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Geometric construction of third order iterative method for solving nonlinear equations

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Abstract

In this paper, we present a new iterative method which is constructed from approximate cubic equation. The convergence analysis shows that the order of convergence is three. Some numerical examples are given to demonstrate the performance of the new method by comparing with some other well-known methods.

1 Introduction

Solving nonlinear equations by using a particular technique is still an active line of research in numerical analysis. In this paper, we consider iterative method to find a simple root of a nonlinear equation

$$f(x) = 0, \tag{1}$$

where, for an open interval D, $f\colon D \subseteq R \to R$ is a scalar function.

There are many techniques to solve equation (1), one of which is by using

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geometric construction. Considering the equation of a straight line

$$y = ax + b, \tag{2}$$

and then deriving the unknowns a and b by imposing the tangency conditions

$$y(x_n) = f(x_n),\ y'(x_n) = f'(x_n), \tag{3}$$

we get the tangent line

$$y(x) = f(x_n) + f'(x_n)(x - x_n), \tag{4}$$

to the graph of f at $(x_n, f(x_n))$. The zero of the tangent line is x_{n+1}, the next iterative is Newton's method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, \ldots \tag{5}$$

It is known that Newton's method converges quadratically to a simple root [1].

If we consider the parabola

$$x^2 + ax + by + c = 0, \tag{6}$$

and we impose the tangency conditions

$$y(x_n) = f(x_n),\ y'(x_n) = f'(x_n),\ y''(x_n) = f''(x_n), \tag{7}$$

we obtain

$$y(x) - f(x_n) = f'(x_n)(x - x_n) + \frac{f''(x_n)}{2}(x - x_n)^2. \tag{8}$$

The point of intersection of (8) with he x-axis, gives the irrational Halley's method [2,3]

$$x_{n+1} = x_n - \frac{2}{1 + \sqrt{1 + 2L(x_n)}}\frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, \ldots \tag{9}$$

$$\text{where } L(x_n) = \frac{f(x_n)f''(x_n)}{f'^2(x_n)}. \tag{10}$$

If we consider the parabola

$$ay^2 + y + bx + c = 0, \tag{11}$$

by using the imposition of conditions (7) and taking the intersection of (11) with the $x-$axis as the next iterate, gives the Chebyshev's method [3]

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}\Big(1 + \frac{L(x_n)}{2}\Big), \quad n = 0, 1, 2, \ldots \tag{12}$$

If we consider the hyperbola

$$axy + y + bx + c = 0, \tag{13}$$

by using the imposition of conditions (7) and taking the intersection of (13) with $x-$axis as the next iterate, gives the Halley's method [4]

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}\Big(\frac{2}{2 - L(x_n)}\Big), \quad n = 0, 1, 2, \ldots \tag{14}$$

In this paper, we construct iterative method by admitting geometric derivation with the help of possible cubic curve.

2 The methods

Let us consider the equation in x and y of the form

$$y = x^3 + ax^2 + bx + c, \tag{15}$$

and we impose the tangency conditions

$$y(x_n) = f(x_n),\ y'(x_n) = f'(x_n),\ y''(x_n) = f''(x_n), \tag{16}$$

then we get the tangent line

$$y(x) - f(x_n) = (x - x_n)^3 + \frac{f''(x_n)(x - x_n)^2}{2} + f'(x_n)(x - x_n). \tag{17}$$

The point x_{n+1} where the graph of (17) intersects with x-axis gives us $y(x_{n+1}) = 0$, then (17) becomes

$$-f(x_n) = (x_{n+1} - x_n)^3 + \frac{f''(x_n)(x_{n+1} - x_n)^2}{2} + f'(x_n)(x_{n+1} - x_n), \tag{18}$$

and rearranging (18) to be

$$x_{n+1} = x_n - \frac{f(x_n)}{(x_{n+1} - x_n)^2 - \frac{f''(x_n)(x_{n+1} - x_n)}{2} + f'(x_n)}, \tag{19}$$

we see that equation (19) is an implicit equation. In order to implement this problem, we replace x_{n+1} on the right-hand side of the equation by y_n where $y_n = x_n - \frac{f(x_n)}{f'(x_n)}$. Therefore,

$$x_{n+1} = x_n - \frac{f(x_n)}{(y_n - x_n)^2 - \frac{f''(x_n)(y_n - x_n)}{2} + f'(x_n)}. \tag{20}$$

The order of convergence of this method is analyzed in the next section.

3 Analysis of convergence

Theorem 1. For an open interval I, let $\alpha \in I$ be a simple zero of a sufficiently differentiable function $f\colon I \subseteq R \to R$. If x_0 is sufficiently close to α, then the method defined by (20) converges cubically to α.

Proof. Let $e_n = x_n - \alpha$. Using Taylor expansion around $x = x_n$ and taking $f(\alpha) = 0$, we obtain

$$f(x_n) = f'(\alpha)[e_n + C_2 e_n^2 + C_3 e_n^3 + O(e_n^4)], \tag{21}$$

where $C_k = \frac{f^{(k)}(\alpha)}{k! f'(\alpha)}$, $k = 2, 3, 4, \ldots$ Moreover, we have

$$f'(x_n) = f'(\alpha)[1 + 2C_2 e_n + 3C_3 e_n^2 + 4C_4 e_n^3 + O(e_n^4)], \tag{22}$$

and

$$f''(x_n) = f'(\alpha)[2C_2 + 6C_3 e_n + 12C_4 e_n^2 + 20C_5 e_n^3 + O(e_n^4)]. \tag{23}$$

Dividing (21) by (22), we obtain

$$\frac{f(x_n)}{f'(x_n)} = e_n - C_2 e_n^2 + 2(c_2^2 - c_3) e_n^3 + O(e_n^4), \tag{24}$$

and hence,

$$y_n = \alpha + C_2 e_n^2 - 2(c_2^2 - c_3) e_n^3 + O(e_n^4). \tag{25}$$

From (25) and $x_n = e_n + \alpha$, we obtain

$$y_n - x_n = -e_n + C_2 e_n^2 - 2(C_2^2 - C_3) e_n^3 + O(e_n^4). \tag{26}$$

From (22), (23) and (26), we obtain

$$(y_n - x_n)^2 + \frac{f''(x_n)(y_n - x_n)}{2} + f'(x_n) = 1 + C_2 e_n + (1 + C_2^2)e_n^2 + (-2C_4 + 5C_2C_3 - 2C_2^3 - 2C_2)e_n^3 + O(e_n^4). \quad (27)$$

Substituting (21) and (27) in x_{n+1} in (20), we obtain

$$e_{n+1} = (1 + C_2^2 - C_3)e_n^3 + O(e_n^4). \quad (28)$$

This means that the method defined by (20) is cubically convergent. We see that the new method has the same order as Chebyshev's method and Halleys method. Per iteration, the new method required three evaluations of the function. We consider the definition of efficiency index [5] as $p^{1/w}$, where p is the order of the method and w is the number of functions evaluations per iteration required by the method. The new method has an efficiency index equal to $\sqrt[3]{3} \approx 1.442$, which is equal to the those of the Chebyshevs method and Halleys method and better than Newtons method $\sqrt{2} \approx 1.414$.

4 Numerical Examples

In this section, we employ formula (20) to solve some nonlinear equations and compare the results with Newton's method (NM), Chebyshev's method (CM) and Halley's method (HM). In Table 1, we display the number of iteratives (IT) required such that $|f(x_n)| \leq 10^{-14}$.

We use the following functions and their approximate roots as a test.
$f_1(x) = x^3 + 4x^2 - 10, \quad \alpha = 1.3652300134140969.$
$f_2(x) = x^2 - e^x - 3x + 2, \quad \alpha = 0.25753028543986084.$
$f_3(x) = xe^{x^2} - \sin^2 x + 3\cos x + 5, \quad \alpha = -1.2076478271309189.$
$f_4(x) = \cos(x) - x, \quad \alpha = 0.73908513321516067.$
$f_5(x) = \sin^2 x - x^2 + 1, \quad \alpha = 1.4044916482153411.$
$f_6(x) = x^2 + \sin(x/5) - 1/4, \quad \alpha = 0.4099920179891371.$
$f_7(x) = x^2 + \sin^2 x + e^{x^2 \cos x \sin x} - 28, \quad \alpha = 4.6221041635528383.$
$f_8(x) = x^{-x} + \cos(x), \quad \alpha = 1.7461395304080124.$

Table 1 : Comparison of various iterative methods

$function$	x_0	NM	CM	HM	$equation$ 20
f_1	0.8	8	4	4	4
f_2	0	4	3	3	3
f_3	-1	8	4	3	3
f_4	1	4	3	3	3
f_5	1.2	5	3	3	3
f_6	0.15	5	5	4	4
f_7	3.5	8	4	3	3
f_8	1.5	4	3	3	3

From the results in Table 1, the methods HM and equation (20) have the same results and they appear more rapidly convergent than the method CM for some cases. Moreover, for all of the cases, the methods HM, CM and equation (20) converge more rapidly than NM.

5 Conclusion

In this paper, we presented a new iterative method constructed by approximating the cubic equation $y = x^3 + ax^2 + bx + c$. We proved that the order of convergence of the method is three. We compared the new method to Newton's method, Chebyshev's method and Halley's method and we found that the new method yields better performance than Newton's method, and is comparable to Halley's method and Chebyshev's method.

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