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On the Diophantine Equation $p^x + 7^y = z^2$, where p is prime and x, y, z are non-negative integers

Apisit Pakapongpun, Bunthita Chattae

Department of Mathematics
Faculty of Science
Burapha University
Chonburi 20131, Thailand

email: bunthita@buu.ac.th

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Abstract

In this paper, we examine the solutions (if any) of the Diophantine equation $p^x + 7^y = z^2$, where p is prime and x, y, z are non-negative integers.

1 Introduction

In 1844, Catalan [1] conjectured that $(3, 2, 2, 3)$ is the unique solution (a, b, x, y) for the Diophantine equation $a^x - b^y = 1$, where a, b, x and y are integers such that $min\{a, b, x, y\} > 1$. Later in 2004, the Catalan's conjecture was proved by Mihailescu [2].

The Diophantine equation of the form $a^x + b^y = c^2$ has been studied by many researchers. In 2011, Suvarnamani [3] studied the Diophantine equation $2^x + p^y = z^2$, where p is a prime number and x, y and z are non-negative integers. In 2012, Tatong and Suvarnamani [4] studied the Diophantine equation $p^x + p^y = z^2$, where p is a prime number and x, y and z are non-negative integers. In 2013, Sroysang [5] showed that the Diophantine

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equation $5^x + 7^y = z^2$, where x, y and z are non-negative integers, has no non-negative integer solution. Later in 2014, Sroysang [6] also showed that the Diophantine equation $7^x + 31^y = z^2$, where x, y and z are non-negative integers, has no non-negative integer solutions.

In this paper, inspired by [3, 4, 5] and [6], we study the Diophantine equation $p^x + 7^y = z^2$, where p is a prime number satisfying some conditions and x, y and z are non-negative integers.

2 Preliminaries

Lemma 2.1. *[7] The Diophantine equation $p^x + 1 = z^2$ has no positive integer solution for any prime $P > 3$.*

Lemma 2.2. *If $5^x \equiv 5 \pmod 6$, then x is odd.*

Proof. Suppose that x is even. Since $5 \equiv -1 \pmod 6$, $5^x \equiv 1 \pmod 6$. This is a contradiction. □

Lemma 2.3. *If $a \equiv 3 \pmod 7$ and x is odd, then $a^x \equiv 3 \pmod 7$ or $a^x \equiv 5 \pmod 7$ or $a^x \equiv 6 \pmod 7$.*

Proof. Since x is odd, x can be written as either $x = 6n - 1$ or $x = 6n - 3$ or $x = 6n - 5$, where $n \in \mathbb{N}$. Since $p \equiv 3 \pmod 7$, $p^{6n-6} = p^{6(n-1)} \equiv 1 \pmod 7$.
If $x = 6n - 1$, then $p^x = p^{6n-1} \equiv p^5 \equiv 3^5 \equiv 5 \pmod 7$.
If $x = 6n - 3$, then $p^x = p^{6n-3} \equiv p^3 \equiv 3^3 \equiv 6 \pmod 7$.
If $x = 6n - 5$, then $p^x = p^{6n-5} \equiv p \equiv 3 \pmod 7$. □

Lemma 2.4. *[3] The Diophantine equation $p^x + 7^y = z^2$, where p is a prime number and $p \equiv 2 \pmod 6$, has one solution $(x, y, z) = (3, 0, 3)$ in non-negative integers.*

3 Results

Throughout this paper, x, y and z are non-negative integers and p is a prime number. We explore the solutions of the Diophantine equation $p^x + 7^y = z^2$, where p is prime, in different cases of p.

Theorem 3.1. *The Diophantine equation $p^x + 7^y = z^2$, where p is a prime number and $p \equiv 1 \pmod 6$, has no non-negative integer solution.*

Proof. Case 1: x or y is zero. Then, by Lemma 2.1, the Diophantine equation $p^x + 7^y = z^2$ has no positive integer solution. Hence, there is no non-negative integer solution. Case 2: $x \geq 1$ and $y \geq 1$. Then z is even. Thus $z^2 \equiv 0 \pmod 6$ or $z^2 \equiv 4 \pmod 6$. Since $p \equiv 1 \pmod 6$, $p^x \equiv 1 \pmod 6$ and $7^y \equiv 1 \pmod 6$. Hence $z^2 \equiv 2 \pmod 6$, which is a contradiction. □

Theorem 3.2. *The Diophantine equation $p^x + 7^y = z^2$, where p is a prime number and $p \equiv 3 \pmod 6$, has two solutions $(x, y, z) = (1, 0, 2)$ and $(x, y, z) = (2, 1, 4)$ in non-negative integers.*

Proof. If $p \equiv 3 \pmod 6$, then the equation becomes $3^x + 7^y = z^2$.
Case 1: $x = 0$. We have $1 + 7^y = z^2$.
If $y = 0$, then $z^2 = 2$, which is impossible.
If $y = 1$, then $z^2 = 8$, which is impossible.
If $y > 1$, then the Diophantine equation $z^2 - 7^y = 1$ has no non-negative solution by the Catalan's conjecture.
Case 2: $y = 0$. We have $3^x + 1 = z^2$
If $x = 0$, then $z^2 = 2$, which is impossible.
If $x = 1$, then $z^2 = 4$ and so $z = 2$.
If $x > 1$, then the Diophantine equation $z^2 - 3^x = 1$ has no non-negative solution by the Catalan's conjecture.
Case 3: $x \geq 1$ and $y \geq 1$.
Subcase 3.1: x and y are odd. Then $3^x + 7^y \equiv (-1)^x + (-1)^y \equiv 2 \pmod 4$ but $z^2 \equiv 0, 1 \pmod 4$. A contradiction.
Subcase 3.2: x is a positive integer and y is even. Then $y = 2s$, for some positive integer $s \geq 1$. Then $3^x + 7^{2s} = z^2$ and so $3^x = z^2 - 7^{2s} = (z - 7^s)(z + 7^s)$. Thus, there exist non-negative integer a, b such that $3^a = z - 7^2$ and $3^b = z + 7^s$, where $a < b$ and $x = a + b$. Therefore, $2 \cdot 7^s = 3^a(3^{b-a} - 1)$. This implies that $a = 0$ and so $2 \cdot 7^s = 3^b - 1 = 2 \cdot (3^{b-1} + 3^{b-2} + \cdots + 3 + 1)$. Thus $7^s = 3^{b-1} + 3^{b-2} + \cdots + 3 + 1$, which is impossible.
Subcase 3.3: y is a positive integer and x is even. Then $x = 2k$ for some positive integer $k \geq 1$. Then $3^{2k} + 7^y = z^2$ and so $7^y = z^2 - 3^{2k} = (z - 3^k)(z + 3^k)$. Thus there exist non-negative integers a, b such that $7^a = z - 3^k$ and $7^b = z + 3^k$, where $a < b$ and $y = a + b$. Therefore, $2 \cdot 3^k = 7^a(7^{b-a} - 1)$. This implies that $a = 0$ and so $2 \cdot 3^k = 7^b - 1 = 6 \cdot (7^{b-1} + 7^{b-2} + \cdots + 7 + 1)$. Then $3^{k-1} = 7^{b-1} + 7^{b-2} + \cdots + 7 + 1$. So $k = 1, b = 1$ and consequently $(x, y, z) = (2, 1, 4)$. □

Theorem 3.3. *The Diophantine equation $p^x + 7^y = z^2$,, where p is a prime number that $p \equiv 5 \pmod 6$ and $p \equiv 3 \pmod 7$, has no solution in non-negative integers.*

Proof. Suppose $p \equiv 3 \pmod 7$.
Case 1: x or y is zero. Then, by Lemma 2.1, the Diophantine equation $p^x + 7^y = z^2$ has no positive integer solution. Hence, there is no non-negative integer solution.
Case 2: $x \geq 1$ and $y \geq 1$. Then z is even and so $z^2 \equiv 0 \pmod 6$ or $z^2 \equiv 4 \pmod 6$. Since $7^y \equiv 1 \pmod 6$, $p^x = z^2 - 7^y \equiv 0 - 1 \equiv 5 \pmod 6$ or $p^x = z^2 - 7^y \equiv 4 - 1 \equiv 3 \pmod 6$. But $p \equiv 5 \pmod 6$ so $p^x \equiv 1 \pmod 6$ or $p^x \equiv 5 \pmod 6$. It follows that $p^x \equiv 5 \pmod 6$. By Lemma 2.2, x is odd. Since $p \equiv 3 \pmod 7$, $p^x \equiv 3 \pmod 7$ or $p^x \equiv 5 \pmod 7$ or $p^x \equiv 6 \pmod 7$ by Lemma 2.3. Therefore, $z^2 = p^x + 7^y \equiv 3 \pmod 7$ or $z^2 \equiv 5 \pmod 7$ or $z^2 \equiv 6 \pmod 7$. This is a contradiction since z is even and so $z^2 \equiv 0, 1, 2, 4 \pmod 7$. □

Theorem 3.4. *The Diophantine equation $p^x + 7^y = z^2$, where p is a prime number such that $p \equiv 5 \pmod 6$ and $p \equiv 5 \pmod 7$, has no solution in non-negative integers.*

Proof. The proof is similar to that of Theorem 3.3. □

Theorem 3.5. *The Diophantine equation $p^x + 7^y = z^2$, where p is a prime number that $p \equiv 5 \pmod 6$ and $p \equiv 6 \pmod 7$, has no solution in non-negative integers.*

Proof. The proof is similar to that Theorem 3.3. □

Remark. For $p < 500$, there are some non-negative integer solutions as follows:

1. For $p \equiv 5 \pmod 6$ and $p \equiv 1 \pmod 7$, the non-negative integer solution (p, x, y, z) is $(29, 1, 1, 6($.

2. For $p \equiv 5 \pmod 6$ and $p \equiv 2 \pmod 7$, the non-negative integer solutions (p, x, y, z) are $(233, 1, 3, 24)$ and $(317, 1, 1, 18)$.

3. For $p \equiv 5 \pmod 6$ and $p \equiv 4 \pmod 7$, the non-negative integer solution (p, x, y, z) is $(137, 1, 1, 12)$.

4 Conclusions

We found that the Diophantine equation $p^x + 7^y = z^2$ has no non-negative integer solution in the 4 following cases:
1) when $p \equiv 1 \pmod 6$ (Theorem 3.1),
2) when $p \equiv 5 \pmod 6$ and $p \equiv 3 \pmod 7$ (Theorem 3.3),
3) when $p \equiv 5 \pmod 6$ and $p \equiv 5 \pmod 7$ (Theorem 3.4), and
4) when $p \equiv 5 \pmod 6$ and $p \equiv 6 \pmod 7$ (Theorem 3.5).
In the case $p \equiv 2 \pmod 6$, the equation has exactly one solution $(x, y, z) = (3, 0, 3)$ in non-negative integer (Lemma 2.4) and and in the case $p \equiv 3 \pmod 6$ the equation has two non-negative integer solutions $(1, 0, 2)$ and $(2, 1, 4)$ (Theorem 3.2). In addition, the equation also has solutions in the 3 following cases:
i) when $p \equiv 5 \pmod 6$ and $p \equiv 1 \pmod 7$,
ii) when $p \equiv 5 \pmod 6$ and $p \equiv 2 \pmod 7$, and
iii) when $p \equiv 5 \pmod 6$ and $p \equiv 5 \pmod 7$.

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