots, q_n), \ldots, S^n(S^n(r_n, p_1, \ldots, p_n)), q_1, \ldots, q_n)) \\
= \; & f_i(S^n(r_1, S^n(p_1, q_1, \ldots, q_n), \ldots, S^n(p_n, q_1, \ldots, q_n)), \ldots, \\
& S^n(r_n, S^n(p_1, q_1, \ldots, q_n), \ldots, S^n(p_n, q_1, \ldots, q_n))) \\
= \; & S^n(f_i(r_1, \ldots, r_n), S^n(p_1, q_1, \ldots, q_n), \ldots, S^n(p_n, q_1, \ldots, q_n)).
\end{aligned}$$

This completes the proof. □

From Theorem 2.3, we say that the algebra clone$_{sing_n}(\tau_n)$ belongs to a variety of all superassociative algebras since it satisfies the superassociative law.

For $t \in W_{\tau_n}^{sing_n}(X_n)$ and $\alpha, \beta \in sing_n$, we define a term t that arises from a singular mapping β as follows:

(1) $t_\beta := f_i(x_{\beta(\alpha(1))}, \ldots, x_{\beta(\alpha(n))})$ if $t = f_i(x_{\alpha(1)}, \ldots, x_{\alpha(n)})$.

(2) $t_\beta := f_i((t_1)_\beta, \ldots, (t_n)_\beta)$ if $t = f_i(t_1, \ldots, t_n)$.

Observe that, for all $\beta \in sing_n$, t_β is again an element in $W_{\tau_n}^{sing_n}(X_n)$ if t is. The following theorem gives some properties of an operation S^n and a singular mapping on clone$_{sing_n}(\tau_n)$.

Theorem 2.4. *Let $t, s_1, \ldots, s_n$ be arbitrary singular full terms and let β be a mapping in $sing_n$. Then*

$$S^n(t_\beta, s_1, \ldots, s_n) = S^n(t, s_{\beta(1)}, \ldots, s_{\beta(n)}) = S^n(t, s_1, \ldots, s_n)_\beta.$$

Proof. We first show that $S^n(t_\beta, s_1, \ldots, s_n)$ and $S^n(t, s_{\beta(1)}, \ldots, s_{\beta(n)})$ are the same. For this, we prove on the complexity of a singular full term t. If $t = f_i(x_{\alpha(1)}, \ldots, x_{\alpha(n)})$, by the definition of terms induced by $\beta \in sing_n$, we have $S^n(f_i(x_{\alpha(1)}, \ldots, x_{\alpha(n)})_\beta, s_1, \ldots, s_n) = S^n(f_i(x_{\beta(\alpha(1))}, \ldots, x_{\beta(\alpha(n))}), s_1, \ldots, s_n) = f_i(s_{\beta(\alpha(1))}, \ldots, s_{\beta(\alpha(n))}) = S^n(f_i(x_{\alpha(1)}, \ldots, x_{\alpha(n)}), s_{\beta(1)}, \ldots, s_{\beta(n)})$. Assume now that $S^n((t_k)_\beta, s_1, \ldots, s_n) = S^n(t_k, s_{\beta(1)}, \ldots, s_{\beta(n)})$ for all $1 \leq k \leq n$. Note that $S^n(f_i(t_1, \ldots, t_n)_\beta, s_1, \ldots, s_n) = S^n(f_i(t_1, \ldots, t_n), s_{\beta(1)}, \ldots, s_{\beta(n)})$ since

$$\begin{aligned}
& f_i(S^n((t_1)_\beta, , s_1, \ldots, s_n), \ldots, S^n((t_n)_\beta, , s_1, \ldots, s_n)) \\
= \; & f_i(S^n(t_1, s_{\beta(1)}, \ldots, s_{\beta(n)}), \ldots, S^n(t_n, s_{\beta(1)}, \ldots, s_{\beta(n)})).
\end{aligned}$$

Now we prove that $S^n(t_\beta, s_1, \ldots, s_n) = S^n(t, s_1, \ldots, s_n)_\beta$ by induction on the depth of a term $t \in W_{\tau_n}^{sing_n}(X_n)$. First, if $t = f_i(x_{\alpha(1)}, \ldots, x_{\alpha(n)})$, we obtain

$$S^n(f_i(x_{\alpha(1)}, \ldots, x_{\alpha(n)})_\beta, s_1, \ldots, s_n) = S^n(f_i(x_{\beta(\alpha(1))}, \ldots, x_{\beta(\alpha(n))}), s_1, \ldots, s_n).$$

Since $f_i(s_{\beta(\alpha(1))}, \ldots, s_{\beta(\alpha(n))})$ equals $S^n(f_i(x_{\alpha(1)}, \ldots, x_{\alpha(n)}), s_1, \ldots, s_n)_\beta$, the result follows in this case. Inductively, suppose that $t = f_i(t_1, \ldots, t_n)$ is a composite term and $S^n((t_k)_\beta, s_1, \ldots, s_n) = S^n(t_k, s_1, \ldots, s_n)_\beta$ for all $1 \leq k \leq n$. Then

$S^n(f_i(t_1, \ldots, t_n)_\beta, s_1, \ldots, s_n) = f_i(S^n((t_1)_\beta, s_1, \ldots, s_n), \ldots, S^n((t_n)_\beta, s_1, \ldots, s_n))$.

Since $S^n((t_k)_\beta, s_1, \ldots, s_n) = S^n(t_k, s_1, \ldots, s_n)_\beta$ for all $1 \leq k \leq n$,

$$\begin{aligned} &f_i(S^n((t_1)_\beta, , s_1, \ldots, s_n), \ldots, S^n((t_n)_\beta, , s_1, \ldots, s_n)) \\ &= f_i(S^n(t_1, s_1, \ldots, s_n)_\beta, \ldots, S^n(t_n, s_1, \ldots, s_n)_\beta). \end{aligned}$$

The proof of the theorem is complete. □

Note that Theorem 2.4 provides a nice connection between the superposition S^n and a term t_β which arises from a singular mapping β on $\bar{n}$. This means that we can distribute an arbitrary mapping into a term under superposition.

3 Final remarks

In this work, we gave singular full terms generated by full transformations in $T_n \setminus S_n$. We applied a superposition operation S^n of type $(n+1)$ on the set of all singular full terms of type τ_n. However, the problems related to the generating system, the freeness, and hyperidentities derived by singular full terms of an algebra $\text{clone}_{sing_n}(\tau_n)$ remain open.

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