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The Duffing Equation-A Trigonometric Point Of View

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Abstract

In this paper, we give an approximate solution to the undamped and unforced Duffing equation. We illustrate the obtained results with Physics applications.

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1 Introduction

The Duffing oscillator equation

$$\ddot{u} + pu + qu^3 = 0 \tag{1.1}$$

is well known and it has many important applications in Physics and Engineering (in the case when $q = 0$, we obtain the linear ode $\ddot{u} + pu = 0$ whose solution is expressed through trigonometric functions). Salas, Castillo, and Pinzon [1] gave a solution of the Duffing equation by means of elementary functions. The aim of this paper is to show how the Duffing equation can be approximated by means of trigonometric functions in a reasonable way. The general solution to the Duffing equation (1.1) is given by

$$u(t) = c_0 \mathrm{cn}\left(\sqrt{p + qc_0^2}t + c_1, \frac{qc_0^2}{2(p + qc_0^2)}\right). \tag{1.2}$$

The values of the constants c_0 and c_1 are determined from the initial conditions

$$u(0) = u_0 \text{ and } u'(0) = \dot{u}_0. \tag{1.3}$$

In order to solve (1.1) in an elementary way, we consider the equation

$$u(t) = \frac{v(t)}{\sqrt{1 + \lambda v^2(t)}}, \tag{1.4}$$

where $v = v(t)$ is the solution to some *linear* ode $\ddot{v} + \omega^2 v = 0$ and λ is some constant to be determined. The ode (1.1) may be written in the form

$$\begin{array}{l} \dot{u}^2 + pu^2 + \frac{q}{2}u^4 - C = 0, \\ C = \dot{u}_0^2 + pu_0^2 + \frac{q}{2}u_0^4. \end{array} \tag{1.5}$$

Plugging equation (1.4) into the ode (1.5) and using the identity

$$\dot{v}^2 = D - \alpha v^2,\ D = \dot{v}_0^2 + \alpha v_0^2 \tag{1.6}$$

we get

$$\begin{array}{l} \dot{u}^2 + pu^2 + \frac{q}{2}u^4 - C = \\ \frac{1}{2(\lambda v(t)^2+1)^3}\left[\begin{array}{c} -2(C - D) + 2\left(p - 3C\lambda - \omega^2\right)v(t)^2 + \\ \left(q - 6C\lambda^2 + 4p\lambda\right)v(t)^4 + \lambda\left(-2C\lambda^2 + 2p\lambda + q\right)v(t)^6 \end{array}\right] \end{array} \tag{1.7}$$

We choose the values of D, ω and λ so that

$$C - D = 0,\ p - 3C\lambda - \omega^2 = 0 \text{ and } q - 6C\lambda^2 + 4p\lambda = 0. \tag{1.8}$$

From system (1.8) we obtain

$$\begin{array}{l} D=C=\dot{u}_0^2+pu_0^2+\frac{q}{2}u_0^4,\ \omega=\pm\sqrt{p-3C\lambda}.\\ v_0=\pm\frac{u_0}{\sqrt{1-u_0^2\lambda}},\ \dot{v}_0=\pm\frac{\dot{u}_0}{\left(1-u_0^2\lambda\right)^{3/2}}.\\ 2u_0^2\left(2pu_0^2+qu_0^4+2\dot{u}_0^2\right)\lambda^3-\left(8pu_0^2+3qu_0^4+6\dot{u}_0^2\right)\lambda^2+4p\lambda+q=0. \end{array} \tag{1.9}$$

On the other hand, the solution to the i.v.p.

$$\ddot{v}+\omega^2 v=0,\ v(0)=v_0 \text{ and } v'(0)=\dot{v}_0 \tag{1.10}$$

reads

$$v(t)=\frac{\dot{v}_0}{\omega}\sin(\omega t)+v_0\cos(\omega t). \tag{1.11}$$

Thus, an approximate analytical solution to the i.v.p.

$$\ddot{u}+pu+qu^3=0,\ u(0)=u_0 \text{ and } u'(0)=\dot{u}_0 \tag{1.12}$$

is given by

$$u(t)=\frac{\frac{\dot{v}_0}{\omega}\sin(\omega t)+v_0\cos(\omega t)}{\sqrt{1+\lambda[\frac{\dot{v}_0}{\omega}\sin(\omega t)+v_0\cos(\omega t)]^2}}. \tag{1.13}$$

The values of ω, v_0, $\dot{v}_0$ and λ can be determined from (1.9).
Remark 1. We can obtain a more precise approximation assuming thee equation

$$u(t)=\frac{v(t)}{\sqrt{1+\lambda\cdot v^2(t)+\mu\cdot v^4(t)}}. \tag{1.14}$$

In this case, $v(t)=\frac{\dot{v}_0}{\omega}\sin(\omega t)+v_0\cos(\omega t)$, where

$$\begin{array}{l} v_0=\pm\sqrt{\frac{1-u_0^2\lambda\pm\sqrt{u_0^4(\lambda^2-4\mu)-2u_0^2\lambda+1}}{2u_0^2\mu}},\ \ \dot{v}_0=-\frac{\dot{u}_0\left(v_0^2\lambda+v_0^4\mu+1\right)^{3/2}}{v_0^4\mu-1},\ \ u_0\neq 0.\\ \omega=\sqrt{p-3C\lambda},\mu=\frac{4p\lambda+q-6C\lambda^2}{10C},C=pu_0^2+\frac{qu_0^4}{2}+\dot{u}_0^2.\\ 31\left(2pu_0^2+qu_0^4+2\dot{u}_0^2\right)^2\lambda^3-62p\left(2pu_0^2+qu_0^4+2\dot{u}_0^2\right)\lambda^2+\\ (32p^2-14pqu_0^2-7q^2u_0^4-14q\dot{u}_0^2)\lambda+8pq=0. \end{array} \tag{1.15}$$

In the case when $u_0=0$ and $\dot{u}_0\neq 0$ the parameter values will be

$$\begin{array}{l} v_0=0,\ \dot{v}_0=\dot{u}_0,\ \omega=\sqrt{p-3\dot{u}_0^2\lambda},\ \mu=\frac{4p\lambda+q-6\dot{u}_0^2\lambda^2}{10\dot{u}_0^2}.\\ 62\dot{u}_0^4\lambda^3-62p\dot{u}_0^2\lambda^2+\left(16p^2-7q\dot{u}_0^2\right)\lambda+4pq=0. \end{array} \tag{1.16}$$

Remark 2. An approximate trigonometric solution to the i.v.p

$$\ddot{x}+f(x)=0,\ x(0)=x_0,\ x'(0)=0 \tag{1.17}$$

is given by

$$x_{\text{trigo}}(t) = \frac{\sqrt{2}\sqrt{2p+qx_0^2}x_0 \cos\left(\sqrt{p+\frac{3q\left(px_0^2+\frac{qx_0^4}{2}\right)}{2\left(2p+qx_0^2\right)}}t\right)}{\sqrt{4p+3qx_0^2-qx_0^2\cos^2\left(\sqrt{p+\frac{3q\left(px_0^2+\frac{qx_0^4}{2}\right)}{2\left(2p+qx_0^2\right)}}t\right)}}. \tag{1.18}$$

The solution is periodic with period $T = 2\pi/\left(\sqrt{p+\frac{3q\left(px_0^2+\frac{qx_0^4}{2}\right)}{2\left(2p+qx_0^2\right)}}\right)$.

Remark 3. Making use of equation (1.14), we obtain the following approximate trigonometric solution to the i.v.p. n(1.17) :

$$z_{\text{trigo}}(t) = \frac{v_0 \cos(wt)}{\sqrt{1+v_0^2\lambda\cos^2(wt)+\mu\cos^4(wt)}}, \text{ where} \tag{1.19}$$

$$\begin{aligned} &w = \tfrac{1}{4}\sqrt{5x_0 - \tfrac{65x_0^3\lambda}{2}},\ \mu = \tfrac{v_0^4\left(3x_0^2\lambda\left(4-13x_0^2\lambda\right)+7\right)}{65x_0^4}. \\ &v_0 = \pm\frac{\sqrt{2}x_0}{\sqrt{1-\lambda x_0^2 \pm \frac{\sqrt{\lambda x_0^2\left(221\lambda x_0^2-178\right)+37}}{\sqrt{65}}}}, \text{ Period}= \tfrac{2\pi}{w}. \\ &5239x_0^6\lambda^3 - 2418x_0^4\lambda^2 - 349x_0^2\lambda + 168 = 0. \end{aligned} \tag{1.20}$$

2 Applications.

2.1 Nonlinear Conservative Oscillators.

Conservative single-degree-of-freedom nonlinear oscillators are modeled by second-order autonomous ordinary differential equations of the form (1.17), where x and t are the non-dimensional displacement and time, respectively. Here, $f(x)$ is the nonlinear function and A is the initial oscillation amplitude. We assume that the nonlinear function $f(x)$ is an odd function and satisfies $xf(x) > 0$ for $x \in [-A, A]$, $x \neq 0$, where A is the oscillation amplitude. The conservative nonlinear restoring force is given by $F(x) = -f(x)$. The motion is assumed to be periodic and the problem is to determine the angular frequency of oscillation, ω, and corresponding solution, x, as a function of time, t, the system parameters and the oscillation amplitude A [2]. The function $f(x)$ may be approximated by means of a cubic or quintic polynomial by using either Chebyshev polynomials or by means of minimization techniques.

Example. Let $f(x) = x\,|x|$. We have $f(x) \approx \frac{5A}{16}x + \frac{35}{48A}x^3$. An approximate trigonometric solution is obtained from (1.18) with $p = \frac{5}{16}A$ and $q = \frac{35}{48A}$. Let $x_0 = A$. Then

$$x_{\text{trigo}}(t) = \frac{\sqrt{\frac{26}{33}}A\cos\left(\frac{1}{8}\sqrt{55A}t\right)}{\sqrt{1-\frac{7}{33}\cos^2\left(\frac{1}{8}\sqrt{55A}t\right)}}.$$

Using formulas (1.19)-(1.20) we get a more accurate approximation. Let us compare these solutions with the numerical solution using Runge-Kutta algorithm. Let $A = 1$. The approximate value for the period is $T = 16\pi/\sqrt{55}$. The two approximate solutions are (See Fig 1-a and Fig 1-b):

$$x_{\text{trigo}}(t) = \frac{0.887625\cos(0.927025t)}{\sqrt{1.-0.212121\cos^2(0.927025t)}},\ \text{Error on } [-T,T] = 0.02.$$

$$z_{\text{trigo}}(t) = \frac{0.895532\cos(0.918645t)}{\sqrt{1-0.20981\cos^2(0.918645t)+0.011788\cos^4(0.918645t)}},\ \text{Error on } [-T,T] = 0.007.$$

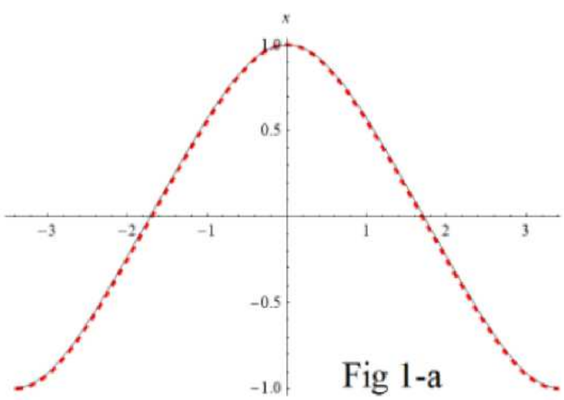

Fig 1-a

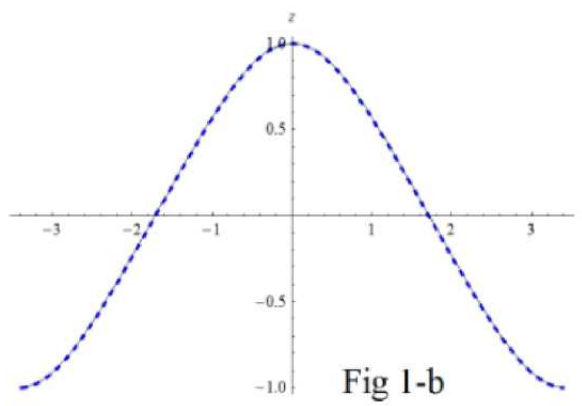

Fig 1-b

References

[1] Alvaro H. Salas, Jairo E. Castillo, Jorge E. Pinzon, Solution of Duffing's differential equation by means of elementary functions and its application to a non-linear electrical circuit, Int. j. Math. Comput. Sci., **17,** no. 1, 277–287.

[2] A. Beléndez et al., Solutions for Conservative Nonlinear Oscillators Using an Approximate Method Based on Chebyshev Series Expansion of the Restoring Force, Acta Physica Polonica A, **130,** no. 3, (2016), 667–678.

[3] Alvaro H. Salas, Samir A. El-Tantawy, Analytical Solutions of Some Strong Nonlinear Oscillators, IntechOpen, https://www.intechopen.com/online-first/76936