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New Conjugate Direction Method for Unconstrained Optimization

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Abstract

The aim of this article is to introduce a new conjugate direction method (CDM) which satisfies a conjugate and descent property to solve an unconstrained optimization problem. The objective function is differentiable and the gradient (first derivative of the objective function) is Lipchitz continuous. The new direction is computed based on the last direction with a new parameter as a coefficient of the next gradient of the given objective function. We consider Numerical examples, with the help of the MATLAB program, using the new method compared with the Fletcher-Reeves conjugate gradient (F-R-CGM) method to verify the effectiveness of the new CDM.Our results are reported as a table containing total iterations and function evaluation.

1 Introduction

Many researchers modified CDM to CGM in particular, depending on the last gradient and the last direction because this method is easy to implement

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(depends only on the first derivative of the objective function). In [1], the CGM was modified to make its direction always in a descent direction to solve a large scale unconstrained optimization problem. In [2], the authors proposed the modified hybrid CGM and proved the descent property and the global convergence. In [3], the authors proposed a new nonlinear CGM based on inexact line search to solve unconstrained optimization problem. In [4], the authors proposed a modified PRP CGM by mixing the steepest descent algorithm together with CGM to solve unconstrained optimization problem. In [5], the authors proposed a new CGM based on conjugate descent formula. In [6], the author proposed a new technique of hybrid nonlinear CGM to guarantee the descent property at every iteration. Finally, in [7], the authors developed the CGM as a special case of the more general iterated for solving a system of linear equations. In this paper, we propose a new CDM which guarantees the conjugate and the descent property for all the feasible directions to solve the unconstrained optimization problem.,

2 CDM Algorithm

The general direction algorithm is as follows:

1. Given a starting point $x_0 \in R^n$, $eps > 0,\ j = 0$
2. Compute $\nabla f\left(x_j\right) = h_j$
3. Compute the direction d_j such that $h_j^T d_j < 0$
4. If $\|h_j\| < eps$, stop otherwise go to step 5
5. Compute $\mu_j > 0$ such that $f\left(x_j + \mu_j d_j\right) < f(x_j)$, and set $x_{j+1} = x_j + \mu_j d_j$
6. Compute d_{j+1} by some CDM such that $d_{j+1}^T G d_k = 0$, $k = 1, 2, \ldots, j$
7. $j = j + 1$ and go to 4

3 CGM Algorithm

The property of this algorithm that is all its directions are conjugate as in step 6 of the above algorithm, and hence the CGM is one of these methods. For the quadratic case, the CGM algorithm is as follows:

1. Given a starting point $x_0 \in R^n$, $eps > 0,\ j = 0$.
2. Compute $\nabla f\,(x_j) = h_j$.
3. $d_j = -h_j$
4. If $\|h_j\| < eps$, stop otherwise go to step 5.
5. Compute $\mu_j > 0$ such that $f\,(x_j + \mu_j d_j) < f(x_j)$, and set $x_{j+1} = x_j + \mu_j d_j$.
6. Compute the next direction by one of the conjugate gradient formula like
   $d_{j+1} = -h_j + \frac{h_{j+1}^T h_{j+1}}{h_j^T h_j} d_j$ (Fletcher-Reeves formula)
7. $j = j + 1$ and go to 4

The CDM does not give an explicit procedure to compute the next direction. However, in CGM, there are more than one formula to compute the next direction.

4 New CDM

Consider the differentiable quadratic objective function $f\,(x) = \frac{1}{2}x^T Gx + b^T x + c$, where $x, b \in R^n,\ c \in R$, and G symmetric and positive definite matrix.

Note that, $h\,(x) = Gx$ and by the property of CGM [8] we have $h_j^T h_i = 0$, *i=0,1,. . . ,j-1.*

Consider the following direction

$$d_{j+1} = \ d_j - \theta_k h_{j+1}, \tag{4.1}$$

where $d_0 = -h_0$ and $\theta_k \in R$ is the conjugate coefficients, k=1,2,. . . ,j. Let us suppose that $\theta_k = 0,\ k = 1, 2, \ldots, j-1$.Hence equation (4.1) becomes

$$d_{j+1} = \ d_j - \theta_j h_{j+1} \tag{4.2}$$

The problem is to find the formula of θ_j such that the direction defined in equation (4.1) is a descent direction $h_{j+1}^T d_{j+1} < 0$, and the conjugate direction is $d_{j+1}^T G d_j = 0$.

To determine θ_j by definition of conjugate we have the following equation

$$d_{j+1}^T G d_j = 0 \tag{4.3}$$

Substituting equation (2) into equation (3), we get the following equation:

$$d_{j+1}^T G d_k = (d_j - \theta_j h_{j+1})\, G d_j = 0, k = 0, 1, 2, \ldots, j$$
$$d_{j+1}^T G d_k = d_j^T G d_j - \theta_j h_{j+1}^T G d_j = 0$$

Hence, $\theta_j = \frac{d_j^T G d_j}{h_{j+1}^T G d_j}$. Therefore, using equation (2), we have the following equation:

$$d_{j+1} = \ d_j - \frac{d_j^T G d_j}{h_{j+1}^T G d_j} h_{j+1} \tag{4.4}$$

Equation (4.4) represents the new CDM based on the CGM.

Lemma 4.1. *The direction given in equation (4) is a descent direction*

Proof. By definition of the descent property
$h_{j+1}^T d_{j+1} = h_{j+1}^T \left(d_j - \frac{d_j^T G d_j}{h_{j+1}^T G d_j} h_{j+1} \right)$
$h_{j+1}^T d_{j+1} = h_{j+1}^T d_j - \frac{d_j^T G d_j}{h_{j+1}^T G d_j} h_{j+1}^T h_{j+1}$, by the principal theorem of CDM [8], the first part is equal to zero, and
$h_{j+1}^T d_{j+1} = -\frac{d_j^T G d_j}{h_{j+1}^T G(\frac{x_{j+1} - x_j}{\mu_j})} \left\| h_{j+1} \right\|^2 = -\frac{\mu_j d_j^T G d_j}{h_{j+1}^T (G x_{j+1} - G x_j)} \left\| h_{j+1} \right\|^2$
$h_{j+1}^T d_{j+1} = -\frac{\mu_j d_j^T G d_j}{h_{j+1}^T (h_{j+1} - h_j)} \left\| h_{j+1} \right\|^2$ By the property theorem of the CGM, the positive definiteness, and $\mu_j > 0$, we have
$h_{j+1}^T d_{j+1} = -\frac{\mu_j d_j^T G d_j}{\left\| h_{j+1} \right\|^2} \left\| h_{j+1} \right\|^2 = -\mu_j d_j^T G d_j < 0$. Consequently, d_{j+1} is descent direction.

4.1 New Conjugate Direction Algorithm

In this part, we give the algorithm of the new CDM as follows:

1. Given a starting point $x_0 \in R^n$, $eps > 0$, $j = 0$
2. Compute $\nabla f\,(x_j) = h_j$
3. $d_j = -h_j$
4. If $\|h_j\| < eps$, stop otherwise go to step 5

5. Compute $\mu_j > 0$ such that $f(x_j + \mu_j d_j) < f(x_j)$, and set $x_{j+1} = x_j + \mu_j d_j$

6. Compute the next direction by equation (4)

7. $j = j + 1$ and go to 4

8. Numerical Results

In this part, many standard functions are tested by the new method. We report the results as a table of number of iterations and function evaluations. The list of test functions is as follows 8:

1. Cube function.

2. Extended Rosenbrock function.

3. Wood function

4. Powell Singular function, 5. Trigonometric function, 6. Cosine function

5. Perturbed Tri diagonal quadratic function

Table 1: Comparison between the CGM and new CDM

f.no	**Starting point**	**Dim**	**CGM Iter**	**CGM f min**	**New CDM Iter**	**New CDM f min**
1	[0,0]	2	33	0.2459	34	0.2247
2	[0,0]	2	32	0.0850	34	0.0429
3	[0,0,0,0]	4	48	0.0043	51	0.0003
4	[1,1,1,1]	4	48	0.1762	51	0.1737
5	[1,1,1,1]	4	8	$1.1596X10^{(-11)}$	7	$1.79884X10^{(-12)}$
6	[1,1,1,1]	4	7	-3	8	-3
7	[0.5,...,0.5]	12	66	$3.4107X10^{(-7)}$	68	$2.6148X10^{(-7)}$

From Table 1, it is clear that the new method can solve the unconstrained optimization problems better than the CG method with respect to function evaluation, whereas the CG method is better than the new method with respect to the total iterations.

5 Conclusion

A new CDM was proposed and a new coefficient of gradient is derived to obtain the conjugate and descent directions. The new Algorithm was implemented for a standard function and the results were compared with the CG method. The new method was effective in solving the unconstrained optimization problem.

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