International Journal of Mathematics and
Computer Science, **17**(2022), no. 2, 801–813

Semigroups in which the radical of every $(\mathbf{m}, \mathbf{n})$-ideal is a subsemigroup

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(Received November 1, 2021, Accepted December 1, 2021)

Abstract

For a subset A of a semigroup S, the radical of A, denoted $\sqrt{A}$, is the set of all x in S such that $x^n \in A$ for some positive integer n. In order to extend the results obtained by Sanborisoot and Changphas [4], we give a characterization when the radical of every (m, n)-ideal of S is a subsemigroup of S.

1 Introduction

Let S be a semigroup. For $a_1, \ldots, a_n \in S$, let $\langle a_1, \ldots, a_n \rangle$ denote the subsemigroup of S generated by $\{a_1, \ldots, a_n\}$; i.e., $\langle a_1, \ldots, a_n \rangle$ is the intersection of the subsemigroups containing $a_1, \ldots, a_n$ of S. A nonempty subset A of S is called a *left* (respectively, *right*) *ideal* of S if $SA \subseteq A$ (respectively, $AS \subseteq A$). If A is both a left and a right ideal of S, then A is called a *two-sided ideal* (or *ideal*) of S. A subsemigroup B of S is called a *bi-ideal* of S if $BSB \subseteq B$. A nonempty subset Q of S is called a *quasi-ideal* of S if $QS \cap SQ \subseteq Q$. It is known that every quasi-ideal is a bi-ideal [5]. Let m, n be nonnegative integers. A subsemigroup A of S is called an (m, n)*-ideal* of S if $A^m S A^n \subseteq A$

Key words and phrases: Semigroup, subsemigroup, radical, ideal, (m, n)-ideal, (m, n)-quasi-ideal.
AMS (MOS) Subject Classifications: 20M17.
ISSN 1814-0432, 2022, http://ijmcs.future-in-tech.net

[2]. Here, let $A^0S = S = SA^0$. A subsemigroup Q of a semigroup S is called an (m, n)-*quasi-ideal* of S if $Q^mS \cap SQ^n \subseteq Q$ [3]. Here, let $Q^0S = S = SQ^0$.

For a subset A of a semigroup S, let $\sqrt{A}$ denote the *radical* [1] of A; i.e.,

$$\sqrt{A} = \{x \in S \mid x^n \in A \text{ for some positive integer } n\}.$$

In [1], Bogdanović and Ćirić characterized semigroups in which the radical of every ideal (right ideal, bi-ideal, subsemigroup) is a subsemigroup (ideal, bi-ideal, right ideal) of S. Moreover in [4], the authors considered semigroups in which the radical of every quasi-ideal is a subsemigroup of S. The aim of this paper is to characterize semigroups in which the radical of every (m, n)-quasi-ideal (subsemigroup, ideal, (m, n)-ideal) is a subsemigroup (ideal, (m, n)-ideal, (m, n)-quasi-ideal) of S. The results obtained in [1] and [4] then become special cases.

2 Main results

Let $\mathbb{N} = \{1, 2, 3, \ldots\}$ denote the set of all positive integers. First, we have the following theorem.

Theorem 2.1. *Let S be a semigroup and let m, n be nonnegative integers. Then the radical of every (m, n)-ideal of S is a subsemigroup of S if and only if*

$$(\forall a, b \in S\ \forall k, l \in \mathbb{N}\ \exists i \in \mathbb{N})\ (ab)^i \in \{a^k, b^l\}^m S \{a^k, b^l\}^n.$$

Proof. Assume that the radical of every (m, n)-ideal of S is a subsemigroup of S. Let $a, b \in S$ and $k, l \in \mathbb{N}$. Let $A = \{a^k, b^l\}^m S \{a^k, b^l\}^n$. Consider

$$\begin{aligned} A^2 &= (\{a^k, b^l\}^m S \{a^k, b^l\}^n)(\{a^k, b^l\}^m S \{a^k, b^l\}^n) \\ &= \{a^k, b^l\}^m (S \{a^k, b^l\}^n \{a^k, b^l\}^m S) \{a^k, b^l\}^n \\ &\subseteq \{a^k, b^l\}^m S \{a^k, b^l\}^n \\ &= A. \end{aligned}$$

Then, we have A is a subsemigroup of S. Moreover, observe that

$$\begin{aligned} A^m S A^n &\subseteq ASA \\ &= (\{a^k, b^l\}^m S \{a^k, b^l\}^n) S (\{a^k, b^l\}^m S \{a^k, b^l\}^n) \\ &= \{a^k, b^l\}^m (S \{a^k, b^l\}^n S \{a^k, b^l\}^m S) \{a^k, b^l\}^n \\ &\subseteq \{a^k, b^l\}^m S \{a^k, b^l\}^n. \\ &= A. \end{aligned}$$

Then A is an (m,n)-ideal of S. By assumption, $\sqrt{A}$ is a subsemigroup of S. From $a^{k(m+n)+1}, b^{l(m+n)+1} \in A$, then $a, b \in \sqrt{A}$, whence $ab \in \sqrt{A}$. Hence there exists $i \in \mathbb{N}$ such that $(ab)^i \in A$.

Conversely, assume that

$$(\forall a, b \in S\ \forall k, l \in \mathbb{N}\ \exists i \in \mathbb{N})\ (ab)^i \in \{a^k, b^l\}^m S \{a^k, b^l\}^n.$$

Let A be an (m,n)-ideal of S and let $a, b \in \sqrt{A}$. Let $k, l \in \mathbb{N}$ be such that $a^k, b^l \in A$. By hypothesis, $(ab)^i \in \{a^k, b^l\}^m S \{a^k, b^l\}^n$ for some $i \in \mathbb{N}$. From

$$\{a^k, b^l\}^m S \{a^k, b^l\}^n \subseteq A^m S A^n \subseteq A,$$

we have $ab \in \sqrt{A}$. Therefore $\sqrt{A}$ is a subsemigroup of S. □

Theorem 2.2. *Let S be a semigroup and let m, n be nonnegative integers. Then the radical of every subsemigroup of S is an (m,n)-ideal of S if and only if* (1) *and* (2) *hold:*

(1) $(\forall a, b \in S\ \forall k, l \in \mathbb{N}\ \exists i \in \mathbb{N})\ (ab)^i \in \langle a^k, b^l \rangle$;

(2) $(\forall a_1, \ldots, a_m, b_1, \ldots, b_n, c, u \in S\ \forall k_1, \ldots, k_m, l_1, \ldots, l_n \in \mathbb{N})$

$$(c = a_1 \ldots a_m u b_1 \ldots b_n) \Rightarrow (\exists i \in \mathbb{N})\ c^i \in \langle a_1^{k_1}, \ldots, a_m^{k_m}, b_1^{l_1}, \ldots b_n^{l_n} \rangle.$$

Proof. Assume that the radical of every subsemigroup of S is an (m,n)-ideal of S. Let $a, b \in S$ and $k \in \mathbb{N}$. Since $\langle a^k, b^l \rangle$ is a subsemigroup of S, we have $\sqrt{\langle a^k, b^l \rangle}$ is an (m,n)-ideal of S by assumption. As $a, b \in \sqrt{\langle a^k, b^l \rangle}$, we get $ab \in \sqrt{\langle a^k, b^l \rangle}$, and so $(ab)^i \in \langle a^k, b^l \rangle$ for some $i \in \mathbb{N}$. This shows that (1) holds. Now, let $a_1, \ldots, a_m, b_1, \ldots, b_n,\ c, u \in S$ be such that $c = a_1 \ldots a_m u b_1 \ldots b_n$. For $k_1, \ldots, k_m,\ l_1, \ldots, l_n \in \mathbb{N}$, put

$$B = \langle a_1^{k_1}, \ldots, a_m^{k_m}, b_1^{l_1}, \ldots, b_n^{l_n} \rangle.$$

Then, by assumption, $\sqrt{B}$ is an (m,n)-ideal of S. From $a_1, \ldots, a_m, b_1, \ldots, b_n \in \sqrt{B}$, we obtain that

$$c \in \sqrt{B}^m S \sqrt{B}^n \subseteq \sqrt{B}.$$

Then there is $i \in \mathbb{N}$ such that $c^i \in B$. Thus (2) holds.

Conversely, assume (1) and (2) hold. Let A be a subsemigroup of S and $a, b \in \sqrt{A}$. Then $a^k, b^l \in A$ for some $k, l \in \mathbb{N}$. By (1), we have that $(ab)^i \in \langle a^k, b^l \rangle \subseteq A$ for some $i \in \mathbb{N}$. We conclude that $ab \in \sqrt{A}$. Therefore

$\sqrt{A}$ is a subsemigroup of S. Now, let $c \in \sqrt{A}^m S \sqrt{A}^n$. Then there are $u \in S$ and $a_1, \ldots, a_m, b_1, \ldots, b_n \in \sqrt{A}$ such that

$$c = a_1 \ldots a_m u b_1 \ldots b_n.$$

There exist $k_1, \ldots, k_m, l_1, \ldots, l_n \in \mathbb{N}$ such that $a_1^{k_1}, \ldots, a_m^{k_m}, b_1^{l_1}, \ldots, b_n^{l_n} \in A$. Thus there exists $i \in \mathbb{N}$ such that

$$c^i \in \langle a_1^{k_1}, \ldots, a_m^{k_m}, b_1^{l_1}, \ldots, b_n^{l_n} \rangle \subseteq A$$

by assumption. Therefore $c \in \sqrt{A}$. □

Theorem 2.3. *Let S be a semigroup and let m, n be nonnegative integers. Then the radical of every (m,n)-ideal of S is an ideal of S if and only if*

$$(\forall a, b, u, v \in S\ \forall k, l \in \mathbb{N}\ \exists i, j \in \mathbb{N})\ (au)^i, (vb)^j \in \{a^k, b^l\}^m S \{a^k, b^l\}^n.$$

Proof. Assume that the radical of every (m,n)-ideal of a semigroup S is an ideal of S. Let $a, b, u, v \in S$ and let $k, l \in \mathbb{N}$. Define

$$A = \{a^k, b^l\}^m S \{a^k, b^l\}^n.$$

Then $a, b \in \sqrt{A}$ and A is an (m,n)-ideal of S. By assumption, $\sqrt{A}$ is an ideal of S. Thus $au \in \sqrt{A} S \subseteq \sqrt{A}$ and $vb \in S\sqrt{A} \subseteq \sqrt{A}$. Hence $(au)^i, (vb)^j \in A$ for some $i, j \in \mathbb{N}$.

Conversely, assume that for every $a, b, u, v \in S$ and $k, l \in \mathbb{N}$,

$$(au)^i, (vb)^j \in \{a^k, b^l\}^m S \{a^k, b^l\}^n$$

for some $i, j \in \mathbb{N}$. Let A be an (m,n)-ideal of S. We want to show that $\sqrt{A}$ is an ideal of S. Let $a, b \in \sqrt{A}$ and $u, v \in S$. Then $a^k, b^l \in A$ for some $k, l \in \mathbb{N}$. By assumption,

$$(au)^i, (vb)^j \in \{a^k, b^l\}^m S \{a^k, b^l\}^n$$

for some $i, j \in \mathbb{N}$. Since $\{a^k, b^l\}^m S \{a^k, b^l\}^n \subseteq A^m S A^n \subseteq A$, we have $au, vb \in \sqrt{A}$. This proves that $\sqrt{A}$ is an ideal of S. □

Theorem 2.4. *Let S be a semigroup and let m, n be nonnegative integers. Then the radical of every ideal of S is an (m,n)-ideal of S if and only if* (1) *and* (2) *hold:*

(1) $(\forall a, b \in S\ \forall k, l \in \mathbb{N}\ \exists i \in \mathbb{N})\ (ab)^i \in S\{a^k, b^l\}S$;

(2) $(\forall a_1,\dots,a_m,b_1,\dots,b_n,c,u\in S\ \forall k_1,\dots,k_m,l_1,\dots,l_n\in\mathbb{N})$

$$c=a_1\dots a_mub_1\dots b_n\Rightarrow(\exists i\in\mathbb{N})\ c^i\in S\{a_1^{k_1},\dots,a_m^{k_m},b_1^{l_1},\dots,b_n^{l_n}\}S.$$

Proof. Suppose that the radical of every ideal of S is an (m,n)-ideal of S. Let $a,b\in S$ and let $k,l\in\mathbb{N}$. Define the set $A=S\{a^k,b^l\}S$. It is clear that A is an ideal of S. Then, by assumption, $\sqrt{A}$ is an (m,n)-ideal of S. Since $a,b\in\sqrt{A}$, we have $ab\in\sqrt{A}$. That is $(ab)^i\in A$ for some $i\in\mathbb{N}$, and hence (1) follows. Let $a_1,\dots,a_m,b_1,\dots,b_n,c,u\in S$ be such that

$$c=a_1\dots a_mub_1\dots b_n.$$

For any $k_1,\dots,k_m,l_1,\dots,l_n\in\mathbb{N}$, we put the set

$$B=S\{a_1^{k_1},\dots,a_m^{k_m},b_1^{l_1},\dots,b_n^{l_n}\}S.$$

Then B is an ideal of S. By hypothesis, $\sqrt{B}$ is an (m,n)-ideal of S containing $a_1,\dots,a_m,b_1,\dots,b_n$. Since $c=a_1\dots a_mub_1\dots b_n$ and $\sqrt{B}$ is an (m,n)-ideal of S, $c\in\sqrt{B}^mS\sqrt{B}^n\subseteq\sqrt{B}$. Therefore $c^i\in B$ for some $i\in\mathbb{N}$. Consequently, (2) holds.

Conversely, assume that (1) and (2) hold. Let A be an ideal of S and $a,b\in\sqrt{A}$. Then $a^k,b^l\in A$ for some $k,l\in\mathbb{N}$. By (1) and A is an ideal of S,

$$(ab)^i\in S\{a^k,b^l\}S\subseteq SAS\subseteq A$$

for some $i\in\mathbb{N}$. Thus $ab\in\sqrt{A}$. This proves that $\sqrt{A}$ is a subsemigroup of S. Let $c\in\sqrt{A}^mS\sqrt{A}^n$. Then $c=a_1\dots a_mub_1\dots b_n$ for some $u\in S$ and $a_1,\dots,a_m,b_1,\dots,b_n\in\sqrt{A}$. Hence $a_1^{k_1},\dots,a_m^{k_m},b_1^{l_1},\dots,b_n^{l_n}\in A$ for some $k_1,\dots,k_m,l_1,\dots,l_n\in\mathbb{N}$. By (2) and A is an ideal of S, there is $i\in\mathbb{N}$ such that

$$c^i\in S\{a_1^{k_1},\dots,a_m^{k_m},b_1^{l_1},\dots,b_n^{l_n}\}S\subseteq SAS\subseteq A.$$

Consequently, $c\in\sqrt{A}$. □

Theorem 2.5. *Let S be a semigroup and let $m,n\in\mathbb{N}$. Then the radical of every (m,n)-ideal of S is an (m,n)-ideal of S if and only if (1) and (2) hold:*

(1) $(\forall a,b\in S\ \forall k,l\in\mathbb{N}\ \exists i\in\mathbb{N})\ (ab)^i\in\{a^k,b^l\}^mS\{a^k,b^l\}^n$;

(2) $(\forall a_1,\dots,a_m,b_1,\dots,b_n,c,u\in S\ \forall k_1,\dots,k_m,l_1,\dots,l_n\in\mathbb{N})$

$$c=a_1\dots a_mub_1\dots b_n\Rightarrow$$

$$(\exists i\in\mathbb{N})c^i\in\{a_1^{k_1},\dots,a_m^{k_m},b_1^{l_1},\dots,b_n^{l_n}\}^mS\{a_1^{k_1},\dots,a_m^{k_m},b_1^{l_1},\dots,b_n^{l_n}\}^n.$$

Proof. Assume that the radical of every (m,n)-ideal of S is an (m,n)-ideal of S. Let $a,b \in S$ and $k,l \in \mathbb{N}$. Since

$$A = \{a^k, b^l\}^m S \{a^k, b^l\}^n$$

is an (m,n)-ideal of S such that $a,b \in \sqrt{A}$, $\sqrt{A}$ is an (m,n)-ideal of S by assumption. We have that $ab \in \sqrt{A}$, whence (1) follows. Let $a_1,\dots,a_m$, $b_1,\dots,b_n$, $c,u \in S$ be such that $c = a_1 \dots a_m u b_1 \dots b_n$. Let $k_1,\dots,k_m$, $l_1,\dots,l_n \in \mathbb{N}$, and set

$$B = \{a_1^{k_1},\dots,a_m^{k_m},b_1^{l_1},\dots,b_n^{l_n}\}^m S \{a_1^{k_1},\dots,a_m^{k_m},b_1^{l_1},\dots,b_n^{l_n}\}^n.$$

Clearly, B is an (m,n)-ideal of S. Then, by assumption, $\sqrt{B}$ is an (m,n)-ideal of S. Since $a_1,\dots,a_m,b_1,\dots,b_n \in \sqrt{B}$, we have $c \in \sqrt{B}^m S \sqrt{B}^n \subseteq \sqrt{B}$. Thus $c^i \in B$ for some $i \in \mathbb{N}$, and so (2) follows.

Conversely, assume that (1) and (2) follow. Let A be an (m,n)-ideal of S. As in the proof of Theorem 2.1, by (1), we have $\sqrt{A}$ is a subsemigroup of S. Now let $c \in \sqrt{A}^m S \sqrt{A}^n$. Then $c = a_1 \dots a_m u b_1 \dots b_n$ for some $a_1,\dots,a_m,b_1,\dots,b_n \in \sqrt{B}$ and $u \in S$. Thus there are $k_1,\dots,k_m,l_1,\dots,l_n \in \mathbb{N}$ such that $a_1^{k_1},\dots,a_m^{k_m},b_1^{l_1},\dots,b_n^{l_n} \in A$. Then, by hypothesis, we get

$$c^i \in \{a_1^{k_1},\dots,a_m^{k_m},b_1^{l_1},\dots,b_n^{l_n}\}^m S \{a_1^{k_1},\dots,a_m^{k_m},b_1^{l_1},\dots,b_n^{l_n}\}^n$$

for some $i \in \mathbb{N}$. Observe that

$$\{a_1^{k_1},\dots,a_m^{k_m},b_1^{l_1},\dots,b_n^{l_n}\}^m S \{a_1^{k_1},\dots,a_m^{k_m},b_1^{l_1},\dots,b_n^{l_n}\}^n \subseteq A^m S A^n \subseteq A,$$

we obtain that $c \in \sqrt{A}$. This proves that $\sqrt{A}$ is an (m,n)-ideal of S. □

Theorem 2.6. *Let S be a semigroup and let m,n be a nonnegative integer. Then the radical of every (m,n)-quasi-ideal of S is a subsemigroup of S if and only if*

$$(\forall a,b \in S\ \forall k,l \in \mathbb{N}\ \exists i \in \mathbb{N})\ (ab)^i \in \{a^k,b^l\}^m S \cap S\{a^k,b^l\}^n.$$

Proof. Assume that the radical of every (m,n)-quasi-ideal of S is a subsemigroup of S. Let $a,b \in S$ and let $k,l \in \mathbb{N}$. Consider

$$Q = \{a^k,b^l\}^m S \cap S\{a^k,b^l\}^n.$$

We have

$$Q^2 \subseteq (\{a^k,b^l\}^m S \cap S\{a^k,b^l\}^n)S \subseteq (\{a^k,b^l\}^m S)S \subseteq \{a^k,b^l\}^m S.$$

Similarly,

$$Q^2 \subseteq S(\{a^k, b^l\}^m S \cap S\{a^k, b^l\}^n) \subseteq S(S\{a^k, b^l\}^n) \subseteq S\{a^k, b^l\}^n.$$

Then $Q^2 \subseteq Q$ and Q is a subsemigroup of S. Moreover,

$$\begin{aligned} Q^m S \cap SQ^n &\subseteq Q^m S \\ &\subseteq QS \\ &= (\{a^k, b^l\}^m S \cap S\{a^k, b^l\}^n)S \\ &\subseteq (\{a^k, b^l\}^m S)S \\ &\subseteq \{a^k, b^l\}^m S. \end{aligned}$$

Similarly, $Q^m S \cap SQ^n \subseteq S\{a^k, b^l\}^n$. Then $Q^m S \cap SQ^n \subseteq Q$. Hence Q is an (m, n)-quasi-ideal of S. By assumption, $\sqrt{Q}$ is a subsemigroup of S. Observe that $a^{k(m+n)+1}, b^{l(m+n)+1} \in Q$, whence $a, b \in \sqrt{Q}$; hence $ab \in \sqrt{Q}$. Thus there exists $i \in \mathbb{N}$ such that

$$(ab)^i \in \{a^k, b^l\}^m S \cap S\{a^k, b^l\}^n.$$

Conversely, assume that for all $a, b \in S$ and $k, l \in \mathbb{N}$,

$$(ab)^i \in \{a^k, b^l\}^m S \cap S\{a^k, b^l\}^n$$

for some $i \in \mathbb{N}$. Let Q be an (m, n)-quasi-ideal of S and $a, b \in \sqrt{Q}$. Then $a^k \in Q$ and $b^l \in Q$, for some $k, l \in \mathbb{N}$. By hypothesis,

$$(ab)^i \in \{a^k, b^l\}^m S \cap S\{a^k, b^l\}^n$$

for some $i \in \mathbb{N}$. From

$$\{a^k, b^l\}^m S \cap S\{a^k, b^l\}^n \subseteq Q^m S \cap SQ^n \subseteq Q,$$

it follows that $ab \in \sqrt{Q}$. □

Theorem 2.7. *Let S be a semigroup and let m, n be nonnegative integers. Then the radical of every subsemigroup of S is an (m, n)-quasi-ideal of S if and only if (1) and (2) hold:*

(1) $(\forall a, b \in S\ \forall k, l \in \mathbb{N}\ \exists i \in \mathbb{N})\ (ab)^i \in \langle a^k, b^l \rangle$;

(2) $(\forall a_1, a_2, \ldots, a_m, b_1, b_2, \ldots, b_n, c, u, v \in S\ \forall k_1, \ldots, k_m, l_1, l_2, \ldots, l_n \in \mathbb{N})$

$$c = a_1 a_2 \ldots a_m u \wedge c = v b_1 b_2 \ldots b_n \Rightarrow$$

$$(\exists i \in \mathbb{N})\ c^i \in \langle a_1^{k_1}, \ldots, a_m^{k_m}, b_1^{l_1}, b_2^{l_2}, \ldots, b_n^{l_n} \rangle.$$

Proof. Suppose that the radical of every subsemigroup of S is an (m,n)-quasi-ideal of S. Let $a, b \in S$ and let $k, l \in \mathbb{N}$. Define the set $A = \langle a^k, b^l \rangle$. Then, by assumption, $\sqrt{A}$ is a subsemigroup of S. Moreover, $ab \in \sqrt{A}$ because $a, b \in \sqrt{A}$. Then $(ab)^i \in A$ for some $i \in \mathbb{N}$. That is, (1) holds. Let $a_1, \ldots, a_m, b_1, \ldots, b_n, c, u, v \in S$ be such that $c = a_1 \ldots a_m u$ and $c = v b_1 \ldots b_n$. Let $k_1, \ldots, k_m, l_1, \ldots, l_n \in \mathbb{N}$. Set

$$Q = \langle a_1^{k_1}, \ldots, a_m^{k_m}, b_1^{l_1}, \ldots, b_n^{l_n} \rangle.$$

Then $\sqrt{Q}$ is an (m,n)-quasi-ideal of S by hypothesis. Since $a_1, \ldots, a_m, b_1, \ldots, b_n \in \sqrt{Q}$, we have $c \in \sqrt{Q}^m S$ and $c \in S\sqrt{Q}^n$. Hence

$$c \in \sqrt{Q}^m S \cap S\sqrt{Q}^n \subseteq \sqrt{Q}.$$

That is $c^i \in Q$ for some $i \in \mathbb{N}$. Hence (2) holds.

Conversely, assume that (1) and (2) hold. Let Q be a subsemigroup of S. Let $a, b \in \sqrt{Q}$; so that $a^k, b^l \in Q$ for some $k, l \in \mathbb{N}$. By (1), $(ab)^i \in \langle a^k, b^l \rangle \subseteq Q$ for some $i \in \mathbb{N}$, and so $ab \in \sqrt{Q}$. This means that $\sqrt{Q}$ is a subsemigroup of S.
Let $c \in \sqrt{Q}^m S \cap S\sqrt{Q}^n$. Then $c = a_1 \ldots a_m u$ and $c = v b_1 \ldots b_n$, for some $a_1, \ldots, a_m, b_1, \ldots, b_n \in \sqrt{Q}$ and $u, v \in S$. Thus

$$a_1^{k_1}, \ldots, a_m^{k_m}, b_1^{l_1}, \ldots, b_n^{l_n} \in Q,$$

for some $k_1, \ldots, k_m, l_1, \ldots, l_n \in \mathbb{N}$. By (2) and Q is a subsemigroup of S,

$$c^i \in \langle a_1^{k_1}, \ldots, a_m^{k_m}, b_1^{l_1}, \ldots, b_n^{l_n} \rangle \subseteq Q,$$

for some $i \in \mathbb{N}$. Thus $c \in \sqrt{Q}$. □

The following two theorems can be proved in a similar manner to Theorems 2.3 and 2.4.

Theorem 2.8. *Let S be a semigroup and let m, n be nonnegative integers. Then the radical of every (m,n)-quasi-ideal of S is an ideal of S if and only if*

$$(\forall a, b, u, v \in S \ \forall k, l \in \mathbb{N} \ \exists i, j \in \mathbb{N}) \ (au)^i, (vb)^j \in \{a^k, b^l\}^m S \cap S\{a^k, b^l\}^n.$$

Theorem 2.9. *Let S be a semigroup and let m, n be nonnegative integers. Then the radical of every ideal of S is an (m,n)-quasi-ideal of S if and only if* (1) *and* (2) *hold:*

(1) $(\forall a, b \in S\ \forall k, l \in \mathbb{N}\ \exists i \in \mathbb{N})\ (ab)^i \in S\{a^k, b^l\}S$;

(2) $(\forall a_1, \ldots, a_m, b_1, \ldots, b_n, c, u, v \in S\ \forall k_1, \ldots, k_m, l_1, \ldots, l_n \in \mathbb{N})$

$$(c = a_1 \ldots a_m u \wedge c = v b_1 \ldots b_n) \Rightarrow$$
$$(\exists i \in \mathbb{N})\ c^i \in S\{a_1^{k_1}, \ldots, a_m^{k_m}, b_1^{l_1}, \ldots, b_n^{l_n}\}S.$$

Theorem 2.10. *Let S be a semigroup and let m, n be nonnegative integers. Then the radical of every (m, n)-quasi-ideal of S is an (m, n)-quasi-ideal of S if and only if* (1) *and* (2) *hold:*

(1) $(\forall a, b \in S\ \forall k, l \in \mathbb{N}\ \exists i \in \mathbb{N})\ (ab)^i \in \{a^k, b^l\}^m S \cap S\{a^k, b^l\}^n$;

(2) $(\forall a_1, \ldots, a_m, b_1, \ldots, b_n, c, u, v \in S\ \forall k_1, \ldots, k_m, l_1, \ldots, l_n \in \mathbb{N})$

$$c = a_1 \ldots a_m u \wedge c = v b_1 \ldots b_n \Rightarrow$$
$$(\exists i \in \mathbb{N})\ c^i \in \{a_1^{k_1}, \ldots, a_m^{k_m}, b_1^{l_1}, \ldots, b_n^{l_n}\}^m S \cap S\{a_1^{k_1}, \ldots, a_m^{k_m}, b_1^{l_1}, \ldots, b_n^{l_n}\}^n.$$

Proof. Assume that the radical of every (m, n)-quasi-ideal of S is an (m, n)-quasi-ideal of S. Let $a, b \in S$, and let $k, l \in \mathbb{N}$. Set

$$A = \{a^k, b^l\}^m S \cap S\{a^k, b^l\}^n.$$

As the proof of Theorem 2.6, we have A is an (m, n)-quasi-ideal of S with $a, b \in \sqrt{A}$. Then, by assumption, $\sqrt{A}$ is an (m, n)-quasi-ideal of S, and so $ab \in \sqrt{A}$. Hence (1) holds. Let $a_1, \ldots, a_m, b_1, \ldots, b_n, c, u, v \in S$ be such that $c = a_1 \ldots a_m u$ and $c = v b_1 \ldots b_n$. Let $k_1, \ldots, k_m, l_1, \ldots, l_n \in \mathbb{N}$. Set

$$Q = \{a_1^{k_1}, \ldots, a_m^{k_m}, b_1^{l_1}, \ldots, b_n^{l_n}\}^m S \cap S\{a_1^{k_1}, \ldots, a_m^{k_m}, b_1^{l_1}, \ldots, b_n^{l_n}\}^n.$$

Observe that

$$Q \subseteq \{a_1^{k_1}, \ldots, a_m^{k_m}, b_1^{l_1}, \ldots, b_n^{l_n}\}^m S$$

and

$$Q \subseteq S\{a_1^{k_1}, \ldots, a_m^{k_m}, b_1^{l_1}, \ldots, b_n^{l_n}\}^n.$$

Then

$$\begin{aligned} Q^2 &\subseteq (\{a_1^{k_1}, \ldots, a_m^{k_m}, b_1^{l_1}, \ldots, b_n^{l_n}\}^m S)S \\ &\subseteq \{a_1^{k_1}, \ldots, a_m^{k_m}, b_1^{l_1}, \ldots, b_n^{l_n}\}^m S \end{aligned}$$

and

$$\begin{aligned} Q^2 &\subseteq S(S(\{a_1^{k_1}, \ldots, a_m^{k_m}, b_1^{l_1}, \ldots, b_n^{l_n}\})^n) \\ &\subseteq S(\{a_1^{k_1}, \ldots, a_m^{k_m}, b_1^{l_1}, \ldots, b_n^{l_n}\})^n. \end{aligned}$$

Hence $Q^2 \subseteq Q$, and so that Q is a subsemigroup of S. Consider

$$Q^mS \cap SQ^n \subseteq Q^mS \subseteq QS \subseteq \{a_1^{k_1}, \dots, a_m^{k_m}, b_1^{l_1}, \dots, b_n^{l_n}\}^m S.$$

And,

$$Q^mS \cap SQ^n \subseteq SQ^n \subseteq SQ \subseteq S\{a_1^{k_1}, \dots, a_m^{k_m}, b_1^{l_1}, \dots, b_n^{l_n}\}^n.$$

Then $Q^mS \cap SQ^n \subseteq Q$, whence Q is an (m,n)-quasi-ideal of S. By assumption, $\sqrt{Q}$ is an (m,n)-quasi-ideal of S. Since

$$c = a_1 \dots a_m u,\ c = vb_1 \dots b_n, \text{ and } a_1, \dots, a_m, b_1, \dots, b_n \in \sqrt{Q},$$

we have

$$c \in \sqrt{Q}^m S \cap S\sqrt{Q}^n \subseteq \sqrt{Q}.$$

Then $c^i \in Q$ for some $i \in \mathbb{N}$, and so (2) holds.

Conversely, assume that (1) and (2) hold. Let Q be an (m,n)-quasi-ideal of S. By Theorem 2.6 with (1), $\sqrt{Q}$ is a subsemigroup of S. Let $c \in \sqrt{Q}^m S \cap S\sqrt{Q}^n$; then $c = a_1 \dots a_m u$ and $c = vb_1 \dots b_n$ for some $a_1, \dots, a_m, b_1, \dots, b_n \in \sqrt{Q}$ and $u, v \in S$. There exist $k_1, \dots, k_m, l_1, \dots, l_n \in \mathbb{N}$ such that

$$a_1^{k_1}, \dots, a_m^{k_m}, b_1^{l_1}, \dots, b_n^{l_n} \in Q.$$

By (2) and Q is an (m,n)-quasi-ideal of S, there is $i \in \mathbb{N}$ such that

$$\begin{aligned} c^i &\in \{a_1^{k_1}, \dots, a_m^{k_m}, b_1^{l_1}, \dots, b_n^{l_n}\}^m S \cap S\{a_1^{k_1}, \dots, a_m^{k_m}, b_1^{l_1}, \dots, b_n^{l_n}\}^n \\ &\subseteq Q^mS \cap SQ^n \\ &\subseteq Q. \end{aligned}$$

These prove that $\sqrt{Q}$ is an (m,n)-quasi-ideal of S. □

Theorem 2.11. *Let S be a semigroup and let m,n be nonnegative integers. Then the radical of every (m,n)-quasi-ideal of S is an (m,n)-ideal of S if and only if* (1) *and* (2) *hold:*

(1) $(\forall a, b \in S\ \forall k, l \in \mathbb{N}\ \exists i \in \mathbb{N})\ (ab)^i \in \{a^k, b^l\}^m S \cap S\{a^k, b^l\}^n$;

(2) $(\forall a_1, \dots, a_m, b_1, \dots, b_n, c, u \in S\ \forall k_1, \dots, k_m, l_1, \dots, l_n \in \mathbb{N})$

$$c = a_1 \dots a_m u b_1 \dots b_n \Rightarrow (\exists i \in \mathbb{N})$$

$$c^i \in \{a_1^{k_1}, \dots, a_m^{k_m}, b_1^{l_1}, \dots, b_n^{l_n}\}^m S \cap S\{a_1^{k_1}, \dots, a_m^{k_m}, b_1^{l_1}, \dots, b_n^{l_n}\}^n.$$

Proof. Assume that the radical of every (m,n)-quasi-ideal of a semigroup S is an (m,n)-ideal of S. Let $a,b\in S$ and let $k,l\in\mathbb{N}$. Set

$$A=\{a^k,b^l\}^mS\cap S\{a^k,b^l\}^n.$$

From Theorem 2.6, we have that A is an (m,n)-quasi-ideal of S with $a,b\in\sqrt{A}$. Then, by assumption, $\sqrt{A}$ is an (m,n)-ideal of S. Thus $(ab)^i\in A$ for some $i\in\mathbb{N}$. Let $a_1,\dots,a_m,b_1,\dots,b_n,c,u\in S$ be such that

$$c=a_1\dots a_mub_1\dots b_n.$$

Let $k_1,\dots,k_m,l_1,\dots,l_n\in\mathbb{N}$, and we put

$$Q=\{a_1^{k_1},\dots,a_m^{k_m},b_1^{l_1},\dots,b_n^{l_n}\}^mS\cap S\{a_1^{k_1},\dots,a_m^{k_m},b_1^{l_1},\dots,b_n^{l_n}\}^n.$$

As the proof of Theorem 2.10, Q is an (m,n)-quasi-ideal of S. By assumption, $\sqrt{Q}$ is an (m,n)-ideal of S. Since $a_1,\dots,a_m,b_1,\dots,b_n\in\sqrt{Q}$ and $c=a_1\dots a_mub_1\dots b_n$, it follows that $c\in\sqrt{Q}^mS\sqrt{Q}^n\subseteq\sqrt{Q}$. This implies that $c^i\in Q$ for some $i\in\mathbb{N}$; consequently, $c\in\sqrt{Q}$.

Conversely, assume that (1) and (2) hold. Let Q be an (m,n)-quasi-ideal of S. Let $a,b\in\sqrt{Q}$; then $a^k,b^l\in Q$ for some $k,l\in\mathbb{N}$. As Theorem 2.6 and by (1), $\sqrt{Q}$ is a subsemigroup of S. Let $c\in\sqrt{Q}^mS\sqrt{Q}^n$. Then $c=a_1\dots a_mub_1\dots b_n$ for some $u\in S$ and $a_1,\dots,a_m,b_1,\dots,b_n\in\sqrt{Q}$. There exist $k_1,\dots,k_m,l_1,\dots,l_n\in\mathbb{N}$ be such that $a_1^{k_1},\dots,a_m^{k_m},b_1^{l_1},\dots,b_n^{l_n}\in Q$. By (2) and Q is an (m,n)-quasi-ideal of S, there is $i\in\mathbb{N}$ such that

$$\begin{aligned} c^i &\in \{a_1^{k_1},\dots,a_m^{k_m},b_1^{l_1},\dots,b_n^{l_n}\}^mS\cap S\{a_1^{k_1},\dots,a_m^{k_m},b_1^{l_1},\dots,b_n^{l_n}\}^n\\ &\subseteq Q^mS\cap SQ^n\\ &\subseteq Q.\end{aligned}$$

Hence $c\in\sqrt{Q}$. □

Theorem 2.12. *Let S be a semigroup and let m,n be nonnegative integers. The radical of every (m,n)-ideal of S is an (m,n)-quasi-ideal of S if and only if* (1) *and* (2) *hold:*

(1) $(\forall a,b\in S\ \forall k,l\in\mathbb{N}\ \exists i\in\mathbb{N})\ (ab)^i\in\{a^k,b^l\}^mS\{a^k,b^l\}^n$*;*

(2) $(\forall a_1,\dots,a_m,b_1,\dots,b_n,c,u,v\in S\ \forall k_1,\dots,k_m,l_1,\dots,l_n\in\mathbb{N})$

$$c=a_1\dots a_mu\wedge c=vb_1\dots b_n\Rightarrow$$

$$(\exists i\in\mathbb{N})c^i\in\{a_1^{k_1},\dots,a_m^{k_m},b_1^{l_1},\dots,b_n^{l_n}\}^mS\{a_1^{k_1},\dots,a_m^{k_m},b_1^{l_1},\dots,b_n^{l_n}\}^n.$$

Proof. Suppose that the radical of every (m,n)-ideal of S is an (m,n)-quasi-ideal of S. Let $a,b\in S$ and let $k,l\in\mathbb{N}$. Set

$$A=\{a^k,b^l\}^mS\{a^k,b^l\}^n.$$

By Theorem 2.1, A is an (m,n)-ideal of S, and so $\sqrt{A}$ is an (m,n)-quasi-ideal of S by assumption. Since $a,b\in\sqrt{A}$, $ab\in\sqrt{A}$. Thus $(ab)^i\in A$ for some $i\in\mathbb{N}$. This completes the proof of (1). To prove (2), let $a_1,\dots,a_m$, $b_1,\dots,b_n,c,u,v\in S$ be such that $c=a_1\dots a_mu$ and $c=vb_1\dots b_n$. For any $k_1,\dots,k_m,l_1,\dots,l_n\in\mathbb{N}$, we define

$$Q=\{a_1^{k_1},\dots,a_n^{k_m},b_1^{l_1},\dots,b_n^{l_n}\}^mS\{a_1^{k_1},\dots,a_m^{k_m},b_1^{l_1},\dots,b_n^{l_n}\}^n.$$

As in the proof of Theorem 2.5 we obtain that Q is an (m,n)-ideal of S. Then, by assumption, $\sqrt{Q}$ is an (m,n)-quasi-ideal of S. Since $a_1,\dots,a_m,b_1,\dots,b_n\in\sqrt{Q}$ and $\sqrt{Q}$ is an (m,n)-quasi-ideal of S, we have

$$c\in\sqrt{Q}^mS\cap S\sqrt{Q}^n\subseteq\sqrt{Q}$$

That is $c^i\in Q$ for some $i\in\mathbb{N}$. Thus (2) holds.

Conversely, assume that (1) and (2) hold. Let Q be an (m,n)-ideal of S and $a,b\in\sqrt{Q}$. Then $a^k,b^l\in Q$ for some $k,l\in\mathbb{N}$. By (1) and Q is an (m,n)-ideal of S,

$$(ab)^i\in\{a^k,b^l\}^mS\{a^k,b^l\}^n\subseteq Q^mSQ^n\subseteq Q$$

for some $i\in\mathbb{N}$. This shows that $ab\in\sqrt{Q}$. To show that

$$\sqrt{Q}^mS\cap S\sqrt{Q}^n\subseteq\sqrt{Q},$$

let $c\in\sqrt{Q}^mS\cap S\sqrt{Q}^n$. Then $c=a_1\dots a_mu$ and $c=vb_1\dots b_n$ for some $a_1,\dots,a_m,b_1,\dots,b_n\in\sqrt{Q}$ and $u,v\in S$. Now, we have

$$a_1^{k_1},\dots,a_n^{k_m},b_1^{l_1},\dots,b_n^{l_n}\in Q$$

for some $k_1,\dots,k_m,l_1,\dots,l_n\in\mathbb{N}$. By assumption and Q is an (m,n)-ideal of S, there exists $i\in\mathbb{N}$ such that

$$\begin{aligned}c^i &\in \{a_1^{k_1},\dots,a_n^{k_m},b_1^{l_1},\dots,b_n^{l_n}\}^mS\{a_1^{k_1},\dots,a_m^{k_m},b_1^{l_1},\dots,b_n^{l_n}\}^n\\ &\subseteq Q^mSQ^n\\ &\subseteq Q.\end{aligned}$$

Hence $c\in\sqrt{Q}$, and so $\sqrt{Q}$ is an (m,n)-quasi-ideal of S. ☐

Acknowledgment. The second author has received funding from the National Science, Research and Innovation Fund. The Third author was supported by the Research Fund for supporting a Lecturer to admit a student with a high potential to do research on the Expert Program Year 2018.

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