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On the Norms of Circulant Matrics with the Gaussian Pell Numbers

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Abstract

In this paper, we give the formula for the Euclidean norm of circulant matrix with the Guassian Pell numbers. Moreover, we find the upper and lower bounds of the spectral norm of this matrix. Furthermore, we obtain the silver ratio in Gaussian Pell numbers.

1 Introduction

We say that an $n \times n$ matrix C is a circulant matrix if it is of the form

$$C = \begin{bmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-1} \\ c_{n-1} & c_0 & c_1 & \cdots & c_{n-2} \\ c_{n-2} & c_{n-1} & c_0 & \cdots & c_{n-3} \\ \vdots & \vdots & \vdots & & \vdots \\ c_1 & c_2 & c_3 & \cdots & c_0, \end{bmatrix}$$

where the i, j entry of C is $c_{j-i\,(\mathrm{mod}\, n)}$. Denote the circulant matrix C as $C = Cir(c_0, c_1, ..., c_{n-1})$. Circulant matrices have numerous applications in areas such as image processing, coding theory, signal processing and digital image

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disposal. For properties of circulant matrices, we refer the reader to [1] and [2]. Nevertheless, we present some useful ones here; for instance, circulants of the same order commute and so all circulants are normal matrices. Another property is that eigenvalues of $C = Cir(c_0, c_1, ..., c_{n-1})$ are $\lambda_m = \sum_{k=0}^{n-1} c_k \omega^{-mk}$, where $\omega = e^{\frac{2\pi i}{n}}$. Now the Pell and Pell-Lucas sequences, P_n and Q_n are defined by $P_n = 2P_{n-1} + P_{n-2}$ and $Q_n = 2Q_{n-1} + Q_{n-2}$, where $P_0 = 0$ and $P_1 = 1$ and $Q_0 = 2$ and $Q_1 = 2$, respectively. Pell and Pell-Lucas numbers have many elegant properties such as, for $\gamma = 1 + \sqrt{2}$ and $\delta = 1 - \sqrt{2}$,

1. $\lim_{n\to\infty} \frac{P_{n+1}}{P_n} = \gamma$ (silver ratio),
2. $P_n = \frac{\gamma^n - \delta^n}{\gamma - \delta}$ and $Q_n = \gamma^n + \delta^n$,
3. $P_{n+1}^2 + P_n^2 = P_{2n+1}$,
4. $2Q_n + 3P_n = Pn + 2$,
5. $\sum_{n=1}^{m} P_{2n+1} = \frac{1}{2}[2Q_{2m} + 3P_{2m} - P_0]$,
6. $\sum_{n=1}^{m} P_n = \frac{1}{2}[Q_{m+1} - Q_0]$

For more details, one can see [5] and [6]. Later, Serpil and Sinan [3] introduce Gaussian Pell numbers as $GP_n = P_n + iP_{n-1}$, where $i = \sqrt{-1}$. Also Binet's formula for Gaussian Pell numbers is given by

$$GP_n = \frac{\gamma^n - \delta^n}{\gamma - \delta} + i\frac{\gamma\delta^n - \delta\gamma^n}{\gamma - \delta}.$$

For any $n \times n$ matrix $A = (a_{ij})$, the Euclidean norm of A is

$$||A||_E = \left(\sum_{j=1}^{n} \sum_{i=1}^{n} (a_{ij}^2) \right)^{\frac{1}{2}}$$

and the spectral norm of A is

$$||A||_2 = \left(\max_{1 \le i \le n} \lambda_i(A^* A) \right)^{\frac{1}{2}},$$

where $\lambda_i(A^*A)$ are the eigenvalues of A^*A and A^* is the conjugate transpose of A.

In this paper, we compute the Euclidean norm of matrix $GP = Cir(GP_0, GP_1, ...GP_{n-1})$ and find upper and lower bounds of its spectral norm. Also, we show that $\lim_{n\to\infty} \frac{GP_{n+1}}{GP_n} = \gamma = 1 + \sqrt{2}$.

2 Main results

For the first main result, we begin with the silver ratio in Gaussian Pell numbers.

Theorem 2.1. *Let γ be the silver ratio and GP_n be the n^{th} Gaussian Pell number. Then* $\lim_{n\to\infty} \frac{GP_{n+1}}{GP_n} = \gamma$.

Proof. From the properties of limit, we obtain

$$\begin{aligned}
\lim_{n\to\infty} \frac{GP_{n+1}}{GP_n} &= \lim_{n\to\infty} \frac{P_{n+1} + iP_n}{P_n + iP_{n-1}} \\
&= \lim_{n\to\infty} \frac{1 + i\frac{P_n}{P_{n+1}}}{\frac{P_n}{P_{n+1}} + i\frac{P_{n-1}}{P_n}\frac{P_n}{P_{n+1}}} \\
&= \lim_{n\to\infty} \frac{1 + i\frac{1}{\gamma}}{\frac{1}{\gamma} + i\frac{1}{\gamma}\frac{1}{\gamma}} \\
&= \frac{\gamma^2 + \gamma i}{\gamma + i} = \gamma = 1 + \sqrt{2}.
\end{aligned}$$

The following two theorems are about two types of norms of this matrix. First, we start with the Euclidean norm because we will use it for finding bounds of spectral norm.

Theorem 2.2. *The Euclidean norm of the $n \times n$ circulant matrix $GP = C_{ir}(GP_0, GP_1, ..., GP_{n-1})$ is* $||GP||_E = \sqrt{n(\frac{1}{2}P_{2n-2} + 2)}$.

Proof.

$$\begin{aligned}||GP||_E^2 &= n\sum_{s=0}^{n-1}|GP_s|^2 = n\sum_{s=0}^{n-1}(P_s^2 + P_{s-1}^2)\\ &= n\sum_{s=0}^{n-1}P_{2s-1} = n\left(\sum_{s=1}^{n-1}P_{2s-1} + P_{-1}\right)\\ &= n\left(\sum_{s=0}^{n-2}P_{2s+1} + P_{-1}\right) = n\left(\frac{1}{2}(2Q_{2n-4} + 3P_{2n-4} - P_0) + P_{-1}\right)\\ &= n\left(\frac{1}{2}(2Q_{2n-4} + 3P_{2n-4} - P_0) + 2\right) = n\left(\frac{1}{2}P_{2n-2} + 2\right).\end{aligned}$$

Theorem 2.3. *The eigenvalues of GP are the following*

$$\lambda_0 = \frac{1}{2}(Q_n - 2) + i\frac{1}{2}(Q_{n-1} - 4)$$

and

$$\lambda_m = \frac{1}{2\sqrt{2}}\left[(1 - i\delta)\frac{1 - (\gamma\omega^{-m})^n}{1 - \gamma\omega^{-m}} + (1 + i\gamma)\frac{1 - (\gamma\omega^{-m})^n}{1 - \gamma\omega^{-m}}\right],$$

where $m = 1, 2, ..., n-1$.

Proof. Since GP is a circulant matrix, its eigenvalues are $\lambda_m = \sum_{s=0}^{n-1} GP_s e^{\frac{-2\pi i m s}{n}}$. Therefore,

$$\begin{aligned}\lambda_0 &= \sum_{s=0}^{n-1} GP_s = \sum_{s=0}^{n-1}(P_s + iP_{s-1}\\ &= \frac{1}{2}(Q_n - Q_0) + i\frac{1}{2}(Q_{n-1} - Q_0 - P_{-1})\\ &= \frac{1}{2}(Q_n - 2) + i\frac{1}{2}(Q_{n-1} - 4).\end{aligned}$$

For $m \neq 0$,

$$\begin{aligned}
\lambda_m &= \sum_{s=0}^{n-1} GP_s \omega^{-ms} \\
&= \sum_{s=0}^{n-1} \left(\frac{\gamma^s - \delta^s}{\gamma - \delta} + i \frac{\gamma\delta^s - \delta\gamma^s}{\gamma - \delta} \right) \omega^{-ms} \\
&= \frac{1}{2\sqrt{2}} \left[\sum_{s=0}^{n-1} (\gamma\omega^{-m})^s - \sum_{s=0}^{n-1} (\delta\omega^{-m})^s \right] + \frac{i}{2\sqrt{2}} \left[\sum_{s=0}^{n-1} \gamma(\delta\omega^{-m})^s - \sum_{s=0}^{n-1} \delta(\gamma\omega^{-m})^s \right] \\
&= \frac{1}{2\sqrt{2}} \left[(1 - i\delta) \sum_{s=0}^{n-1} (\gamma\omega^{-m})^s + (1 + i\gamma) \sum_{s=0}^{n-1} (\delta\omega^{-m})^s \right] \\
&= \frac{1}{2\sqrt{2}} \left[(1 - i\delta) \frac{1 - (\gamma\omega^{-m})^n}{1 - \gamma\omega^{-m}} + (1 + i\gamma) \frac{1 - (\gamma\omega^{-m})^n}{1 - \gamma\omega^{-m}} \right]
\end{aligned}$$

Theorem 2.4. *For any* $GP = C_{ir}(GP_0, GP_1, ..., GP_{n-1})$, *we have*

$$\sqrt{\frac{1}{2}(P_{2n-2} + 2)} \leq ||GP||_s \leq \sqrt{n(\frac{1}{2}P_{2n-2} + 2)}.$$

Proof. For any $n \times n$ matrix A, we have

$$\frac{1}{\sqrt{n}} ||A||_E \leq ||A||_s \leq ||A||_E.$$

Then

$$\sqrt{\frac{1}{2}(P_{2n-2} + 2)} = \frac{1}{\sqrt{n}} ||GP||_E \leq ||GP||_s \leq ||GP||_E = \sqrt{n(\frac{1}{2}P_{2n-2} + 2)}.$$

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References

[1] P. J. Davis, Circulant Matrices, Wiley, New York, Chichester, Brisbane, 1979.

[2] Z. L. Jiang, Z. X. Zhou, Circulant Matrices, Chengdu Technology University Publishing Company, Chengdu, China, 1999.

[3] S. Halici, S. Oz, On Gaussian Pell and Pell-Lucas numbers, J. Sci. Tech., **6,** no. 1, (2016), 8–18.

[4] V. N. Purav, Silver Ratio and Pell numbers, J. App. Sci. Comp., **5,** no. 7, (2018), 8–13.

[5] Z. Cerin, G. M. Gianella, On Sums of Pell numbers, Acc. Sc. Torino Atti Sc. Fis., **140,** (2016).

[6] T. Koshy, Pell and Pell-Lucas numbers with applications, Springer, 2014.