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On a study of a class of functions associated with a multiplier transformation

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Abstract

Making use of extended multiplier transformation, we introduce a unified class of analytic functions with negative coefficients. Moreover, we establish a necessary and sufficient condition for those functions in a subclass of it.

1 Introduction and Preliminaries

Denote by $\mathbb{U}$ the unit disc of the complex plane, $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ and $H(\mathbb{U})$ the space of holomorphic function $\mathbb{U}$.
Let A denote the class of analytic functions

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

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in the (open) unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ (Note the normalization $f(0) = 0$ and $f'(0) = 1$). The subfamily T consists of functions of the form

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k, \quad a_k \geq 0 \tag{1.2}$$

In addition, define the identity function in A as

$$e(z) = z \tag{1.3}$$

Alamoush and Darus [1] defined the differential operator $D^{n,\lambda}_{\alpha,\beta,\mu_1,\mu_2}$ given by

$$D^{n,\lambda}_{\alpha,\beta,\mu_1,\mu_2} f(z) = z + \sum_{k=2}^{\infty} \Upsilon^m a_k z^k \tag{1.4}$$

where

$$\Upsilon = \left(\frac{a + (\alpha-\beta)(\lambda+\mu_2-\mu_1)(k-1)+b}{a+b}\right)^m a_k z^k \tag{1.5}$$

and $a, b \geq 0$, $a + b \neq 0$, $\alpha > \beta \geq 0$, $\lambda > \mu_2 \leq \mu_1, m \in \mathbb{N}_0$.

The operator generalizes some known operators in the literature. For $0 \leq \delta \leq 1$, $\alpha > \beta \geq 0$, $\lambda > \mu_2 \leq \mu_1$, $\xi \geq 1$, $0 \leq \eta \leq 1, m \in \mathbb{N}_0$ and $f \in A$, we define a new generalized subclass $SI_m(\xi, \eta, \delta)$ satisfying the following inequality:

$$\Re\left(\frac{z\left(D^{m,\lambda}_{\alpha,\beta,\mu_2,\mu_1} f(z)\right)'}{\eta\left(D^{m,\lambda}_{\alpha,\beta,\mu_2,\mu_1} f(z)\right) + \xi e(z)}\right) > \delta, \quad for\ z \ \in\ U \tag{1.6}$$

where $e(z)$ is defined by (1.3). This extends classes studied in [2] and [3]. For example, class $SI_m(1, 0, \delta) \equiv SI_m(\delta)$. Also, for $\alpha - \beta = 1$, $a + b = 1$, $\lambda = \mu_1$ and $\mu_2 = \frac{\alpha+\lambda}{\gamma+\beta}$ in (1.6), we have $SI_m(0, 1, \delta) \equiv K^m_{\alpha,\lambda,\gamma,\beta}(n, \delta)$. Moreover, the work generalizes the class of function studied in [1] and [5]. The class $SI_m(\xi, \eta, \delta)$ is introduced in the study. Furthermore, the closure properties based on certain integral transforms for functions $f \in SI_m(\xi, \eta, \delta)$ were discussed.

2 Main Results

2.1 Coefficient Estimate

Theorem 2.1. *Let $0 \leq \delta \leq 1$, $\xi \geq 1$ and $0 \leq \eta \leq 1$. Suppose the function $f(z)$ is defined by (1.1). Then $f \in SI_m(\xi, \eta, \delta)$ if and only if*

$$\sum_{k=2}^{\infty}[k - \eta(2-\delta)]\Upsilon^m a_k \leq \eta(2-\delta) + \xi - 1. \tag{2.7}$$

Proof: Assume the inequality (2.7) holds and let $|z| = 1$. Then

$$\begin{aligned}\left|\frac{z\left(D^{m,\lambda}_{\alpha,\beta,\mu_2,\mu_1} f(z)\right)'}{\eta\left(D^{m,\lambda}_{\alpha,\beta,\mu_2,\mu_1} f(z)\right) + \xi e(z)}\right| &\leq \left|\frac{1 + \sum_{k=2}^{\infty} k\Upsilon^m a_k z^{k-1} - \eta - \sum_{k=2}^{\infty} \eta\Upsilon^m a_k z^{k-1} - \xi}{\eta + \sum_{k=2}^{\infty} \eta\Upsilon^m a_k z^{k-1} + \xi}\right| \\ &= \frac{(1-\eta-\xi) + \sum_{k=2}^{\infty} k\Upsilon^m |a_k| - \sum_{k=2}^{\infty} \eta\Upsilon^m |a_k|}{(\eta+\xi) + \sum_{k=2}^{\infty} \eta\Upsilon^m |a_k|} \\ &\leq 1 - \delta\end{aligned}$$

which implies (2.7). Hence $f \in SI_m(\xi, \eta, \delta)$. Conversely, assume that $f \in SI_m(\xi, \eta, \delta)$. Then

$$\begin{aligned}&\Re\left(\frac{z\left(D^{m,\lambda}_{\alpha,\beta,\mu_2,\mu_1} f(z)\right)'}{\eta\left(D^{m,\lambda}_{\alpha,\beta,\mu_2,\mu_1} f(z)\right) + \xi e(z)}\right) \\ =\ &\Re\left(\frac{1 + \sum_{k=2}^{\infty} k\Upsilon^m a_k z^{k-1} - \eta - \sum_{k=2}^{\infty} \eta\Upsilon^m a_k z^{k-1} - \xi}{\eta + \sum_{k=2}^{\infty} \eta\Upsilon^m a_k z^{k-1} + \xi}\right) \\ \geq\ &\delta.\end{aligned}$$

Choose the value of z on the real axis so that $\frac{z\left(D^{m,\lambda}_{\alpha,\beta,\mu_2,\mu_1} f(z)\right)'}{\eta\left(D^{m,\lambda}_{\alpha,\beta,\mu_2,\mu_1} f(z)\right)+\xi e(z)}$ is real and simplify to clear the denominator in (2.8) with $z \to 1^-$ so that

$$\sum_{k=2}^{\infty}[k - \eta(2-\delta)]\Upsilon^m a_k \leq \eta(2-\delta) + \xi - 1.$$

Corollary 2.1. *If $f \in SI_m(\xi, \eta, \delta)$, then*

$$|a_k| \leq \frac{\eta(2-\delta) + \xi - 1}{[k - \eta(2-\delta)]\Upsilon^m} z^k.$$

Equality holds for

$$f(z) = z + \frac{\eta(2-\delta)+\xi-1}{[k-\eta(2-\delta)]\Upsilon^m} z^k.$$

Corollary 2.2. *[2] Let* $\alpha - \beta = 1$, $a + b = 1$, $\lambda = \mu_1$ *and* $\mu_2 = \frac{\alpha+\lambda}{\gamma+\beta}$ *for* $\alpha, \beta, \lambda, \gamma \geq 0$. *Then* $f \in SI_m(0,1,\delta)$ *is given by*

$$\sum_{k=2}^{\infty}(k-2+\delta)\left[\left(\frac{\alpha+\lambda}{\gamma+\beta}\right)[(k-1)+1]\right]^m a_k \leq 1-\delta.$$

Theorem 2.2. *The set* $SI_m(\xi,\eta,\delta)$ *is convex.*

Proof: Let

$$f_j(z) = z + \sum_{k=2}^{\infty} a_{k,j} z^k \ (z \in \mathbb{U};\ j = 1,2), \quad z \in \mathbb{U} \tag{2.8}$$

be in the class $SI_m(\xi,\eta,\delta)$. We need to show that the function

$$h(z) = \eta_1 f_1 + \eta_2 f_2$$

is in the class $SI_m(\xi,\eta,\delta)$ with η_1 and η_2 non-negative such that $\eta_1 + \eta_2 = 1$. Since

$$h(z) = z + \sum_{k=2}^{\infty}\left(\eta_1 a_{k_1} + \eta_2 a_{k_2}\right) z^k \ for\ z \in \mathbb{U},$$

$$D^{m,\lambda}_{\alpha,\beta,\mu_2,\mu_1} h(z) = z + \sum_{k=2}^{\infty} \Upsilon^n \left(\eta_1 a_{k_1} + \eta_2 a_{k_2}\right) z^k, \quad for\ z \in \mathbb{U}. \tag{2.9}$$

Differentiating (2.9) with respect to z, we have

$$\left(D^{m,\lambda}_{\alpha,\beta,\mu_2,\mu_1} h(z)\right)' = 1 + \sum_{k=2}^{\infty} \Upsilon^n \left(\eta_1 a_{k_1} + \eta_2 a_{k_2}\right) k z^{k-1} \qquad z \in \mathbb{U}.$$

Thus

$$\Re\left(D^{m,\lambda}_{\alpha,\beta,\mu_2,\mu_1} f(z)\right)' = 1 + \Re\left\{\eta_1 \sum_{k=2}^{\infty} k\Upsilon^m z^{k-1}\right\} + \Re\left\{\eta_2 \sum_{k=2}^{\infty} k\Upsilon^m z^{k-1}\right\}. \tag{2.10}$$

Taking into account $f_1, f_2 \in SI_m(\xi, \eta, \delta)$, we deduce

$$\Re\left\{\eta_k \sum_{k=2}^{\infty} k\Upsilon^m z^{k-1}\right\} > \eta_k(\delta - 1), \quad for \quad k = 1, 2. \tag{2.11}$$

Using (2.11) we get from (2.10)

$$\Re\left(D^{m,\lambda}_{\alpha,\beta,\mu_2,\mu_1} f(z)\right)' > 1 + \eta_1(\delta - 1) + \eta_2(\delta - 1) = \delta.$$

Hence

$$\Re\left(D^{m,\lambda}_{\alpha,\beta,\mu_2,\mu_1} f(z)\right)' > \delta.$$

Therefore the class $SI_m(\xi, \eta, \delta)$ is convex.

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