International Journal of Mathematics and
Computer Science, **17**(2022), no. 2, 679–683

Analytical Solution to the Lagrange Top

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(Received October 8, 2021, Accepted November 9, 2021)

Abstract

In this article, the exact periodic and bounded solutions for the motions of Lagrange top with initial conditions are obtained. These solutions are expressed in terms of the Weierstrass elliptic function.

1 Introduction

Let us consider a symmetric Lagrange top as indicated in Figure 1.

Key words and phrases: Helmholtz equation, Duffing equation, analytical solution, Weiesrstrass elliptic function, Weierstrass invariants, simulation, Wolfram Mathematica.
AMS (MOS) Subject Classifications: 34Axx, 34A05, 34A34, 34A25, 33E05.
ISSN 1814-0432, 2022, http://ijmcs.future-in-tech.net

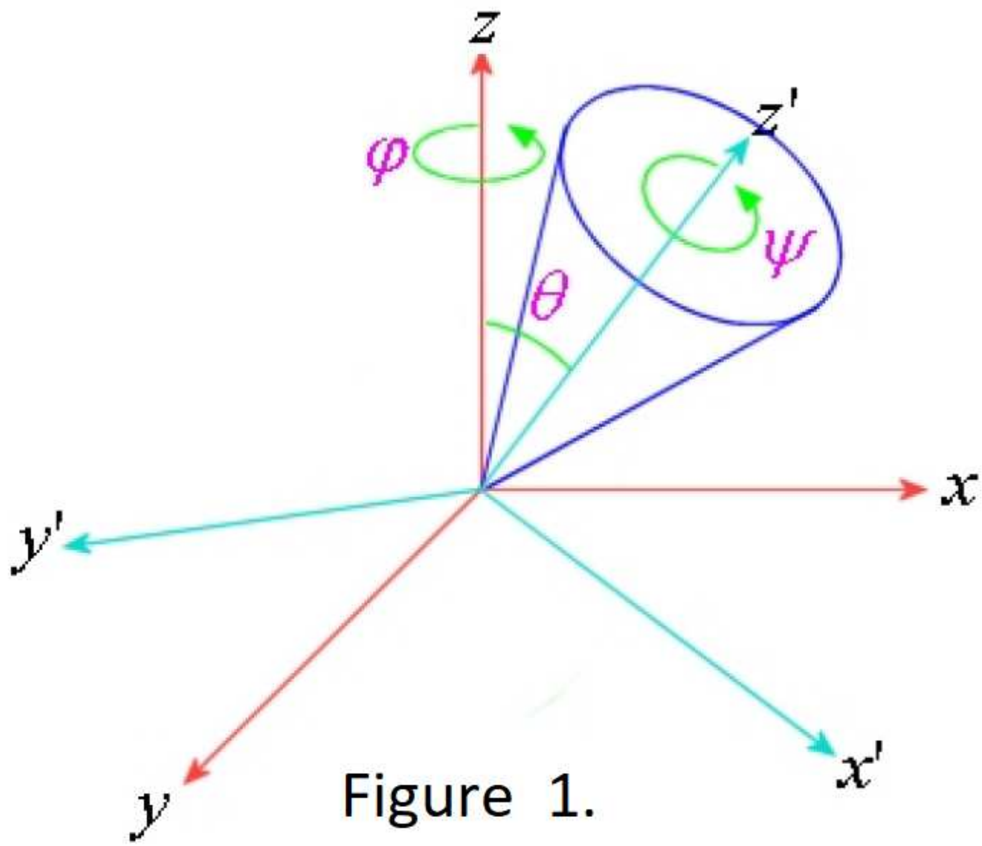

Figure 1.

Assume that it is given in canonical form so that its inertia moments are such that $I_1 = I_2 \neq I_3$ and the top is leaning on its peak O so that O is a fixed point. This system has three degrees of freedom since O coordinates are fixed. In order to study the behavior of this system we will set Euler angles φ, ψ, θ as generalized coordinates. See Figure 1. This problem admits analytical solution and it is known as Euler's problem [1].The aim of this work is to find the closed form solution to this problem in terms of Weiesrtrass elliptic function.

The potential energy of the top reads $U = Mgl\cos\theta$, where l represents the distance form the point O to center of mass of the top. The kinetic energy is obtained by adding the translation kinetic energy of mass center

$E_t = \frac{M}{2}(\dot{\theta}^2 l^2 + \dot{\varphi}^2 l^2 \sin^2\theta)$ and the rotational kinetic energy is given by

$$E_r = \frac{I_1}{2}(\dot{\varphi}^2 \sin^2\theta + \dot{\theta}^2) + \frac{I_3}{2}(\dot{\varphi}\cos\theta + \dot{\psi})^2.$$

Lagrange function is given by the following expression

$$L(\varphi, \psi, \theta, \dot{\varphi}, \dot{\psi}, \dot{\theta}) = \frac{I_1 - Ml^2}{2}(\dot{\theta}^2 + \dot{\varphi}^2 \sin^2\theta) + \frac{I_3}{2}(\dot{\varphi}\cos\theta + \dot{\psi})^2 - MglCos\theta \tag{1.1}$$

Since ψ and φ are cyclic coordinates, $P_\psi = \frac{\partial L}{\partial\dot{\psi}}$ and $P_\varphi = \frac{\partial L}{\partial\dot{\varphi}}$ are the motion integrals. Taking into account the Euler-Lagrange equation [2], the equations of motion have the following form:

$$\begin{pmatrix} I_3(\dot{\varphi}\cos\theta + \dot{\psi}) = M_3 \\ (I'\sin^2\theta + I_3\cos^2\theta)\dot{\varphi} + I_3\dot{\psi}\cos\theta = P_\varphi = M_z \\ I'\ddot{\theta} = I'\dot{\varphi}^2\sin\theta\cos\theta - I_3(\dot{\psi} + \dot{\varphi}\cos\varphi)\dot{\varphi}\sin\theta + Mgl\sin\epsilon \end{pmatrix}, \tag{1.2}$$

where $I' = I_1 + Ml^2$ and M_3, M_z are constants. In order to integrate the system (1.2), we will make use of the fact that the mechanical energy $E = K + U$ of the system is conserved and so

$$E = \frac{I'}{2}(\dot{\theta}^2 + \dot{\varphi}^2\sin^2\theta) + \frac{I_3}{2}(\dot{\varphi}\cos\theta + \dot{\psi})^2 + MglCos\theta \tag{1.3}$$

From the first pair of equations of (1.2), we express $\dot{\varphi}$ and $\dot{\psi}$ in terms of θ. Since $I_3(\dot{\varphi}\cos\theta + \dot{\psi}) = M_3$ and $I'\dot{\varphi}\sin^2\theta + M_3\cos\theta = M_z$, we obtain

$$\dot{\varphi} = \frac{M_z - M_3\cos\theta}{I'\sin^2\theta} \tag{1.4}$$

$$\dot{\psi} = \frac{M_3}{I_3} - \cos\theta\frac{M_z - M_3\cos\theta}{I'\sin^2\theta} \tag{1.5}$$

Using (1.4) and (1.5) with (1.3), leads to

$$E = \frac{I'\dot{\theta}^2}{2} + \frac{M_3^2}{2I_3} + \frac{1}{2I'}\frac{(M_z - M_3\cos\theta)^2}{\sin^2\theta} + Mgl\cos\theta \tag{1.6}$$

Let the effective potential be

$$U_{\text{eff}} = \frac{1}{2I'}\frac{(M_z - M_3\cos\theta)^2}{\sin^2\theta} + Mgl\cos\theta \tag{1.7}$$

and

$$E' = \frac{I'\dot{\theta}^2}{2} + U_{eff}. \tag{1.8}$$

Consequently,

$$E = E' + \frac{M_3^2}{2I_3} \tag{1.9}$$

In view of (1.8) and (1.7), we have

$$\dot{\theta}^2\sin^2\theta = \frac{2}{I'}\{E'\sin^2\theta - \frac{1}{2I'}((M_z - M_3\cos\theta)^2 - Mgl\sin^2\theta\}. \tag{1.10}$$

Let $u = \cos\theta$, $du = -\sin\theta d\theta$. Then, from (1.10),

$$\dot{u}^2 = \left(\frac{2}{I'}E' - \frac{2Mgl}{I'}u\right)(1-u^2) - \left(\frac{M_z}{I'} - \frac{M_3}{I'}u\right)^2 \tag{1.11}$$

Introducing the notations

$$a = \frac{M_3}{I'}\ , b = \frac{M_z}{I'}\ , \alpha = \frac{2}{I'}E'\ \ , \beta = \frac{2Mgl}{I'} \tag{1.12}$$

expression (1.11) takes the form

$$\dot{u}^2 = (1-u^2)(\alpha - \beta u) - (b - au)^2 \tag{1.13}$$

In view of (1.13), $\dot{u}^2$ is a cubic polynomial in u as follows:

$$\dot{u}^2 = \beta u^3 - (\alpha + a^2)u^2 + (2ab - \beta)u + (\alpha - b^2) \tag{1.14}$$

Taking the derivative w.r.t. t and taking into account $\dot{u} \neq 0$, we get the Helmholtz oscillator equation

$$\ddot{u} - n + pu - qu^2 = 0, \tag{1.15}$$

where

$$n = \frac{2ab - \beta}{2}\ ,\ p = (\alpha + a^2), \quad q = \frac{3\beta}{2}. \tag{1.16}$$

Since $u = \cos\theta$, $-1 \leq u \leq 1$, $-\pi \leq \theta \leq \pi$, we will solve the equation (1.15) for the initial conditions

$$u(t=0) = u_0 = \cos\theta_0 = 1 \text{ and } \dot{u}_0 = 0 \tag{1.17}$$

2 Analytical Solution

In this section we will solve the initial value problem

$$\dot{u}^2 = \beta u^3 - (\alpha + a^2)u^2 + (2ab - \beta)u + (\alpha - b^2),\ u(0) = u_0 \text{ and } u'(0) = \dot{u}_0 \quad (2.18)$$

Let

$$u(t) = A + B\wp(t - t_0; g_2, g_3), \quad (2.19)$$

where $\wp(t) = \wp(t - t_0; g_2, g_3)$ is the Weierstrass elliptic function. This function satisfies the nonlinear ode $\wp'(t)^2 = 4\wp^3(t) - g_2\wp(t) - g_3$. The numbers g_2 and g_3 are called the Weierstrass elliptic invariants.

Inserting equation (2.19) into (2.18), gives

$$\begin{aligned} &-aA^2 - 2aAb + b^2 - B^2 g_3 - \alpha - A^2\alpha + A\beta - A^3\beta - \\ &B\left(2aA + 2ab + Bg_2 + 2A\alpha - \beta + 3A^2\beta\right)\wp(t) - \\ &B^2(a + \alpha + 3A\beta)\wp(t)^2 - B^2(-4 + B\beta)\wp(t)^3 = 0. \end{aligned}$$

Equating the coefficients of $\wp^j(t)$ to zero, gives an algebraic system. Solving it using the initial conditions, we obtain the following solutio :

$$\begin{aligned} &A = \tfrac{-a-\alpha}{3\beta},\ B = \tfrac{4}{\beta},\ g_2 = \tfrac{1}{12}\left(a^2 + \alpha^2 + 2\alpha a - 6ab\beta + 3\beta^2\right). \\ &g_3 = \tfrac{1}{432}\left(-2a^3 - 2\alpha^3 - 6\alpha a^2 + 18a^2 b\beta - 6\alpha^2 a - 9a\beta^2 - 36\alpha\beta^2 + 18\alpha ab\beta + 27b^2\beta^2\right). \\ &t_0 = \pm\wp^{-1}\left(\begin{array}{c} \tfrac{1}{12}\left(a + \alpha + 3u_0\beta\right); \tfrac{1}{12}\left((a+\alpha)^2 + 3\beta^2 - 6ab\beta\right), \\ \tfrac{1}{432}\left(-2(a+\alpha)^3 + 18ab\beta(a+\alpha) - 9\left(-3b^2 + a + 4\alpha\right)\beta^2\right) \end{array}\right). \end{aligned}$$

Finally, the solution to the top problem reads

$$\theta(t) = \cos^{-1}\left(-\frac{1}{3\beta}\left(a + \alpha - 12\wp\left(\begin{array}{c} t - t_0; \tfrac{1}{12}\left((a+\alpha)^2 + 3\beta^2 - 6ab\beta\right), \\ \tfrac{1}{432}\left(-2(a+\alpha)^3 + 18ab\beta(a+\alpha) - 9\left(-3b^2 + a + 4\alpha\right)\beta^2\right) \end{array}\right)\right)\right).$$

References

[1] A. N. Nayfeth, D. T. Mook, Non-linear oscillations, John Wiley, New York, 1973.

[2] Claude Gignoux · Bernard Silvestre-Brac, Solved Problems in Lagrangian and Hamiltonian Mechanics, Springer Science+Business Media B. V., 2009.