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Normal Cover of Finite Groups

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Abstract

For a finite group G, the minimal collection of proper subgroups of G whose union is G, is called the cover of G and its order is called the covering number of G. In this article we present a special cover of G which is a collection of proper normal subgroups. This cover is called the normal cover of G, the size of the minimal collection of such cover is called the normal covering number of G and denoted by $\sigma_N(G)$. In addition, we will configure σ_N for certain finite groups, and discuss the properties and the structure description of some groups based on its normal covering number.

1 Introduction

A cover of a group G is the minimal collection of its subgroups whose union is the whole group G. So, if G is a group and $\Sigma = \{H_i \mid H_i < G \ , \ 1 \leq i \leq n\}$ for which $\bigcup_{i=1}^{n} H_i = G$ and subtracting any subgroup from Σ implies that

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$\bigcup_{i=1}^{n} H_i \neq G$, then Σ is a cover of G. The size of minimal covering of a group G is called the covering number of G and denoted by $\sigma(G)$ [2]. According to Neumann [6], a group has a finite covering if and only if it has a finite noncyclic homomorphic image. Therefore, we intend to find the covering number of finite groups if it exists.

It is well known that $\sigma(G) \geq 3$ for any finite group G. The case $\sigma(G) = 3$ holds if and only if G has a homomorphic image to the Klein-Four group, which has been shown in [7]. The groups of covering number 4, and the groups of covering number 5 are characterized in [3] and [4]. In [5], Seyyed obtained $\sigma(G)$ for some affine primitive groups. Moreover, Sun in [8], provided a group cover by left cosets of some subgroups, and showed some properties of such cover when these left cosets are produced by the identity elements and subnormal subgroups.

The current study is devoted to a certain collection of proper subgroups of a group G, whose union is G. This collection contains only proper normal subgroups, and the minimal size of such covering is called the normal covering number of G and denoted by $\sigma_N(G)$. More precisely, we restrict our investigation on certain finite groups to characterize their normal covering numbers.

2 Normal covering of certain finite groups

The ordinary cover of a finite group G is the minimal collection of proper subgroups of G for which their union is G, and the size of such collection is called the covering number of G and denoted by $\sigma(G)$. This covering will be reduced using a selective collection of subgroups as given in the next definition.

Definition 2.1. *Let G be a finite group. The normal cover of G is the minimal collection of proper normal subgroups of G whose union is G. The size of such collection is called the normal covering number of G and denoted by $\sigma_N(G)$. If a group G has no such cover (i.e. G can not be written as a union of proper normal subgroups), then we set $\sigma_N(G) = \infty$.*

Remark 2.2. *Consider a finite group G for which $\sigma(G) = k$, then $\sigma_N(G) \geq k$.*

Proof. Let G be a finite group and $\sigma(G) = k$. Then, G can be written as a union of proper subgroups, and the minimal size of such collection is k. If each subgroup in this collection is normal, then this collection is a normal covering of G. So, $\sigma_N(G) = k$. There are two cases.

Case 1: There is a collection of subgroups such that replacing this collection by normal subgroups, implies $\sigma_N(G) \geq k$

Case 2: There is no such collection for G and hence $\sigma_N(G)$ is unbounded.

□

The main questions here are: which finite group has a normal cover? and if it has, what is the size of such covering?

In the following, we intend to answer the previous questions.

Remark 2.3. *As every proper subgroup of a group $G = \langle x \rangle$ dose not contain x. Then, every cyclic group has no normal cover. Indeed, any cyclic group has no ordinary covering.*

It is clear that, every subgroup of an abelian group is normal. Thus, the proof of the next corollary is obvious and can be omitted.

Corollary 2.4. *If G is a finite abelian noncyclic group, then $\sigma(G) = \sigma_N(G)$.*

The converse is not always true. That is, the condition $\sigma(G) = \sigma_N(G)$ for a finite group does not imply that G is abelian. As an example, the dihedral group $G = D_{12}$ is none abelian group even though $\sigma(G) = \sigma_N(G) = 3$.

Recall that the dihedral group $G = D_{2n}$ can be written as

$$G = \left\langle r, s \mid r^n = s^2 = (r^k s)^2, \text{ for } k = 1, 2, \cdots, n \right\rangle$$

There are n rotations r^k , $k = 1, 2, \ldots, n$ and n reflections $r^k s$, $k = 1, 2, \ldots, n$. Using the previous structure description of D_{2n} one can obtains the next theorem.

Theorem 2.5. *Let $G = D_{2n}$ be the dihedral group of order $2n$, $n \geq 3$. Then the following holds:*

1. *$\sigma(G) = \sigma_N(G) = 3$, for n is even.*

2. *$\sigma(G) = n + 1$ and $\sigma_N(G) = \infty$, for $n \geq 3$ is prime*

3. *$\sigma(G) = k + 1$ and $\sigma_N(G) = \infty$, for $n = km$ where $k > 2$ is prime and m is an odd integer.*

Proof. Let $G = D_{2n}$, $n \geq 3$. Then, $K = \langle r \rangle$ is a subgroup of index 2. Let $M = \{s, rs, r^2s, \ldots, r^{n-1}s\}$ be the set of all reflections. The covering number of G can be computed as follows:

1. If n is even, then M contains an even number of elements. Set $H_1 = \langle r^2, rs \rangle$ and $H_2 = \langle r^2, r^2s \rangle$. Each of these subgroups is normal of order n and $H_1 \cap H_2 = \{e, r^2, r^4\}$ which is not H_1 or H_2. Moreover, $K \cup H_1 \cup H_2 = G$. This implies that $\delta(G) = \sigma_N(G) = 3$.

2. If n is an odd prime, then the only proper subgroup of G that contains $x \in M$ is $H_i = \{e, x\}$. Therefore, there are n such subgroups, for which $\bigcap_{i=1}^{n} H_i = \{e\}$ and $\bigcup_{i=1}^{n} H_i = M$. This implies that $\delta(G) = n + 1$. Moreover, the only proper none trivial normal subgroup of G is K which is not a covering G. Hence, $\sigma_N(G) = \infty$.

3. If $n = km$ for a prime $k > 2$ and an odd integer m where $k \leq m$, then the subgroups $H_i = \left\langle r^k, r^i s \right\rangle$, $i = 1, 2, \cdots, k$ are covering M each of these subgroups is of order $2m$. In addition, $\bigcap_{i=1}^{k} H_i = \left\langle r^k \right\rangle$. Therefore, K and $\{H_i \mid i = 1, \ldots, k\}$ are covers of G. Hence, $\sigma(G) = k + 1$. Moreover, The proper none trivial normal subgroups of G are subsets of K. Clearly, these subgroups are not the cover of G. So, $\sigma_N(G) = \infty$.

□

Corollary 2.6. *Let $G = D_{2n}$ be the dihedral group of order $2n$ for n is even and Σ be a normal cover of G. Then $G/H \cong Z(G)$ for all $H \in \Sigma$.*

Proof. Let $G = D_{2n}$ be the dihedral group of order $2n$ for n is even. Then G has a normal cover Σ of minimal size 3 (Theorem 2.5). In fact, $Z(G) = \{e, r^{n/2}\} \cong \mathbb{Z}_2$. Moreover, $[G, H] = 2 = |G/H|$ for all $H \in \Sigma$. □

Recall that, the generalized quaternion group GQ_{2n} is a non-abelian 2-generator group of order $2n = 2(2^i)$ for a positive integer i , $i \geq 3$. The group presentation in terms of rules and generators can be given by:

$$GQ_{2n} = \left\langle r, s \mid r^n = s^4 = e,\ r^{n/2} = s^2,\ rs = sr \right\rangle$$

Theorem 2.7. *Let $G = GQ_{2n}$ be the generalized quaternion group of order $2n = 2(2^i)$ for a positive integer i , $i \geq 3$. Then $\sigma_N(G) = 3$.*

Proof. Let $G = GQ_{2n}$ be the generalized quaternion group of order $2n = 2(2^i)$ for a positive integer i , $i \geq 3$. Then G has $i + 4$ normal subgroups. Select the subgroups $H_1 = \langle r \rangle$, $H_2 = \langle r^{-2}, s \rangle$ and $H_3 = \langle r^{-2}, sr^{-1} \rangle$, each of these subgroups is of index 2. Moreover, $\bigcap_{i=1}^{3} H_i = \langle r \rangle$ and $\bigcup_{i=1}^{3} H_i = G$. Hence, $\Sigma = \{H_1, H_2, H_3\}$ is a normal cover of G of minimal size. □

Recall that, the quasidihedral group QD_{2n} is a non-abelian 2-generator group of order $2n = 2(2^i)$ for a positive integer i , $i \geq 3$. The group presentation in terms of rules and generators can be given by:

$$QD_{2n} = \left\langle r, s \mid r^n = s^2 = e,\ r^{n/2} = s^2,\ rs = sr \right\rangle$$

Theorem 2.8. *Let $G = QD_{2n}$ be the quasidihedral group of order $2n = 2(2^i)$ for a positive integer i , $i \geq 3$. Then $\sigma_N(G) = 3$.*

Proof. Let $G = QD_{2n}$ be the quasidihedral group of order $2n = 2(2^i)$ for a positive integer i , $i \geq 3$. Then G has $i + 4$ normal subgroups. Set $H_1 = \langle r \rangle$, $H_2 = \langle r^{-2}, s \rangle$ and $H_3 = \langle r^{-2}, sr^{-1} \rangle$, each of these subgroups is of index 2. Moreover, $\bigcap_{i=1}^{3} H_i = \langle r \rangle$ and $\bigcup_{i=1}^{3} H_i = G$. Hence, $\Sigma = \{H_1, H_2, H_3\}$ is a normal cover of G of minimal size. □

Remark 2.9. *Let G be the group GQ_{2n} or the group QD_{2n} each of order $2n = 2(2^i)$ for a positive integer i , $i \geq 3$. Let Σ be the normal cover of G as assumed in Theorem 2.7 and Theorem 2.8. Then $G/H \cong \mathbb{Z}_2$, for all $H \in \Sigma$ and so $G/H \cong Z(G) = \left\langle r^{n/2} \right\rangle$.*

Recall that, the nondihedral group ND_{2n} is a non-abelian 2-generator group of order $2n = 2(2^i)$ for a positive integer i, $i \geq 4$. The group presentation in terms of rules and generators can be given by:

$$ND_{2n} = \left\langle r, s \mid r^n = s^2 = e,\ sr = r^{n+1} \right\rangle \cong \mathbb{Z}_n \rtimes \mathbb{Z}_2$$

Theorem 2.10. *Let $G = ND_{2n}$ be the nondihedral group of order $2n = 2(2^i)$ for a positive integer i , $i \geq 4$. Then $\sigma_N(G) = 3$.*

Proof. Let $G = ND_{2n}$ be the nondihedral group of order $2n = 2(2^i)$ for a positive integer i , $i \geq 4$. Then G has $3(i - 1)$ normal subgroups. Set $H_1 = \langle r \rangle$, $H_2 = \langle r^{-2}, s \rangle$ and $H_3 = \langle r^{-2}, sr^{-1} \rangle$, each of these subgroups is of index 2. Moreover, $\bigcap_{i=1}^{3} H_i = \langle r^2 \rangle$ and $\bigcup_{i=1}^{3} H_i = G$. Hence, $\Sigma = \{H_1, H_2, H_3\}$ is a normal cover of G of minimal size. □

Since each subgroup in the normal covering of $G = ND_{2n}$ is of index 2. Then, the next result is a direct consequence of Theorem 2.10.

Corollary 2.11. *Let $G = ND_{2n}$ be the nondihedral group of order $2n = 2(2^i)$ for a positive integer i , $i \geq 4$ and $\Sigma = \{H_1, H_2, H_3\}$ be the normal cover of G. Then $G/H_i \cong \langle s \rangle$ for $i = 1, 2, 3$.*

Lemma 2.12. *The group $G = \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$ (p prime) has a normal cover. Moreover, $\sigma_N(G) = p + 1$.*

Proof. Let $G = \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p \cong \langle a, b, c \mid a^p = b^p = c^p = [a, b] = [a, c] = [b, c] = e \rangle$. Then, the group G has $2p^2 + 2p + 4$ subgroups (Remark 3.6 [1]), each is normal. The trivial group, $p^2 + p + 1$ subgroups of order p, $p^2 + p + 1$ subgroups of order p^2 and the improper subgroup G of order p^3. The subgroups of order p^2 are:

- $\langle ac^i, bc^j \rangle$, $i, j = 1, 2, \ldots, p$,
- $H_i = \langle ab^i, c \rangle$, $i = 1, 2, \ldots, p$ and
- $H_{p+1} = \langle b, c \rangle$.

Choosing the subgroups H_i , $i = 1, 2, \ldots, p + 1$. The intersection of these subgroups is $\langle c \rangle$ where $|\langle c \rangle| = p$. Therefore,

$$\begin{aligned} \left| \bigcup_{i=1}^{p+1} H_i \right| &= \sum_{i=1}^{p+1} |H_i| - (p+1) \left| \bigcap_{i=1}^{p+1} H_i \right| + |\langle c \rangle| \\ &= (p+1)p^2 - (p+1)p + p \\ &= p^3 + p^2 - p^2 - p + p \\ &= p^3 \\ &= |G| \end{aligned}$$

This shows that these subgroups form a cover of G. Thus $\sigma_N(G) \leq p+1$. In fact, any other cover of G is at least of size p^2. Hence, $\sigma_N(G) = p + 1$. □

Let G be a p-group of order p^n for a prime p. Then the number of normal subgroups of G of order p^{n-1} is one of the following: $p+1$, p^2+p+1, p^3+p^2+p+1, $\ldots, p^{n-1}+p^{n-2}+\cdots+p+1$. The number of such subgroups depends on the number of the group generators m, if these groups are noncyclic, then $m \geq 2$. Therefore, we can give $N(G)$ the number of normal subgroups of order p^{n-1} of any noncyclic p-group G of order p^n of m-generators as follows: $N(G) = p^{m-1} + p^{m-2} + \cdots + 1$. As a consequence, one can prove the next theorem.

Theorem 2.13. *Every noncyclic p-group G has a normal cover and* $\sigma_N(G) = p+1$.

Proof. Every noncyclic p-group G of order p^n has at least two generators. So, the number of the normal subgroups of G of order p^{n-1} is at least $N(G) = p^{2-1}+1 = p+1$. Set Σ to be the collection of all such subgroups. Then, none of these subgroups is contained in the others, otherwise they are identically equal. Moreover, $\left|\bigcap_{H\in\Sigma} H\right| = p^{n-2}$. Therefore, $\left|\bigcup_{H\in\Sigma} H\right| = (p+1)p^{n-1} - (p+1)p^{n-2} + p^{n-2} = p^n$. Thus, $\sigma_N(G) = |\Sigma| \leq p+1$. Clearly, any other system of normal subgroups of order less than p^{n-1} that needs to cover G should contains at least p^2 subgroups. Hence, $\sigma_N(G) = p+1$. □

Corollary 2.14. *Let G be a p-group and* Σ *be the normal cover of G. Then* $G/H \cong \mathbb{Z}_p$ *for all* $H \in \Sigma$.

Proof. If G is a p-group of order p^n, then Σ is a collection of $p+1$ normal subgroups each of order p^{n-1} (Theorem 2.13). So, if $H \in \Sigma$, then $[G : H] = \frac{p^n}{p^{n-1}} = p$. Thus, $G/H \cong \mathbb{Z}_p$ for all $H \in \Sigma$. Hence the claim. □

Remark 2.15. *Note that, not every noncyclic group which has a normal cover is a p-group. For example, let* $G = D_{12}$ *be the dihedral group of order 12, which can be written as* $G = \langle r, s \mid r^6 = e = (r^i s)^2\ \forall i = 1, 2, \ldots, n\rangle$. *Clearly, G is a noncyclic group. Furthermore, the subgroups* $H_1 = \{e, r^2, r^4, s, r^2s, r^4s\}$, $H_2 = \{e, r^2, r^4, rs, r^3s, r^5s\}$ *and* $H_3 = \langle r\rangle$, *are proper normal subgroups each of index 2 and* $H_1 \cup H_2 \cup H_3 = G$, *then* $\sigma_N(G) = 3$. *On the other hand G is not a p-group.*

In [2], Cohn conjectured that the covering number of any solvable group has the form $p^m + 1$, where p is a prime and m a positive integer. This conjecture is not always true.

Remark 2.16. *Not every solvable group G has ordinary cover. Consider the group* $G = \mathbb{Z}_5 \times S_3$ *which is solvable and has 10 none trivial proper subgroups, four of them are normal subgroups, but there is no collection of these subgroups is a cover of G. So* $\sigma(G) = \sigma_N(G) = \infty$.

Remark 2.17. *There exists a non-solvable group G that has normal cover and* $\sigma_N(G) = 3$. *As an example* $G = A_5 \rtimes Q_8$, *the group G is non-solvable of order 480, and G has 3 proper normal subgroups* $H_1 \cong A_5 \rtimes \mathbb{Z}_5$, $H_2 \cong A_5 \rtimes \mathbb{Z}_5$ *and* $H_3 \cong \mathbb{Z}_4 \times A_5$, *where* $\bigcap_{i=1}^{3} H_i \cong \mathbb{Z}_2 \rtimes A_5$ *and* $\bigcup_{i=1}^{3} H_i = G$. *Moreover, G* *has a homomorphic image $\phi(G)$ isomorphic to the Klein-Four group and* $\ker\phi \cong \mathbb{Z}_2 \rtimes A_5$.

Remark 2.18. *The alternating group $G = A_n$ has no normal cover for any $n \geq 3$. As G is a simple group for all $n \neq 4$. For the case $n = 4$, the group G has only one non-trivial proper normal subgroup of index 3.*

Remark 2.19. *The Symmetric group $G = S_n$ has no normal cover for any $n \geq 3$. Since, G has one maximal normal subgroup H of index 2, and the other normal subgroups if exist are subgroups of H. See Table 1.*

Table 1: The subgroups description of S_n

n	Number of subgroups	Proper normal subgroups
3	6	$H \cong \mathbb{Z}_3$
4	30	$H_1 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ and A_4
5	165	$H = A_5$
6	1455	$H = A_6$
7	11300	$H = A_7$
8	151221	$H = A_8$
9	1694723	$H = A_9$

Remark 2.20. *The Mathieu groups M_{11}, M_{12}, M_{22} and M_{23} have no normal cover. In fact, any simple group has no normal cover.*

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