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On classification of groups of order p^4, where p is an odd prime

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Abstract

The classification of p-groups is a difficult task. It is also important because these groups have certain axioms and properties of which one can have further investigation for the other finite groups. In fact, there is no complete classification for p-groups, and the only complete classification has been done for certain cases of prime numbers or for certain nilpotency classes. In this article, we give a complete description of p-groups of order p^4, where p is an odd prime. In addition, we discuss many other algebraic properties. Significantly, computer calculations were of an extensive use in this work.

1 Introduction

The classification of all isomorphism types of finite groups of a known order has attracted the interest of mathematicians since the first determination of cyclic groups and groups of order 4 and 6 by Cayley [6]. The groups of order

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p^3 were independently determined in 1893 by Hölder [8], Cole and Glover [7] and Young [14]. In [12], a complete list of groups of order 3^5 was calculated. In Burnside [5], the groups of order p^3 and p^4 were presented by defining a set of relations, containing no indeterminate symbols, such that in each case a set of generating operations have been chosen to satisfy these relations. It is finally necessary to verify that the relations actually defined a group of order p^m. We use a distinct rule in which the classification of groups of order p^4 (p is an odd prime) will be considered, by finding the structure description of each group, using the group generators and a list of relations. The motivation behind this work is that similar calculations could be applied for other p-groups of different orders.

Certainly, one can go much further with the help of computers using certain computer software to yield imperative results, clear discussion and avoiding miscalculations. Therefore, many results in this article were investigated using certain computer calculations. These calculations were done by GAP (Groups, Algorithm and Programming). So, most of our results are the result of computations rather than relying on ordinary algebraic tools.

2 Notations and preliminaries

Our notations are fairly standard. The cyclic group generated by an element x of finite order n will be denoted by $\langle x\rangle \cong \mathbb{Z}_n$. The semidirect product of groups G and H will be denoted by $G \rtimes H$. For a group G, the center of G will be denoted by $Z(G)$ and the derived set of G will be denoted by G'.

Theorem 2.1. *(Sylow Theorem)[9] If p is a prime number and p^k divides $|G|$, then G has a subgroup of order p^k.*

Lemma 2.2. *[13] Let G be a group of nilpotency class two. Then, for $x, y, z \in G$ and $n \in \mathbb{Z}$,*

1. $[x, yz] = [x, y][x, z]$ *and* $[xy, z] = [x, z][y, z]$
2. $[x^n, y] = [x, y^n] = [x, y]^n$ *and* $(xy)^n = x^n y^n [y, x]^{\binom{n}{2}}$.

Lemma 2.3. *[13] Let $G = \langle a, b\rangle$ be a group of nilpotency class two. Then*

1. $G' = \langle [a, b]\rangle$.
2. *If G' is finite of order m, then $\langle a\rangle \cap Z(G) = \langle a^m\rangle$ and $\langle b\rangle \cap Z(G) = \langle b^m\rangle$.*

Lemma 2.4. *[10] Let H and K be two subgroups of G. Then $HK = \{hk \mid h \in H,\ k \in K\}$ is a subgroup of G if and only if $HK = KH$.*

Corollary 2.5. *A subgroup H of a group G is permutable if and only if HK is a subgroup of G, for any $K \leq G$.*

Proof. If a subgroup H is permutable in G, then $HK = KH$, for any subgroup K of G. By Lemma 2.4, it follows that $HK \leq G$ for any subgroup K of G. Conversely, let HK be a subgroup of G for a subgroup H and any subgroup K. From the same lemma, $HK = KH$ which is true for any subgroup K. Then H is permutable. □

Recall that, two subgroups H and K of a group G are conjugate, if there exists $g \in G$ such that $gHg^{-1} = K$.

Corollary 2.6. *[10] Let H be a subgroup of a finite group G. Then the total number of distinct conjugates of H in G, counting H itself, is $[G : N_G(H)]$.*

Definition 2.7. *[2] A group G is said to be capable if there exists a group H such that $G \cong H/Z(H)$ or, equivalently, G is isomorphic to the inner automorphism group of a group H.*

Capable groups were first studied by Baer [3], who showed the following:

Remark 2.8. *A finite abelian group is capable if and only if it is a product of cyclic groups of orders $n_1, n_2, \cdots, n_k$, where n_i divides n_{i+1} and $n_{k-1} = n_k$.*

Recall that, the epicenter $Z^*(G)$ of a group G is defined to be the smallest normal subgroup N of G such that G/N is a central quotient of a group.

Theorem 2.9. *[4] A group G is capable if and only if $|Z^*(G)| = 1$.*

The capability of 2-generators p-groups of nilpotency class 2 has been studied by Bacon and Kappe [2] who asserted that certain classes of these groups are capable (see the next corollary). In addition, they showed that, if G is a capable finite group of nilpotency class 2, with a minimal generating set $\{a_1, a_2, \cdots, a_n\}$, such that $o(a_i) = p^{n_i}$, $n_i \geq n_{i+1}$, then $n_1 = n_2$. Also, they showed that the converse is not true; i.e., $n_1 = n_2$ is not sufficient for G to be a capable group. In this article, the obtained results meet these contributions. Although, the groups concluded in this article are of nilpotency class 1,2 or 3, and the size of the minimal generating set is 1,2,3 or 4.

Corollary 2.10. *[2] Let G be a 2-generator p-group of nilpotency class 2, where p is an odd prime. Then G is capable if and only if G is isomorphic to one of the following structures:*

- $(\langle c\rangle \times \langle a\rangle) \rtimes \langle b\rangle$, *where* $[a,b]=c,\ [a,c]=[b,c]=e$ *and* $o(a)=p^{\alpha},\ o(b)=p^{\beta},\ o(c)=p^{\gamma},\ \alpha,\beta,\gamma \in \mathbb{N}$ *with* $\alpha \geq \beta \geq \gamma \geq 1$.
- $\langle a\rangle \rtimes \langle b\rangle$, *where* $[a,b]=a^{p^{\alpha-\gamma}}$ *and* $o(a)=p^{\alpha},\ o(b)=p^{\beta},\ \alpha,\beta,\gamma \in \mathbb{N}$ *with* $\alpha \geq 2\gamma\ ,\ \beta \geq \gamma \geq 1$.

with $\alpha = \beta$.

Remark 2.11. *There is no general upper bound on the index of the center of a finite group in terms of the order of its derived subgroup. The following theorem showed that there is a such bound for all capable groups.*

Theorem 2.12. *[11] There exists a function $B(n)$ defined on the natural numbers such that if G is finite and capable, then $|G:Z(G)| \leq B(|G'|)$.*

3 Main results

In this section, we discuss our main results. Some of the proofs given in this section were conducted numerically. We use certain `GAP` codes to deduce many of the next contributions.

The next theorem gives the number of all groups of order p^4, where p is an odd prime number. The next parts are destined to discuss each group of this family. We intend to investigate various notions in group theory.

Theorem 3.1. *For an odd prime p, there are 15 groups of order p^4, up to isomorphism.*

3.1 Abelian groups of order p^4

Obviously, every abelian group G of order p^4 has the following axioms: $|G| = |Z(G)| = p^4$ and $|G'| = 1$. So, every abelian group G of order p^4 is nilpotent of class 1, solvable, $|Z(G)| = |G| = p^4\ ,\ |G'| = 1$ and every subgroup of G is normal.

Remark 3.2. *There are five non-isomorphic structures of all abelian groups of order p^4. Therefore, any abelian group of order p^4, for an odd prime p is isomorphic to one of the following:*

1. $G_1 = \left\langle a \mid a^{p^4} = e \right\rangle \cong \mathbb{Z}_{p^4}$.
2. $G_2 = \left\langle a,\ b \mid a^{p^2} = b^{p^2} = [a,b] = e \right\rangle \cong \mathbb{Z}_{p^2} \times \mathbb{Z}_{p^2}$.
3. $G_3 = \left\langle a,\ b \mid a^{p^3} = b^p = [a,b] = e \right\rangle \cong \mathbb{Z}_{p^3} \times \mathbb{Z}_p$.
4. $G_4 = \left\langle a,\ b,\ c \mid a^{p^2} = b^p = c^p = [a,b] = [a,c] = [b,c] = e \right\rangle \cong \mathbb{Z}_{p^2} \times \mathbb{Z}_p \times \mathbb{Z}_p$.
5. $G_5 = \langle a,\ b,\ c,\ d \mid a^p = b^p = c^p = d^p = [a,b] = [a,c],\ [a,d],\ [b,c],\ [b,d],\ [c,d] = e\rangle$ $\cong \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$.

In the next subsections, we will study the subgroups classification of each abelian group of order p^4.

3.1.1 The Group G_1

Remark 3.3. *The group*

$$G_1 = \left\langle a \mid a^{p^4} = e \right\rangle \cong \mathbb{Z}_{p^4},$$

has 5 subgroups, these subgroups are: $H_{i+1} = \left\langle a^{p^i} \right\rangle$, $i = 0, 1, 2, 3, 4$. *The order of each subgroup is* $|H_{i+1}| = p^{4-i}$, $i = 0, 1, 2, 3, 4$.

Proof. This result could be considered as an immediate consequence of Theorem 2.1, where G_1 should have p-subgroups of orders p , p^2 , p^3 in addition to the trivial subgroup of order $p^0 = 1$ and the whole group of order p^4. □

Corollary 3.4. *The group* G_1 *is not capable.*

Proof. Let $G = G_1$. Then G is a non-trivial cyclic group. Hence it is not capable. Moreover, the epicenter of G is $Z^*(G) = \langle a \rangle$. □

3.1.2 The Group G_2

Remark 3.5. *The group*

$$G_2 = \left\langle a,\ b \mid a^{p^2} = b^{p^2} = [a,b] = e \right\rangle \cong \mathbb{Z}_{p^2} \times \mathbb{Z}_{p^2},$$

has $p^2 + 3p + 5$ *subgroups.*

Proof. Let $G = G_2$, which is a p-group of order p^4. Using Theorem 2.1, the group G has p-subgroups of orders p , p^2 , p^3 and p^4 in addition to the trivial subgroup $\{e\}$. Next, we count the subgroups of G of each order:

- The p-subgroups of order p, which are $\langle a^p\ b^{ip}\rangle \cong \mathbb{Z}_p$, $i = 1, 2, \cdots, p$ and $\langle b^p\rangle \cong \mathbb{Z}_p$.
- The p-subgroups of order p^2, which are $\langle a\ b^i\rangle \cong \mathbb{Z}_{p^2}$, $i = 1, 2, \cdots, p^2$, $\langle a^p\ b^j\rangle \cong \mathbb{Z}_{p^2}$, $j = 1, 2, \cdots, p-1$, $\langle b\rangle \cong \mathbb{Z}_{p^2}$ and $\langle a^p,\ b^p\rangle \cong \mathbb{Z}_p \times \mathbb{Z}_p$.
- The p-subgroups of order p^3, which are $\langle a^p,\ a^i b\rangle \cong \mathbb{Z}_{p^2} \times \mathbb{Z}_p$, $i = 0, 1, 2, \cdots, p-1$ and $\langle b^p,\ a\rangle \cong \mathbb{Z}_{p^2} \times \mathbb{Z}_p$.
- Finally, the only Sylow p-subgroup of G is G itself.

From above, the group G has $(p+1)+(p^2+p+1)+(p+1)+1+1 = p^2+3p+5$ subgroups. □

Corollary 3.6. *The group $G = G_2$ is capable.*

Proof. Let $G = G_2 \cong \mathbb{Z}_k \times \mathbb{Z}_k$, where $k = p^2$. Since G is an abelian group and it is a product of cyclic groups $\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2}$ in which $n_1 = k$ divides $n_2 = k$ and $n_1 = n_2$. Then Remark 2.8 asserts that G is a capable group.

Also, `GAP` calculations indicate that $Z^*(G) = \{e\}$. So, $|Z^*(G)| = 1$, which satisfies the axioms of Theorem 2.9. □

3.1.3 The Group G_3

Remark 3.7. *The group*

$$G_3 = \left\langle a,\ b \mid a^{p^3} = b^p = [a, b] = e \right\rangle \cong \mathbb{Z}_{p^3} \times \mathbb{Z}_p,$$

has $3p + 5$ subgroups.

Proof. Let $G = G_3$, which is a p-group of order p^4. Using Theorem 2.1, the group G has p-subgroups of orders p , p^2 , p^3 and p^4 in addition to the trivial subgroup. In the following, we count the number of subgroups of G for each order:

- The p-subgroups of order p, which are $\left\langle a^{p^2}\ b^i \right\rangle \cong \mathbb{Z}_p$, $i = 1, 2, \cdots, p$ and $\langle b\rangle \cong \mathbb{Z}_p$.
- The p-subgroups of order p^2, which are $\langle a^p\ b^i\rangle \cong \mathbb{Z}_{p^2}$, $i = 1, 2, \cdots, p$ and $\left\langle a^{p^2},\ b \right\rangle \cong \mathbb{Z}_p \times \mathbb{Z}_p$.
- The p-subgroups of order p^3, which are $\langle a\ b^i\rangle \cong \mathbb{Z}_{p^3}$, $i = 1, 2, \cdots, p$ and $\langle a^p,\ b\rangle \cong \mathbb{Z}_{p^2} \times \mathbb{Z}_p$.

- Finally, the only Sylow p-subgroup of G of order p^4 is G itself.

From above, the group G has $(p+1)+(p+1)+(p+1)+1+1 = 3p+5$ subgroups.

□

Corollary 3.8. *The group $G = G_3$ is not capable.*

Proof. Let $G = G_3$. Then, G is abelian group, which is a direct product of cyclic groups $\mathbb{Z}_{p^3}$ and $\mathbb{Z}_p$. Certainly, p divides p^3, but $p \neq p^3$. Therefore, Remark 2.8 implies that G is not a capable group. □

3.1.4 The Group G_4

Remark 3.9. *The group*

$$G_4 = \left\langle a,\ b,\ c \mid a^{p^2} = b^p = c^p = [a,b] = [a,c] = [b,c] = e \right\rangle \cong \mathbb{Z}_{p^2} \times \mathbb{Z}_p \times \mathbb{Z}_p,$$

has $4p^2+3p+5$ subgroups.

Proof. Let $G = G_4$, which is a p-group of order p^4. Similar to the previous remarks, Theorem 2.1 asserts that the group G has p-subgroups of orders p , p^2 , p^3 and p^4 in addition to the trivial subgroup. Next, we count the subgroups of G for each order:

- The p-subgroups of order p, which are $\langle a^p\ b^i\ c^j\rangle \cong \mathbb{Z}_p$, $i,j = 1,2,\cdots,p$, $\langle b\ c^i\rangle = \mathbb{Z}_p$, $i = 1,2,\cdots,p$ and $\langle c\rangle \cong \mathbb{Z}_p$.
- The p-subgroups of order p^2, which are $\langle a\ b^i\ c^j\rangle \cong \mathbb{Z}_{p^2}$, $i,j = 1,2,\cdots,p$, $\langle c,\ a^p\rangle \cong \mathbb{Z}_p\times\mathbb{Z}_p$, $\langle b, a^p\rangle \cong \mathbb{Z}_p\times\mathbb{Z}_p$, $\langle b\ a^{ip},\ c\rangle \cong \mathbb{Z}_p\times\mathbb{Z}_p$, $i = 1,2,\cdots,p$, $\langle c\ a^{ip},\ b\rangle \cong \mathbb{Z}_p \times \mathbb{Z}_p$, $i = 1,2,\cdots,p-1$ and $\langle b^i\ c,\ a^p\ c^j\rangle \cong \mathbb{Z}_p \times \mathbb{Z}_p$, $i = 1,2,\cdots,p-1$ and $j = 1,2,\cdots,p$.
- The p-subgroups of order p^3, which are $\langle a^p,\ b,\ c\rangle \cong \mathbb{Z}_p\times\mathbb{Z}_p\times\mathbb{Z}_p$, $\langle a\ b\ c^i,\ b\ c^j\rangle \cong \mathbb{Z}_{p^2} \times \mathbb{Z}_p$, $i,j = 1,2,\cdots,p$ and $\langle a\ b^i,\ c\rangle \cong \mathbb{Z}_{p^2} \times \mathbb{Z}_p$, $i = 1,2,\cdots,p$.
- Finally, the only Sylow p-subgroup of G of order p^4 is G itself.

From above calculations, it follows that the group G has $(p^2+p+1)+(2p^2+p+1)+(p^2+p+1)+1+1 = 4p^2+3p+5$ subgroups.

□

Corollary 3.10. *The group $G = G_4$ is not capable.*

Proof. Let $G = G_4$. Then G is abelian group, which is a direct product of cyclic groups $\mathbb{Z}_p\,,;\mathbb{Z}_p$ and $\mathbb{Z}_{p^2}$. In fact, p divides p and p^2, but $p \neq p^2$. Therefore, Remark 2.8 indicates that G is not a capable group. □

3.1.5 The Group G_5

Remark 3.11. *The group*

$$\begin{aligned} G_5 &= \langle a,\ b,\ c,\ d \mid a^p = b^p = c^p = d^p = [a,b] = [a,c],\ [a,d],\ [b,c],\ [b,d],\ [c,d] = e\rangle \\ &\cong \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p, \end{aligned}$$

has $p^4 + 3p^3 + 4p^2 + 3p + 5$ *subgroups.*

Proof. Let $G = G_5$, which is a p-group of order p^4. Similar to the previous arguments used for the abelian groups, Theorem 2.1 asserts that the group G has p-subgroups of orders p , p^2 , p^3 and p^4 in addition to the trivial subgroup. Next, we count the subgroups of G for each order:

- The p-subgroups of order p, which are $\left\langle a\ b^i\ c^j\ d^k \right\rangle \cong \mathbb{Z}_p$, $i,j,k = 1,2,\cdots,p$, $\langle b\ c^i\ d^j\rangle \cong \mathbb{Z}_p$, $i,j = 1,2,\cdots,p$, $\langle c\ d^i\rangle \cong \mathbb{Z}_p$, $i = 1,2,\cdots,p$ and $\langle d\rangle \cong \mathbb{Z}_p$.

- The p-subgroups of order p^2, which are $\left\langle a\ c^i\ d^j,\ b\ c^k\ d^m \right\rangle \cong \mathbb{Z}_p \times \mathbb{Z}_p$, $i,j,k,m = 1,2,\cdots,p$, $\left\langle a\ b^i\ d^j,\ c\ d^k \right\rangle \cong \mathbb{Z}_p \times \mathbb{Z}_p$, $i,j,k = 1,2,\cdots,p$, $\langle a\ b^i\ c^j,\ d\rangle \cong \mathbb{Z}_p \times \mathbb{Z}_p$, $i,j = 1,2,\cdots,p$, $\langle b\ d^i,\ c\ d^j\rangle \cong \mathbb{Z}_p \times \mathbb{Z}_p$, $i,j = 1,2,\cdots,p$, $\langle b\ c^i,\ d\rangle \cong \mathbb{Z}_p \times \mathbb{Z}_p$, $i = 1,2,\cdots,p$ and $\langle c,\ d\rangle \cong \mathbb{Z}_p \times \mathbb{Z}_p$.

- The p-subgroups of order p^3, which are $\left\langle a\ d^i,\ b\ d^j,\ c\ d^k \right\rangle \cong \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$, $i,j,k = 1,2,\cdots,p$, $\langle a\ c^i,\ b\ c^j,\ d\rangle \cong \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$, $i,j = 1,2,\cdots,p$, $\langle a\ b^i,\ d,\ c\rangle \cong \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$, $i = 1,2,\cdots,p$ and $\langle b,\ c,\ d\rangle \cong \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$.

- There is only one p-subgroup of order p^4, which is G itself.

From above, the group G has $(p^3 + p^2 + p + 1) + (p^4 + p^3 + 2p^2 + p + 1) + (p^3 + p^2 + p + 1) + 1 + 1 = p^4 + 3p^3 + 4p^2 + 3p + 5$ subgroups. □

Corollary 3.12. *The group* $G = G_5$ *is capable.*

Proof. Let $G = G_5 \cong \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \mathbb{Z}_{n_3} \times \mathbb{Z}_{n_4}$, where $n_i = p$ for all $i = 1,2,3,4$. Then n_i divides n_{i+1} and $n_3 = n_4$. Using Remark 2.8 implies that the group G is a capable group.

Moreover, we can find the epicentre of the group G using `GAP`, which is $Z^*(G) = \{e\}$, and this implies that G is a capable group (See Theorem 2.9). □

3.2 Non-abelian groups of order p^4

In this part, we will consider all non-abelian groups of order p^4. Using the next remark and the same iterations used on abelian groups, we study the classification of these groups.

Remark 3.13. *There are 10 non-abelian groups of order p^4, which are:*

1. $F_1 = \left\langle a,\ b \mid a^{p^2} = b^p = [a,[a,b]] = [b,[a,b]] = [a,b]^p = e \right\rangle \cong (\mathbb{Z}_{p^2} \times \mathbb{Z}_p) \rtimes \mathbb{Z}_p.$
2. $F_2 = \left\langle a,\ b \mid a^{p^2} = b^{p^2} = e,\ [b,a] = b^p \right\rangle \cong \mathbb{Z}_{p^2} \rtimes \mathbb{Z}_{p^2}.$
3. $F_3 = \left\langle a, b \mid a^{p^3} = b^p = e,\ [b,a] = a^{p^2} \right\rangle \cong \mathbb{Z}_{p^3} \rtimes \mathbb{Z}_p$. *(See Theorem 3.2, [1]).*
4. $F_4 = \langle a,\ b,\ c \mid a^p = b^p = c^p = d^p = [a,c] = [b,c] = [a,d] = [b,d] = [c,d] = e \rangle$
   $\cong \mathbb{Z}_p \times ((\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes \mathbb{Z}_p)$ *where* $d = [a,b]$.
5. $F_5 = \left\langle a,\ b,\ c \mid a^{p^2} = b^p = c^p = [a,c] = [b,c] = e,\ [b,a] = a^p \right\rangle \cong \mathbb{Z}_p \times (\mathbb{Z}_{p^2} \rtimes \mathbb{Z}_p).$
6. $F_6 = \left\langle a,\ b,\ c \mid a^{p^2} = b^p = c^p = [a,b] = [a,c] = e,\ [c,b] = a^p \right\rangle \cong (\mathbb{Z}_{p^2} \times \mathbb{Z}_p) \rtimes \mathbb{Z}_p.$
7. $F_7 = \langle a,\ b \mid a^p = b^p = c^p = [a,c]^p = [b,c] = e,\ [a,[a,c]] = [b,[a,c]] = e \rangle$
   $\cong (\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p) \rtimes \mathbb{Z}_p$ *where* $c = [a,b]$.
8. $F_8 = \left\langle a,\ b \mid a^{p^2} = b^p = [a,b]^p = [b,[a,b]] = e,\ [a,[a,b]] = a^p \right\rangle \cong (\mathbb{Z}_{p^2} \rtimes \mathbb{Z}_p) \rtimes \mathbb{Z}_p.$
9. $F_9 = \left\langle a,\ b \mid a^{p^2} = b^p = [a,b]^p = [a,[a,b]] = e,\ [b,[a,b]] = a^p \right\rangle \cong (\mathbb{Z}_{p^2} \times \mathbb{Z}_p) \rtimes \mathbb{Z}_p.$
10. $F_{10} = \left\langle a,\ b \mid a^{p^2} = b^p = [a,b]^p = [a,[a,b]] = e,\ [b,[a,b]] = a^{2p} \right\rangle \cong (\mathbb{Z}_{p^2} \times \mathbb{Z}_p) \rtimes \mathbb{Z}_p.$

3.2.1 The Group F_1

Let $G = F_1$, where

$$F_1 = \left\langle a,\ b \mid a^{p^2} = b^p = [a,[b,a]] = [b,[b,a]] = [a,b]^p = e \right\rangle \cong (\mathbb{Z}_{p^2} \times \mathbb{Z}_p) \rtimes \mathbb{Z}_p$$

Lemma 3.14. *The group $G = F_1$ is nilpotent of class 2.*

Proof. Since G is a p-group, then it is nilpotent. Using $[a,b,a] = e$, implies that $[a,b]a = a[a,b]$ and similarly, $[a,b]\ b = b\ [a,b]$. Thus, $[a,b]\ g = g\ [a,b]$ for all $g \in G$. Therefore, $[G', G] = \{e\}$. Hence G is a nilpotent group of class 2. ◻

Corollary 3.15. *The group $G = F_1$ is not capable.*

Proof. Let $G = F_1$. The minimal generating set of G is $\{a, b\}$ with $p^\alpha = o(a) = p^2 \neq o(b) = p^\beta = p$, and since G is nilpotent of class 2. Then G does not satisfy Corollary 2.10. Hence G is not capable. On the other hand, the epicenter of G using `GAP` is $Z^*(G) = \langle a^p \rangle$. □

Remark 3.16. *Let $G = F_1$. Then $|G'| = p$.*

Proof. The group $G = F_1$ is non-abelian nilpotent group of class 2. Then $o([a, b]) > 1$ and $o([a, b]) \leq min\{o(a), o(b)\}$ (see Lemma 2.2 [2]). Therefore, $1 < o([a, b]) \leq min\{o(a), o(b)\} = min\{p^2, p\}$, and hence $|G'| = p$.

This result satisfies Lemma 2.3. □

Remark 3.17. *The group F_1 has $3p^2 + 3p + 5$ subgroups.*

Proof. Let $G = F_1$. Then, the group G is a p-group of order p^4. Theorem 2.1 asserts that the group G has p-subgroups of orders p , p^2 , p^3 and p^4 in addition to the trivial subgroup. Next, we count the subgroups of G for each order.

- There are $p^2 + p + 1$ p-subgroups of order p, which are $\langle a^p\ b^i\ [a, b]^j \rangle \cong \mathbb{Z}_p$, $i, j = 1, 2, \cdots, p$, $\langle b\ [a, b]^i \rangle \cong \mathbb{Z}_p$, $i = 1, 2, \cdots, p$ and $\langle [a, b] \rangle \cong \mathbb{Z}_p$.
- There are $2p^2+p+1$ p-subgroups of order p^2, which are $\langle a^p\ [a, b]^i,\ b\ [a, b]^j \rangle \cong \mathbb{Z}_p \times \mathbb{Z}_p$, $i, j = 1, 2, \cdots, p$, $\langle a^p\ b^i,\ [a, b] \rangle \cong \mathbb{Z}_p \times \mathbb{Z}_p$, $i = 1, 2, \cdots, p$, $\langle b,\ [a, b] \rangle \cong \mathbb{Z}_p \times \mathbb{Z}_p$ and $\langle a\ b^i\ [a, b]^j \rangle \cong \mathbb{Z}_{p^2}$, $i, j = 1, 2, \cdots, p$.
- There are $p + 1$ p-subgroups of order p^3, which are $\langle a\ b^i,\ [a, b] \rangle \cong \mathbb{Z}_{p^2} \times \mathbb{Z}_p$, $i = 1, 2, \cdots, p$ and $\langle a^p,\ b,\ [a, b] \rangle \cong \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$.
- There is only one p-subgroup of order p^4, which is G itself.

From above, the group $G = F_1$ has $(p^2 + p + 1) + (2p^2 + p + 1) + (p + 1) + (1) + (1) = 3p^2 + 3p + 5$ subgroups. □

Remark 3.18. *Let $G = F_1$. Then $|Z(G)| = p^2$.*

Proof. Let $G = F_1$. Then $|Z(G)| = 1, p, p^2, p^3$ or p^4, the case p^4 is omitted, because G is non-abelian group. Note that, the group G is nilpotent of class 2. Then $[a^p, y] = [a, y]^p$ (see Lemma 2.2 [13]). Since $o([a, y]) = p$, $[a^p, y] = [a, y]^p = e$ and so $(a^p)^{-1}\ y^{-1}\ a^p\ y = e$, Thus, a^p commutes with all $y \in G$. Therefore, $a^p \in Z(G)$ which implies that $\langle a^p \rangle \subseteq Z(G)$. Since $a^p \notin G'$ and G is non-abelian group, $|G'| < |Z(G)| < |G| = p^4$. Thus $|Z(G)| = p^2$ or p^3. Using Remark 3.17, both of the p-subgroups of G of order p^3 are contained b but $b \notin \mathbb{Z}(G)$. Hence $|Z(G)| = p^2$. □

Corollary 3.19. *The group $G = F_1$ has $p(p-1)$ permutable not normal subgroups.*

Proof. Let $G = F_1$ and $H_{i,j} = \langle a^p\ [a,b]^i,\ b\ [a,b]^j \rangle\ ,\ i = 1, 2, \cdots, p-1\ ,\ j = 1, 2, \cdots, p$.

To show that $H_{i,j}$ is permutable for all such i and j, it is enough to show that any subgroup of this class commutes with all other subgroups in this class too, because the other subgroups are normal. So, Let $H = \langle a^p\ [a,b]^m,\ b\ [a,b]^n \rangle$ and $K = \langle [a,b] \rangle$. Then

$$HK = \left\langle a^p\ b\ [a,b]^{(m+1) \mod p},\ a^p\ b^2\ [a,b]^{(n+1) \mod p} \right\rangle,$$

which is a subgroup of G of order p^2. So, Corollary 2.5 implies that H is a permutable subgroup. Finally, we need to show that H is not normal. So choose $x = a^p\ b\ [a,b]^m \in H$ and $a \in G$. Then $a\ x\ a^{-1} = a^p\ b\ [a,b]^{m+1} \notin H$. Then $a\ H\ a^{-1} \neq H$, implies that H is not normal subgroup. □

3.2.2 The Group F_2

Let $G = F_2$, where

$$F_2 = \left\langle a,\ b \mid a^{p^2} = b^{p^2} = e,\ [b,a] = b^p \right\rangle \cong \mathbb{Z}_{p^2} \rtimes \mathbb{Z}_{p^2}$$

Lemma 3.20. *Let $G = F_2$. Then G is nilpotent of class 2.*

Proof. Note that $o([a,b]) = p$ and G' is not trivial, as $[a,b] = b^{-p} \in Z(G)$. So $[G,G] = G' \neq \{e\}$ and $[G', G] = \{e\}$. □

Corollary 3.21. *The group $G = F_2$ is capable.*

Proof. The group G is a 2-generator p-group of nilpotency class 2, and the minimal generating set of G is $\{a, b\}$ with $o(a) = o(b) = p^2$. Then, Corollary 2.10 asserts that G is a capable group. □

Remark 3.22. *Let $G = F_2$. Then $|G'| = p$.*

Proof. The group G is a non-abelian nilpotent group of class 2. Lemma 2.3 indicates that $G' = \langle [a,b] \rangle = \langle b^{-p} \rangle$. Hence $|G'| = p$. □

Remark 3.23. *Let $G = F_2$. Then $|Z(G)| = p^2$.*

Proof. Let $G = F_2$. Then the nilpotency class of G is 2. Therefore, $[a^p, g] = [a, g]^p = e$ (Lemma 2.2). This implies that $a^{-p}y^{-1}a^p y^{-1} = e$ for all $g \in G$. Thus $a^p \in \mathbb{Z}(G)$. So $\langle a^p \rangle \subseteq Z(G)$. Similarly, $\langle b^p \rangle \subseteq Z(G)$. Since $\langle a^p \rangle \cap \langle b^p \rangle = \{e\}$ and G is non-abelian, $p^2 \leq |Z(G)| \leq p^3$. On the other hand, $[a^p, b^p] = e$. Let $x = a^p$ and $y = b^p$. Then $Z(G) = \langle x, y \mid x^p = y^p = [x, y] = e \rangle \cong \mathbb{Z}_p \times \mathbb{Z}_p$. □

Remark 3.24. *The group F_2 has $p^2 + 3p + 5$ subgroups.*

Proof. Let $G = F_2$. The group G is a p-group of order p^4. Theorem 2.1 asserts that the group G has p-subgroups of orders p , p^2 , p^3 and p^4 in addition to the trivial subgroup. Next, we count the subgroups of G for each order.

- There are $p + 1$ p-subgroups of order p, which are $\langle a^p\ b^{ip} \rangle \cong \mathbb{Z}_p$, $i = 1, 2, \cdots, p$ and $\langle b^p \rangle \cong \mathbb{Z}_p$.
- There are $p^2 + p + 1$ p-subgroups of order p^2, which are $\langle a^p,\ b^p \rangle \cong \mathbb{Z}_p \times \mathbb{Z}_p$, $\langle a\ b^i \rangle \cong \mathbb{Z}_{p^2}$, $i = 1, 2, \cdots, p^2$, $\langle a^p\ b^i \rangle \cong \mathbb{Z}_{p^2}$, $i = 1, 2, \cdots, p-1$ and $\langle b \rangle \cong \mathbb{Z}_{p^2}$.
- There are $p + 1$ p-subgroups of order p^3, which are $\langle a\ b^i,\ [a, b] \rangle \cong \mathbb{Z}_{p^2} \times \mathbb{Z}_p$, $i = 1, 2, \cdots, p$ and $\langle a^p,\ b \rangle \cong \mathbb{Z}_{p^2} \times \mathbb{Z}_p$.
- There is only one p-subgroup of order p^4, which is G itself.

From above, the group G has $(p+1)+(p^2+p+1)+(p+1)+(1)+(1) = p^2+3p+5$ subgroups. □

Corollary 3.25. *The group $G = F_2$ has p^2 permutable not normal subgroups.*

Proof. Clearly, the p-subgroups of order p^3 are normal, since they are of index p. Also, the p-subgroups of order p are normal, because these subgroups generated by central elements a^p and b^p. The subgroups $K_i = \langle a^p\ b^i \rangle \cong \mathbb{Z}_{p^2}$, $i = 1, 2, \cdots, p-1$ and $K_p = \langle b \rangle \cong \mathbb{Z}_{p^2}$ are normal, because $G/N_G(K_i) = \{e\}$ for all $i = 1, 2, \cdots, p$, thus the subgroup K_i is the only conjugacy class subgroup of itself for all $i = 1, 2, \cdots, p$. Finally, Let $H_i = \langle ab^i \rangle$, $i = 1, 2, \cdots, p^2$. Then $|G/N_G(H_i)| = p$, therefore the number of the conjugacy class subgroups of H_i is p, which indicates that H_i is not normal subgroup for all $i = 1, 2, \cdots, p^2$. To show that H_i is permutable for all such i, we use Corollary 2.5, and so it is enough to show that $H_i H_j \leq G$ for $i, j = 1, 2, \cdots, p^2$ and $i \neq j$. Since $[b, a] = b^p$ (by the structure description of F_2), $(a\ b^i)(a\ b^j) = a^2\ b^{p+i+j (\mod p^2)}$. Therefore, $H_i\ H_j = H \leq G$ for all $i, j = 1, 2, \cdots, p^2$ and $i \neq j$. □

3.2.3 The Group F_3

Let $G = F_3$, where

$$F_3 = \left\langle a,\ b \mid a^{p^3} = b^p = e,\ [b,a] = a^{p^2} \right\rangle \cong \mathbb{Z}_{p^3} \rtimes \mathbb{Z}_p$$

Remark 3.26. *The group F_3 has $3p + 5$ subgroups.*

Proof. Let $G = F_3$. The group G is a p-group of order p^4. Theorem 2.1 asserts that the group G has p-subgroups of orders p , p^2 , p^3 and p^4 in addition to the trivial subgroup. Next, we give the number of subgroups of G for each order.

- There are $p+1$ p-subgroups of order p, which are $\left\langle a^{p^2}\ b^i \right\rangle \cong \mathbb{Z}_p$, $i = 1, 2, \cdots, p$ and $\langle b\rangle \cong \mathbb{Z}_p$.
- There are $p+1$ p-subgroups of order p^2, which are $\langle a^p\ b^i\rangle \cong \mathbb{Z}_{p^2}$, $i = 1, 2, \cdots, p$ and $\left\langle a^{p^2},\ b \right\rangle \cong \mathbb{Z}_p \times \mathbb{Z}_p$.
- There are $p+1$ p-subgroups of order p^3, which are $\langle a\ b^i\rangle \cong \mathbb{Z}_{p^3}$, $i = 1, 2, \cdots, p$ and $\langle a^p\ b\rangle \cong \mathbb{Z}_{p^2} \times \mathbb{Z}_p$.
- There is only one p-subgroup of order p^4, which is G itself.

From above, the group $G = F_3$ has $(p + 1) + (p + 1) + (p + 1) + 2 = 3p + 5$ subgroups. ☐

Lemma 3.27. *Let $G = F_3$. Then G is nilpotent of class 2.*

Proof. Let $G = F_3$. Then $o([a,b]) = o(a^{p^2}) = p$ which implies that G' is not empty. Then $\gamma_1 = [G,G] = G' \neq \{e\}$. Since $a^{-p^2} = a^p \in Z(G)$, $[a,b] \in Z(G)$. Then $\gamma_2 = [G', G] = \{e\}$. ☐

Corollary 3.28. *The group $G = F_3$ is not capable.*

Proof. The proof follows the same line of proof of Corollary 3.15. In fact, one can use `GAP` to find the epicenter of G, which is $Z^*(G) = \langle a^p, [a,b]\rangle \neq \{e\}$. ☐

Corollary 3.29. *Let $G = F_3$. Then $|G'| = p$.*

Proof. The group $G = F_3$ is non-abelian nilpotent group of class 2. Therefore, Lemma 2.3 implies that $G' = \langle [a,b]\rangle$ and so $|G'| = p$. ☐

Remark 3.30. *Let $G = F_3$. Then $|Z(G)| = p^2$.*

Proof. Let $G = F_3$. Then G is non-abelian, so $|Z(G)| \leq p^3$. Clearly, $[a, b] \in Z(G)$, which implies that $G' \subseteq Z(G)$. Therefore, $p \leq |Z(G)| \leq p^3$. By Lemma 2.2, we have $[a^p, g] = [a, g]^p = a^{-p^3} = e$ for all $g \in G$. Thus $a^{-p}g^{-1}a^pg = e$, which implies that a^p commute with all $g \in G$. Thus $a^p \in Z(G)$. This reduces $|Z(G)|$ to p^2 or p^3. The only two p-subgroups of G of order p^3 are $K_i = \langle a\ b^i \rangle$, $i = 1, 2, \cdots, p$ and $K_{i+1} = \langle a^p\ b \rangle$, and none of these subgroups is the center. Hence $|Z(G)| = p^2$. Moreover, using GAP, the center of G is $Z(G) = \langle a^p\ b^p \rangle = \langle a^p \rangle$. □

Corollary 3.31. *The group $G = F_3$ has p permutable not normal subgroups.*

Proof. As in Corollary 3.25. We find that the subgroups $H_i = \left\langle a^{p^2}\ b^i \right\rangle \cong \mathbb{Z}_p$, $i = 1, 2, \cdots, p$ are the only subgroups of G which are permutable not normal. Therefore, there are p such subgroups. □

3.2.4 The Group F_4

Let $G = F_4$, where

$$\begin{aligned} F_4 &= \langle a,\ b,\ c \mid a^p = b^p = c^p = d^p = [a, c] = [b, c] = [a, d] = [b, d] = [c, d] = e \rangle \\ &\cong \mathbb{Z}_p \times ((\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes \mathbb{Z}_p), \qquad \text{where } d = [b, a]. \end{aligned}$$

Remark 3.32. *The group F_4 has $2p^3 + 4p^2 + 3p + 5$ subgroups.*

Proof. Let $G = F_4$. The group G is a p-group of order p^4. By Theorem 2.1, G has p-subgroups of orders p , p^2 , p^3 and p^4 in addition to the trivial subgroup. Next, we count the subgroups of G for each order.

- There are p^3+p^2+p+1 p-subgroups of order p, which are $\left\langle a\ b^i\ c^j\ [a, b]^k \right\rangle \cong \mathbb{Z}_p$, $i, j, k = 1, 2, \cdots, p$, $\langle b\ c^i\ [a, b]^j \rangle \cong \mathbb{Z}_p$, $i, j = 1, 2, \cdots, p$, $\langle c\ [a, b]^i \rangle \cong \mathbb{Z}_p$, $i = 1, 2, \cdots, p$ and $\langle [a, b] \rangle \cong \mathbb{Z}_p$.
- There are p^3+2p^2+p+1 p-subgroups of order p^2, which are $\left\langle a\ b^i\ [a, b]^j,\ c\ [a, b]^k \right\rangle \cong \mathbb{Z}_p \times \mathbb{Z}_p$, $i, j, k = 1, 2, \cdots, p$, $\langle a\ b^i\ c^j,\ [a, b] \rangle \cong \mathbb{Z}_p \times \mathbb{Z}_p$, $i, j = 1, 2, \cdots, p$, $\langle b\ [a, b]^i,\ c\ [a, b]^j \rangle \cong \mathbb{Z}_p \times \mathbb{Z}_p$, $i, j = 1, 2, \cdots, p$, $\langle b\ c^i,\ [a, b] \rangle \cong \mathbb{Z}_p \times \mathbb{Z}_p$, $i = 1, 2, \cdots, p$ and $\langle c,\ [a, b] \rangle \cong \mathbb{Z}_p \times \mathbb{Z}_p$
- There are p^2+p+1 p-subgroups of order p^3, which are $\langle a\ c^i,\ [a, b],\ b\ c^j \rangle \cong (\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes \mathbb{Z}_p$, $i, j = 1, 2, \cdots, p$, $\langle a\ b^i, a\ b^i\ [a, b], a\ b^i\ c \rangle \cong \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$, $i = 1, 2, \cdots, p$ and $\langle b,\ c,\ [a, b] \rangle \cong \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$.

- There is only one p-subgroup of order p^4; namely, G itself.

From above, $G = F_4$ has $(p^3+p^2+p+1)+(p^3+2p^2+p+1)+(p^2+p+1)+1+1 = 2p^3 + 4p^2 + 3p + 5$ subgroups. □

Lemma 3.33. *Let $G = F_4$. Then G is nilpotent of class 2.*

Proof. Since G is a p-group, G is nilpotent. To show that G is of nilpotency class 2, we need to show that $\gamma_2(G) = [G', G] = \{e\}$. Clearly, $|G'| \geq o([a, b]) = p$ and $[a, b] \in Z(G)$. From Item 3.2.4 (Subgroups of order p in Remark 3.32), one observes that $G' = \langle [a, b] \rangle$. Therefore, $\gamma_2 = \{e\}$. □

Corollary 3.34. *The group $G = F_4$ is capable.*

Proof. The group G is a 2-generator p-group of nilpotency class 2, and the minimal generating set of G is $\{a, b, c\}$ with $o(a) = o(b) = o(c) = p$ ($\alpha = \beta = 1$). Then Corollary 2.10 asserts that G is capable group. In addition, `GAP` calculations indicate that, the epicenter of G is $Z^*(G) = \langle e \rangle$. □

Corollary 3.35. *Let $G = F_4$. Then $|G'| = p$.*

Proof. The group $G = F_4$ is a non-abelian group of nilpotency class 2, and $o([a, b]) = o([b, a]) = p$. Then, by Lemma 2.3, $G' = \langle [a, b] \rangle$ and so $|G'| = p$. □

Remark 3.36. *Let $G = F_4$. As shown above, the derived subgroup of G is of order p. Since $c \in Z(G)$ and $o(c) = p$ where $\langle c \rangle \cap G' = \{e\}$, $Z(G) = \langle [a, b], c \rangle$, and implies that the center of the group G is of order p^2.*

Corollary 3.37. *Let $G = F_4$. Then there are $2p^2 + 3p + 5$ normal subgroups of G.*

Proof. The following list indicates the normal subgroups of G. (See Remark 3.32)

1. The trivial subgroup and the group itself.
2. The subgroups of order p of the following structures:
   (a) $\langle c\ [a, b]^i \rangle \cong \mathbb{Z}_p$, $i = 1, 2, \cdots, p$.
   (b) $\langle [a, b] \rangle \cong \mathbb{Z}_p$.
3. The subgroups of order p^2 of the following structures:

(a) $\langle a\ b^i\ c^j,\ [a,b]\rangle \cong \mathbb{Z}_p \times \mathbb{Z}_p$, $i,j = 1,2,\cdots,p$.

(b) $\langle b\ c^i,\ [a,b]\rangle \cong \mathbb{Z}_p \times \mathbb{Z}_p$, $i = 1,2,\cdots,p$.

(c) $\langle c,\ [a,b]\rangle \cong \mathbb{Z}_p \times \mathbb{Z}_p$

4. All subgroups of order p^3.

Therefore, there are $(2)+(p+1)+(p^2+p+1)+(p^2+p+1) = 2p^2+3p+5$ normal subgroups of G. The other subgroups are not permutable. $\square$

3.2.5 The Group F_5

Let $G = F_5$, where

$$F_5 = \left\langle a,\ b,\ c \mid a^{p^2} = b^p = c^p = [a,c] = [b,c] = e,\ [b,a] = a^p \right\rangle \cong \mathbb{Z}_p \times (\mathbb{Z}_{p^2} \rtimes \mathbb{Z}_p)$$

Lemma 3.38. *Let $G = F_5$. Then G is nilpotent of class 2.*

Proof. The proof is similar to that of Lemma 3.33. $\square$

Corollary 3.39. *The group $G = F_5$ is not capable.*

Proof. Similar to the proof of Corollary 3.15. On the other hand, one can use `GAP` to find the epicenter of G, which is $Z^*(G) = \langle a^p\rangle$. $\square$

Remark 3.40. *Let $G = F_5$. Then $|G'| = p$.*

Proof. The proof is similar to that of Corollary 3.35. Moreover, `GAP` calculations state that $G' = \langle [a,b]\rangle \cong \mathbb{Z}_p$. $\square$

Remark 3.41. *Let $G = F_5$. Then $|Z(G)| = p^2$.*

Proof. The proof is similar to that of Remark 3.36. Moreover, `GAP` calculations state that $Z(G) = \langle a^p\ c,\ [a,b]\rangle \cong \mathbb{Z}_p \times \mathbb{Z}_p$. $\square$

Remark 3.42. *The group F_5 has $4p^2+3p+5$ subgroups.*

Proof. Let $G = F_5$. The group G is a p-group of order p^4. By Theorem 2.1, G has p-subgroups of orders p , p^2 , p^3 and p^4 in addition to the trivial subgroup. Next, we count the subgroups of G for each order.

- There are p^2+p+1 p-subgroups of order p, which are $\langle a^p\ b\ c^i\ [a,b]^j\rangle \cong \mathbb{Z}_p$, $i,j = 1,2,\cdots,p$, $\langle a^p\ c\,[a,b]^i\rangle \cong \mathbb{Z}_p$, $i = 1,2,\cdots,p$. and $\langle [a,b]\rangle \cong \mathbb{Z}_p$.

- There are $2p^2+p+1$ p-subgroups of order p^2, which are $\langle a\ b^i\ c^j,\ [a,b]\rangle \cong \mathbb{Z}_{p^2}$, $i,j = 1,2,\cdots,p$, $\langle a^p\ b\ [a,b]^i,\ c\ [a,b]^j\rangle \cong \mathbb{Z}_p \times \mathbb{Z}_p$, $i,j = 1,2,\cdots,p$, $\langle a^p\ b\ c^i,\ [a,b]\rangle \cong \mathbb{Z}_p \times \mathbb{Z}_p$, $i = 1,2,\cdots,p$ and $\langle a^p\ c,\ [a,b]\rangle \cong \mathbb{Z}_p \times \mathbb{Z}_p$.
- There are p^2+p+1 p-subgroups of order p^3, which are $\langle a\ b^i,\ c\ [a,b]\rangle \cong \mathbb{Z}_{p^2} \times \mathbb{Z}_p$, $i = 1,2,\cdots,p$, $\langle a\ c^i,\ bc^j\ [a,b]\rangle \cong \mathbb{Z}_{p^2} \rtimes \mathbb{Z}_p$, $i,j = 1,2,\cdots,p$ and $\langle b,\ c,\ [a,b]\rangle \cong \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$.
- There is only one p-subgroup of order p^4, which is G itself.

From above, the group $G = F_5$ has $(p^2+p+1) + (2p^2+p+1) + (p^2+p+1) + 1 + 1 = 4p^2+3p+5$ subgroups. □

Corollary 3.43. *The group $G = F_5$ has $2p^2$ permutable not normal subgroups.*

Proof. Using the subgroups list in Remark 3.42, it can be easily seen that the subgroups $H_{i,j} = \langle a^p\ b\ c^i\ [a,b]^j\rangle \cong \mathbb{Z}_p$, $i,j = 1,2,\cdots,p$ and the subgroups $K_{i,j} = \langle a^p\ b\ [a,b]^i,\ c\ [a,b]^j\rangle \cong \mathbb{Z}_p \times \mathbb{Z}_p$, $i,j = 1,2,\cdots,p$, are permutable not normal subgroups and so there are $2p^2$ such subgroups. □

3.2.6 The Group F_6

Let $G = F_6$, where

$$F_6 = \left\langle a,\ b,\ c \mid a^{p^2} = b^p = c^p = [a,b] = [a,c] = e,\ [b,c] = a^p \right\rangle \cong (\mathbb{Z}_{p^2} \times \mathbb{Z}_p) \rtimes \mathbb{Z}_p$$

Lemma 3.44. *Let $G = F_6$. Then G is nilpotent of class 2.*

Proof. The proof is similar to that of Lemma 3.33. □

Corollary 3.45. *The group $G = F_6$ is not capable.*

Proof. We follow the proof of Corollary 3.15 and use `GAP` to find the epicenter of G; that is $Z^*(G) = \langle a^p\rangle$. □

Remark 3.46. *Let $G = F_6$. Then $|G'| = p$.*

Proof. The proof is similar to that of Corollary 3.35. Moreover, $G' = \langle [b,c]\rangle \cong \mathbb{Z}_p$ □

Remark 3.47. *Let $G = F_6$. Then $|Z(G)| = p^2$.*

Proof. The proof is similar to that of Remark 3.36. Moreover, $Z(G) = \langle a\ b^p\ c^p \rangle = \langle a \rangle \cong \mathbb{Z}_{p^2}$. □

Remark 3.48. *The group F_6 has $3p^2 + 3p + 5$ subgroups.*

Proof. The subgroups of G for each order are:

- There are p^2+p+1 p-subgroups of order p, which are $\langle a^p\ b\ c^i\ [b,c]^j \rangle \cong \mathbb{Z}_p$, $i,j = 1,2,\cdots,p$, $\langle a^p\ c\ [b,c]^i \rangle \cong \mathbb{Z}_p$, $i = 1,2,\cdots,p$ and $\langle [b,c] \rangle \cong \mathbb{Z}_p$.
- There are $p^2 + p + 1$ p-subgroups of order p^2, which are $\langle a\ b^i\ c^j \rangle \cong \mathbb{Z}_{p^2}$, $i,j = 1,2,\cdots,p.$, $\langle a^p,\ b\ c^i \rangle \cong \mathbb{Z}_p \times \mathbb{Z}_p$, $i = 1,2,\cdots,p$ and $\langle a^p,\ c \rangle \cong \mathbb{Z}_p \times \mathbb{Z}_p$.
- There are $p^2 + p + 1$ p-subgroups of order p^3, which are $\langle a\ b^i,\ c \rangle \cong \mathbb{Z}_{p^2} \rtimes \mathbb{Z}_p$, $i = 1,2,\cdots,p-1$, $\langle a^p\ b\ c^i,\ a\ c^j \rangle \cong \mathbb{Z}_{p^2} \rtimes \mathbb{Z}_p$, $i = 1,2,\cdots,p$ and $j = 1,2,\cdots,p-1$;, $\langle a^p,\ b\ c^i \rangle \cong \mathbb{Z}_{p^2} \times \mathbb{Z}_p$, $i = 1,2,\cdots,p$, $\langle a,\ c^2 \rangle \cong \mathbb{Z}_{p^2} \times \mathbb{Z}_p$ and $\langle a^p\ c,\ b \rangle \cong (\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes \mathbb{Z}_p$.
- The only p-subgroup of order p^4 is G itself.

From above, the group $G = F_6$ has $(p^2+p+1)+(p^2+p+1)+(p^2+p+1)+1+1 = 3p^2 + 3p + 5$ subgroups. □

Corollary 3.49. *The group $G = F_6$ has $p^2 + p$ not permutable subgroups, and all of the other subgroups are normal.*

Proof. As $\{e\}$ and G are normal subgroups, it remains to determine the normality of non-trivial subgroups. Using the subgroups list obtained in Remark 3.48, one can note that the subgroups $H_{i,j} = \langle a^p\ b\ c^i\ [b,c]^j \rangle \cong \mathbb{Z}_p$, $i,j = 1,2,\cdots,p$ and $K_i = \langle a^p\ c\ [b,c]^i \rangle \cong \mathbb{Z}_p$, $i = 1,2,\cdots,p$ are not permutable subgroups because $H_{i,j}K_m$ is not a subgroup of G for any $i,j,m = 1,2,\cdots,p$. Therefore, they are no normal subgroups. On the other hand, any subgroup of order p^3 is of index p which implies that any subgroup of order p^3 is normal. Also, the subgroup $\langle a^p\ [b,c]^2 \rangle \cong \mathbb{Z}_p$ is the derived subgroup, and so it is normal. Finally, from the same list (Remark 3.48), any subgroup of order p^2 is characteristic in G, because each subgroup of order p^2 is conjugate to itself. Then these subgroups are normal. So the group G has $2p^2 + 2p + 5$ normal subgroups and $p^2 + p$ not permutable subgroups. □

3.2.7 The Group F_7

Let $G = F_7$ such that

$$\begin{aligned} F_7 &= \langle a,\ b \mid a^p = b^p = [a,b]^p = [a,[a,b]]^p = [b,[a,b]] = e,\ [a,[a,[a,b]]] = [b,[a,[a,b]]] = e \rangle \\ &\cong (\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p) \rtimes \mathbb{Z}_p \end{aligned}$$

Lemma 3.50. *The group $G = F_7$ is nilpotent of class* 3.

Proof. Let $G = F_7$. Using the structure description of G, we find that $o([a,b]) = p = o([a,[a,b]])$ and $[b,[a,b]] = e$. This implies that $|G'| \geq p^2$ and so $\gamma_2(G) = [G', G] \neq \{e\}$. Also, $[[b,a],a] \in Z(G)$ and $[a,[a,[a,b]]] = [b,[a,[a,b]]] = e$ which implies that $\gamma_3(G) = [\gamma_2(G), G] = \{e\}$. □

Remark 3.51. *The group F_7 for $p \geq 5$ has $p^3 + 3p^2 + 3p + 5$ subgroups.*

Proof. Let $G = F_7$. The group G is a p-group of order p^4. Theorem 2.1 asserts that the group G has p-subgroups of orders p , p^2 , p^3 and p^4 in addition to the trivial subgroup. Next, we count the subgroups of G for each order.

Let $c = [a,b]$ and $d = [a,c]$. The following is the classification of all subgroups of G.

- There are p^3+p^2+p+1 p-subgroups of order p, which are $\left\langle a\ b^i\ c^j\ d^k \right\rangle \cong \mathbb{Z}_p$, $i,j,k = 1,2,\cdots,p$, $\langle b\ c^i\ d^j \rangle \cong \mathbb{Z}_p$, $i,j = 1,2,\cdots,p$, $\langle c\ d^i \rangle \cong \mathbb{Z}_p$, $i = 1,2,\cdots,p$ and $\langle d \rangle \cong \mathbb{Z}_p$.

- There are $2p^2+p+1$ p-subgroups of order p^2, which are $\langle a\ b^i\ c^j,\ d \rangle \cong \mathbb{Z}_p \times \mathbb{Z}_p$, $i,j = 1,2,\cdots,p$, $\langle b\ d^i,\ c\ d^j \rangle \cong \mathbb{Z}_p \times \mathbb{Z}_p$, $i,j = 1,2,\cdots,p$, $\langle b\ c^i,\ d \rangle \cong \mathbb{Z}_p \times \mathbb{Z}_p$, $i = 1,2,\cdots,p$ and $\langle c,\ d \rangle \cong \mathbb{Z}_p \times \mathbb{Z}_p$.

- There are $p+1$ p-subgroups of order p^3, which are $\langle a\ b^i,\ d,\ c \rangle \cong (\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes \mathbb{Z}_p$, $i = 1,2,\cdots,p$ and $\langle b,\ d,\ c \rangle \cong \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$.

- There is only one p-subgroup of order p^4, which is G itself.

From above, the group $G = F_7$ has $(p^3 + p^2 + p + 1) + (2p^2 + p + 1) + (p + 1) + 1 + 1 = p^3 + 3p^2 + 3p + 5$ subgroups. □

Corollary 3.52. *The group $G = F_7$ has $p + 5$ normal subgroups.*

Proof. Let $G = F_7$. Using the subgroups list obtained in Remark 3.51, we find that G has only one normal subgroup of order p which is $Z(G) = \langle d \rangle$ in addition to one normal subgroup of order p^2 which is $G' = \langle c,\ d^2 \rangle$. All subgroups of order p^3 are normal subgroups. Clearly, the trivial subgroup and the group itself are normal subgroups. On the other hand, the other subgroups are not permutable because each subgroup has p conjugate subgroups. Therefore, for $p \geq 5$, the group G has $p^3 + 3p^2 + 2p$ not permutable subgroups and $p + 5$ permutable and normal subgroups. For $p = 3$, there are only 8 permutable and normal subgroups and the other 42 subgroups are not permutable. □

The following remarks can be obtained using `GAP` calculations on the group representation of $G = F_7$.

Remark 3.53. *The derived subgroup of G is $G' = \langle [a, b], [a, [a, b]] \rangle \cong \mathbb{Z}_p \times \mathbb{Z}_p$.*

Remark 3.54. *The center of G is $Z(G) = \langle [a, [a, b]] \rangle$.*

Remark 3.55. *Let $G = F_7$ and $p = 3$. Then G has 50 subgroups.*

Remark 3.56. *The epicenter of G is $Z^*(G) = \{e\}$. This indicates that the group G is capable.*

3.2.8 The Group F_8

Let $G = F_8$, where

$$F_8 = \left\langle a,\ b \mid a^{p^2} = b^p = [a, b]^p = [b, [a, b]] = e,\ [a, [a, b]] = a^p \right\rangle \cong (\mathbb{Z}_{p^2} \rtimes \mathbb{Z}_p) \rtimes \mathbb{Z}_p$$

Lemma 3.57. *The group $G = F_8$ is nilpotent of class 3.*

Proof. Let $G = F_8$. Then $|[a, b]| = |[a, [a, b]]| = p$ indicates that $\gamma_2(G) = [\gamma_1(G), G]$ is of order p, but $[a, [a, b]]g = g[a, [a, b]]$ for all $g \in G$, implies that $\gamma_3(G) = \{e\}$. Therefore, G is nilpotent of class 3. □

Remark 3.58. *Let $G = F_8$ and $p = 3$. Then G has 32 subgroups. Actually, 8 normal and 24 not permutable subgroups.*

Remark 3.59. *The group F_8 for $p \geq 5$ has $3p^2 + 3p + 5$ subgroups.*

Proof. Let $G = F_8$. The group G is a p-group of order p^4. Theorem 2.1 asserts that the group G has p-subgroups of orders p , p^2 , p^3 and p^4 in addition to the trivial subgroup. Next, we count the subgroups of G for each order.

Let $c = [a, b]$ and $d = [a, c]$. Then

- There are $p^2 + p + 1$ p-subgroups of order p, which are $\langle a^p\ b\ c^i\ d^j\rangle \cong \mathbb{Z}_p$, $i, j = 1, 2, \cdots, p$, $\langle a^p\ c\ d^i\rangle \cong \mathbb{Z}_p$, $i = 1, 2, \cdots, p$ and $\langle d\rangle \cong \mathbb{Z}_p$.
- There are $2p^2 + p + 1$ p-subgroups of order p^2, which are $\langle a^p\ b^i\ c^j\ d\rangle \cong \mathbb{Z}_{p^2}$, $i, j = 1, 2, \cdots, p$, $\langle a^p\ b\ d^i,\ c\ d^j\rangle \cong \mathbb{Z}_p \times \mathbb{Z}_p$, $i, j = 1, 2, \cdots, p$, $\langle a^p\ b\ c^i,\ d\rangle \cong \mathbb{Z}_p \times \mathbb{Z}_p$, $i = 1, 2, \cdots, p$ and $\langle a^p\ c,\ d\rangle \cong \mathbb{Z}_p \times \mathbb{Z}_p$.
- There are $p + 1$ p-subgroups of order p^3 which are $\langle a\ b^i\ d,\ c\rangle \cong \mathbb{Z}_{p^2} \rtimes \mathbb{Z}_p$, $i = 1, 2, \cdots, p$ and $\langle b,\ c,\ d\rangle \cong \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$.
- There is only one p-subgroup of order p^4 which is G itself.

From above, the group $G = F_8$ has $(p^2+p+1)+(2p^2+p+1)+(p+1)+1+1 = 3p^2 + 3p + 5$ subgroups. □

Corollary 3.60. *The group $G = F_8$ for $p \geq 5$ has $p^2 + p$ permutable not normal subgroups.*

Proof. Let $G = F_8$. Then the subgroups $H_i = \langle a^p\ c\ d^i\rangle \cong \mathbb{Z}_p$, $i = 1, 2, \cdots, p$ and $Z(G) = \langle d\rangle \cong \mathbb{Z}_p$, are permutable subgroups of order p, where $Z(G)$ is the only normal subgroup of this order. Also, the subgroups $K_{i,j} = \langle a^p\ b\ d^i,\ c\ d^j\rangle \cong \mathbb{Z}_p \times \mathbb{Z}_p$, $i, j = 1, 2, \cdots, p$ and $G' = \langle a^p\ c,\ d\rangle \cong \mathbb{Z}_p \times \mathbb{Z}_p$ are permutable subgroups of order p^2 in which G' is the only normal subgroup of this order. All subgroups of order p^3 are normal since these subgroups are of index p. Thus, the group G has $p + p^2$ permutable and not normal subgroups of order p (p^2), respectively. □

The following remarks can be obtained using `GAP` calculations on the group representation of $G = F_8$.

Remark 3.61. *The center of the group G is $Z(G) = \langle [a, [a, b]]\rangle \cong \mathbb{Z}_p$.*

Remark 3.62. *The epicenter of the group G is $Z^*(G) = Z(G)$. Then group G is not capable.*

Remark 3.63. *The derived subgroup of G is $G' = \langle [a, b],\ [a, [a, b]]\rangle \cong \mathbb{Z}_p \times \mathbb{Z}_p$.*

3.2.9 The Group F_9

Let $G = F_9$, where

$$F_9 = \left\langle a,\ b \mid a^{p^2} = b^p = [a,b]^p = [a,[a,b]] = e,\ [b,[a,b]] = a^p \right\rangle \cong (\mathbb{Z}_{p^2} \times \mathbb{Z}_p) \rtimes \mathbb{Z}_p$$

Lemma 3.64. *The group $G = F_9$ is nilpotent of class 3.*

Proof. The proof is similar to that of Lemma 3.57. □

Remark 3.65. *Let $G = F_9$ and $p = 3$. Then G has 50 subgroups. In fact, 8 of these subgroups are normal and 42 are not permutable subgroups.*

Lemma 3.66. *The group F_9 for $p \geq 5$ has $2p^2 + 3p + 5$ subgroups.*

Proof. Let $G = F_9$. The group G is a p-group of order p^4. Theorem 2.1 asserts that the group G has p-subgroups of orders p , p^2 , p^3 and p^4 in addition to the trivial subgroup. Next, we count the subgroups of G for each order.

Let $c = [b, a]$. Then

- There are $p^2 + p + 1$ p-subgroups of order p which are $\langle a^{ip}\ b\ c^j \rangle \cong \mathbb{Z}_p$, $i, j = 1, 2, \cdots, p$, $\langle a^p\ c^i \rangle \cong \mathbb{Z}_p$, $i = 1, 2, \cdots, p$ and $\langle c \rangle \cong \mathbb{Z}_p$.
- There are $p^2 + p + 1$ p-subgroups of order p^2 which are $\langle a\ b^i\ c^j \rangle \cong \mathbb{Z}_{p^2}$, $i, j = 1, 2, \cdots, p$, $\langle a^p,\ b^i\ c \rangle \cong \mathbb{Z}_p \times \mathbb{Z}_p$, $i = 1, 2, \cdots, p$ and
- $\langle a^p,\ b \rangle \cong \mathbb{Z}_p \times \mathbb{Z}_p$.
- There are $p + 1$ p-subgroups of order p^3, which are $\langle a^p\ b,\ c \rangle \cong (\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes \mathbb{Z}_p$, $\langle a,\ c \rangle \cong \mathbb{Z}_{p^2} \times \mathbb{Z}_p$ and $\langle a\ b^i,\ c \rangle \cong \mathbb{Z}_{p^2} \rtimes \mathbb{Z}_p$, $i = 1, 2, \cdots, p-1$.
- There is only one p-subgroup of order p^4, which is G itself.

From above, the group $G = F_9$ has $(p^2+p+1)+(p^2+p+1)+(p+1)+1+1 = 2p^2 + 3p + 5$ subgroups. □

Corollary 3.67. *The group $G = F_9$ has $p + 5$ normal subgroups.*

Proof. Let $G = F_9$. Then the subgroups $Z(G) = \langle a^p,\ c^p \rangle = \langle a^p \rangle \cong \mathbb{Z}_p$ and $G' = \langle a^p,\ b^p\ c \rangle = \langle a^p\ c \rangle \cong \mathbb{Z}_p \times \mathbb{Z}_p$ are the unique normal subgroups of G of order p and p^2, respectively. On the other hand, any subgroup of order p^3 is of index p and so these subgroups are normal. Therefore, the normal subgroups of G are: the trivial group, the group itself, the center, the derived subgroup and the subgroups of order p^3. □

The following remarks can be obtained using `GAP` calculations on the group representation of $G = F_9$.

Remark 3.68. *The derived subgroup of G is $G' = \langle a^p, [a,b] \rangle \cong \mathbb{Z}_p \times \mathbb{Z}_p$.*

Remark 3.69. *The center of the group G is $Z(G) = \langle a^p \rangle \cong \mathbb{Z}_p$.*

Remark 3.70. *The epicenter of the group G is $Z^*(G) = \langle a^p \rangle$. Then group G is not capable.*

3.2.10 The Group F_{10}

Let $G = F_{10}$, where

$$F_{10} = \left\langle a,\ b \mid a^{p^2} = b^p = [a,b]^p = [a,[a,b]] = e,\ [b,[a,b]] = a^{2p} \right\rangle \cong (\mathbb{Z}_{p^2} \times \mathbb{Z}_p) \rtimes \mathbb{Z}_p$$

Lemma 3.71. *The group $G = F_{10}$ is nilpotent of class 3.*

Proof. Similar to the proof of Lemma 3.57. □

Remark 3.72. *Let $G = F_{10}$ and $p = 3$. Then G has 23 subgroups and 11 of these subgroups are permutable.*

Remark 3.73. *The group F_{10} for $p \geq 5$ has $2p^2 + 3p + 5$ subgroups.*

Proof. Let $G = F_{10}$. The group G is a p-group of order p^4. Theorem 2.1 asserts that the group G has p-subgroups of orders p , p^2 , p^3 and p^4 in addition to the trivial subgroup. Next, we count the subgroups of G for each order.

Let $c = [a,b]$. Then

- There are $p^2 + p + 1$ p-subgroups of order p, which are $\langle a^{ip}\ b\ c^j \rangle \cong \mathbb{Z}_p$, $i, j = 1, 2, \cdots, p$, $\langle a^p\ c^i \rangle \cong \mathbb{Z}_p$, $i = 1, 2, \cdots, p$ and $\langle c \rangle \cong \mathbb{Z}_p$.

- There are $p^2 + p + 1$ p-subgroups of order p^2, which are $\langle a\ b^i\ c^j \rangle \cong \mathbb{Z}_{p^2}$, $i, j = 1, 2, \cdots, p$, $\langle a^p,\ b^i\ c \rangle \cong \mathbb{Z}_p \times \mathbb{Z}_p$, $i = 1, 2, \cdots, p$ and $\langle a^p,\ b \rangle \cong \mathbb{Z}_p \times \mathbb{Z}_p$.

- There are $p + 1$ p-subgroups of order p^3, which are $\langle a^p\ b,\ c \rangle \cong (\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes \mathbb{Z}_p$, $\langle a,\ c \rangle \cong \mathbb{Z}_{p^2} \times \mathbb{Z}_p$ and $\langle a\ b^i,\ c \rangle \cong \mathbb{Z}_{p^2} \rtimes \mathbb{Z}_p$, $i = 1, 2, \cdots, p-1$.

- There is only one p-subgroup of order p^4 which is G itself.

From above, the group $G = F_{10}$ has $(p^2+p+1)+(p^2+p+1)+(p+1)+1+1 = 2p^2+3p+5$ subgroups. □

Corollary 3.74. *The group $G = F_{10}$ has $p+5$ normal subgroups.*

Proof. The proof is similar to that of Corollary 3.67. □

The following remarks can be obtained using `GAP` calculations on the group representation of $G = F_{10}$.

Remark 3.75. *The derived subgroup of G is $G' = \langle a^p, [a,b]\rangle \cong \mathbb{Z}_p \times \mathbb{Z}_p$.*

Remark 3.76. *The centre of the group G is $Z(G) = \langle a^p\rangle \cong \mathbb{Z}_p$.*

Remark 3.77. *The epicentre of the group G is $Z^*(G) = \langle a^p\rangle$. Implies that, G is not capable.*

The last remark indicates that the converse of Theorem 2.12 is not always true. Since, if $G = F_{10}$, we have $|Z(G)| = p$ and $|G'| = p^2$. So $|G : Z(G)| = p^3 \leq |G'|^2$. However, G is not capable group. Therefore, the condition $|G : Z(G)| \leq |G'|^n$ for a natural number n is not a sufficient condition for the capability of the group G.

4 Conclusions

In this article, the structure description of any p-group of order p^4 has been given in terms of the group generators and relations. Using the obtained classifications, we provided some algebraic properties and calculations regarding these groups. Also, we noticed the following remarks:
The number of subgroups of any p-group of order 5^4 is a multiple of 5. Moreover, the number of subgroups of any p-group of order p^4 is congruent 5(mod p).
Although the groups concluded in this article are of nilpotency class 1,2 or 3, the results conducted show that, for a group G of order p^4 with a minimal generating set $\{a_1,\ a_2,\ \cdots,\ a_n\}$, such that $o(a_i) = p^{n_i}$, $n_i \geq n_{i+1}$, and $n_1 = n_2$ seem to be sufficient for G to be a capable group; this conjecture can be configured as an extension of Theorem 4.4 in [2].
The results obtained in this article, asserted that the converse of Theorem 2.12 [11] is not always true; i.e., the existence of an upper bound on the index of the centre of a finite group G in terms of the order of its derived subgroup is not sufficient for G to be a capable group.

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