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inf-hesitant fuzzy interior ideals of semigroups

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Abstract

In this paper, we introduce an inf-hesitant fuzzy interior ideal which is a general concept of an interval-valued fuzzy interior ideal in a semigroup and investigate its properties. Moreover, we characterize intra-regular and simple semigroups in terms of inf-hesitant fuzzy interior ideals.

1 Introduction

As an extension of fuzzy sets introduced Zadeh [9], Torra and Narukawa [8] proposed the concept of a hesitant fuzzy set which overcomes limitations for using fuzzy set theory to deal with vague and imprecise information when different sources of vagueness appear simultaneously. Later, the concept was applied to semigroup and algebraic theory: For example, Jun et al. [2] studied hesitant fuzzy sets based on ideal theory in semigroups. Talee et al. [6] proposed hesitant fuzzy ideals in semigroups with a frontier. Kim et al. [3] studied the concepts and properties of hesitant fuzzy sets based on substructures of groupoids, groups and rings. Julatha and Iampan [1] introduced and and studied sup-types of hesitant fuzzy sets in ternary semigroups. Muhiuddin et al. [5] introduced inf-hesitant fuzzy subalgebras and ideals in BCK/BCI-algebras and investigated their properties.

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In this paper, we introduce an inf-hesitant fuzzy interior ideal which is a generalization of the concept of an interval-valued fuzzy interior ideal in semigroups and investigate its properties. We characterize the inf-hesitant fuzzy interior ideal in terms of interior ideals, fuzzy interior ideals, hesitant fuzzy interior ideals and interval-valued fuzzy interior ideals. Finally, we characterize intra-regular and simple semigroups in terms of inf-hesitant fuzzy interior ideals.

2 Preliminaries

In this section, we first give some basic definitions and results which will be used in our paper.

In what follows, unless otherwise specified, T denotes a nonempty set and S denotes a semigroup. By an **interior ideal** (in short, IId) of S we mean a nonempty subset T of S such that $STS \subseteq T$. A **fuzzy subset** (in short, FS) f [9] of T is a mapping from T into $[0,1]$ where $[0,1]$ is the unit segment of the real line. A FS f of S is called a **fuzzy interior ideal** (in short, FIId) of S if it satisfies: $f(t) \leq f(utv)$ for all $t, u, v \in S$.

Let $\mathcal{D}[0,1] = \{[a^-, a^+] | a^-, a^+ \in [0,1] \,\texttt{and}\, a^- \leq a^+\}$. For two elements $\bar{a} = [a^-, a^+]$ and $\bar{b} = [b^-, b^+]$ in $\mathcal{D}[0,1]$, define the operations $\preceq$, $=$ and $\prec$ for two elements in $\mathcal{D}[0,1]$ as follows : :
(1) $\bar{a} \preceq \bar{b}$ if and only if $a^- \leq b^-$ and $a^+ \leq b^+$, (2) $\bar{a} = \bar{b}$ if and only if $a^- = b^-$ and $a^+ = b^+$, (3) $\bar{a} \prec \bar{b}$ if and only if $\bar{a} \preceq \bar{b}$ and $\bar{a} \neq \bar{b}$. A mapping $\tilde{\imath} : T \to \mathcal{D}[0,1]$ is called an **interval-valued fuzzy set** (in short, IvFS) [10] on T, where $\tilde{\imath}(t) = [\imath^-(t), \imath^+(t)]$ for all $t \in T$, $\imath^-$ and $\imath^+$ are FSs of T such that $\imath^-(t) \leq \imath^+(t)$ for all $t \in T$. An IvFS $\tilde{\imath}$ on S is called an **interval-valued fuzzy interior ideal** (in short, IvFIId) [7] of S if it satisfies: $\tilde{\imath}(t) \preceq \tilde{\imath}(utv)$ for all $t, u, v \in S$.

Torra and Narukawa [8] introduced a **hesitant fuzzy set** (in short, HFS) on T in terms of a mapping $\hbar$ that when applied to T returns a subset of $[0,1]$; that is, $\hbar : T \to \wp([0,1])$ where $\wp([0,1])$ denote the set of all subsets of $[0,1]$. Let $HFS(T) = \{\hbar \mid \hbar : T \to \wp([0,1])\}$, $HFS^*(T) = \{\hbar \in HFS(T) \mid \hbar(t) \neq \emptyset$ for all $t \in T\}$ and $IvFS(T) = \{\tilde{\imath} | \tilde{\imath} : T \to \mathcal{D}[0,1]\}$. Then $IvFS(T) \subseteq HFS^*(T) \subseteq HFS(T)$. A HFS $\hbar$ on S is called a **hesitant fuzzy interior ideal** (in short, HFIId) [6] of S if it satisfies:
$\hbar(t) \subseteq \hbar(utv)$ for all $t, u, v \in S$.

3 Main results

For a HFS $\hbar$ on T and $\Bbbk \in \wp([0,1])$, define the element $\mathrm{INF}\,\Bbbk$ of $[0,1]$ and the subset $\mathrm{INF}\,[\hbar;\Bbbk]$ of T by

$$\mathrm{INF}\,\Bbbk = \begin{cases} \inf \Bbbk & \text{if } \Bbbk \neq \emptyset, \\ 0 & \text{otherwise}, \end{cases} \quad \text{and} \quad \mathrm{INF}\,[\hbar;\Bbbk] = \{t \in T \mid \mathrm{INF}\,\hbar(t) \geq \mathrm{INF}\,\Bbbk\}.$$

Definition 3.1. *A HFS $\hbar$ on S is called an* inf*-hesitant fuzzy interior ideal (in short,* inf*-HFIId) of S if $\mathrm{INF}\,[\hbar;\Bbbk]$ is an IId of S for all $\Bbbk \in \wp([0,1])$ when $\mathrm{INF}\,[\hbar;\Bbbk] \neq \emptyset$.*

Lemma 3.2. *Every IvFIId of S is an* inf*-HFIId of S.*

Proof. Suppose that $\tilde{\imath}$ is an IvFIId of S. Let $\Bbbk \in \wp([0,1])$, $u, v \in S$ and $t \in \mathrm{INF}[\tilde{\imath};\Bbbk]$. Then $\inf \tilde{\imath}(t) \geq \mathrm{INF}\,\Bbbk$. By the hypothesis, we get $\tilde{\imath}(t) \preceq \tilde{\imath}(utv)$ and so $\mathrm{INF}\,\Bbbk \leq \inf \tilde{\imath}(t) \leq \inf \tilde{\imath}(utv)$. Thus $utv \in \mathrm{INF}[\tilde{\imath};\Bbbk]$. Hence, $\mathrm{INF}[\tilde{\imath};\Bbbk]$ is an IId of S. Therefore, $\tilde{\imath}$ is an inf-HFIId of S. $\square$

The converse of Lemma 3.2 is not true as the following example shows:

Examle 3.3. *Let $S = \{s, t, u, v\}$ be the semigroup under the binary operation "$\cdot$" defined as follows:*

$\cdot$	s	t	u	v
s	s	s	s	s
t	s	s	s	s
u	s	s	t	s
v	s	s	t	t

(1) *Define a HFS $\hbar$ on S by $\hbar(s) = \{0.5, 0.6, 0.8, 1\}, \hbar(t) = [0.3, 0.7], \hbar(u) = \emptyset$ and $\hbar(v) = [0.2, 0.9]$. Then $\hbar$ is an* inf*-HFIId of S but is not an IvFIId of S because $\hbar$ is not an IvFS on S.*

(2) *Define an IvFS $\tilde{\imath}$ on S by $\tilde{\imath}(s) = [0.4, 0.6], \tilde{\imath}(t) = [0.4, 0.7], \tilde{\imath}(u) = [0, 0.8]$ and $\tilde{\imath}(v) = [0.2, 0.9]$. Then $\tilde{\imath}$ is an* inf*-HFIId of S but is not an IvFIId of S. In fact, $\tilde{\imath}(sts) = [0.4, 0.6] \prec [0.4, 0.7] = \tilde{\imath}(t)$.*

Lemma 3.2 and Example 3.3 indicate that the concept of an inf-HFIId of a semigroup is a general concept of an IvFIId.

For a HFS $\hbar$ on T, define the FS $\mathrm{F}_\hbar$ of T by $\mathrm{F}_\hbar(t) = \mathrm{INF}\,\hbar(t)$ for all $t \in T$. A HFS $\widehat{\zeta}$ on T is called an **infimum complement** of $\hbar$ on T if $\mathrm{INF}\,\widehat{\zeta}(t) = 1 - \mathrm{INF}\,\hbar(t)$ for all $t \in T$. Let $IC(\hbar)$ be the set of all infimum

complements of $\hbar$. Define the HFS $\hbar^\star$ by $\hbar^\star(t) = \{1 - \text{INF}\,\hbar(t)\}$ for all $t \in T$. Thus we have results as follows:
(1) $\hbar^\star \in IC(\hbar)$;(2) $\text{F}_{\hbar^\star}(t) = 1 - \text{INF}\,\hbar(t)$; (3) $\text{INF}\,\hbar(t) = 1 - (1 - \text{INF}\,\hbar(t)) = 1 - \text{INF}\,\hbar^\star(t)$ and (4) $\text{INF}\,(\hbar^\star)^\star(t) = \text{INF}\,\hbar(t) = \text{F}_\hbar(t)$ when $t \in T$.

Lemma 3.4. *If $\hbar \in HFS^*(S)$ and $\hbar$ is a HFIId of S, then $\widehat{\zeta}$ is an* inf*-HFIId of S for all $\widehat{\zeta} \in IC(\hbar)$.*

Proof. Suppose that $\hbar$ is a HFIId of S and $\widehat{\zeta} \in IC(\hbar)$. Let $\Bbbk \in \wp([0,1])$, $u, v \in S$ and $t \in \text{INF}\,[\widehat{\zeta}; \Bbbk]$. By the hypothesis, we have $\hbar(t) \subseteq \hbar(utv)$. Since $\hbar \in HFS^*(S)$, we get $\text{INF}\,\hbar(t) \geq \text{INF}\,\hbar(utv)$ and then $\text{INF}\,\widehat{\zeta}(utv) = 1 - \text{INF}\,\hbar(utv) \geq 1 - \text{INF}\,\hbar(t) = \text{INF}\,\widehat{\zeta}(t) \geq \text{INF}\,\Bbbk$. Thus, $utv \in \text{INF}\,[\widehat{\zeta}; \Bbbk]$. Hence, $\text{INF}\,[\widehat{\zeta}; \Bbbk]$ is an IId of S. Therefore, $\widehat{\zeta}$ is an inf-HFIId of S. □

Lemma 3.5. *For $\hbar \in HFS(S)$, the following are equivalent.*

(1) $\hbar$ is an inf*-HFIId of S.*

(2) $\text{F}_\hbar$ is a FIId of S.

(3) $\text{INF}\,\hbar(utv) \geq \text{INF}\,\hbar(t)$ for all $t, u, v \in S$.

(4) $\text{INF}\,\widehat{\zeta}(utv) \leq \text{INF}\,\widehat{\zeta}(t)$ for all $t, u, v \in S$ and $\widehat{\zeta} \in IC(\hbar)$.

(5) $\text{INF}\,\hbar^\star(utv) \leq \text{INF}\,\hbar^\star(t)$ for all $t, u, v \in S$.

Proof. (1) ⇒ (2). Suppose that $\hbar$ is an inf-HFIId of S and $t, u, v \in S$. Then $t \in \text{INF}\,[\hbar; \hbar(t)]$ and by the hypothesis, we have that $utv \in \text{INF}\,[\hbar; \hbar(t)]$. Thus $\text{F}_\hbar(t) = \text{INF}\,\hbar(t) \leq \text{INF}\,\hbar(utv) = \text{F}_\hbar(utv)$. Therefore, $\text{F}_\hbar$ is a FIId of S.

(3) ⇒ (1). Suppose that (3) is true. Let $\Bbbk \in \wp([0,1])$, $u, v \in S$ and $t \in \text{INF}\,[\hbar; \Bbbk]$. Thus $\text{INF}\,\hbar(utv) \geq \text{INF}\,\hbar(t) \geq \text{INF}\,\Bbbk$ and so $utv \in \text{INF}[\hbar; \Bbbk]$. Hence, $\text{INF}[\hbar; \Bbbk]$ is an IId of S. Therefore, $\hbar$ is an inf-HFIId of S.

(3) ⇒ (4). Suppose that (3) is true and $\widehat{\zeta} \in IC(\hbar)$. Let $t, u, v \in S$. Then $\text{INF}\,\hbar(utv) \geq \text{INF}\,\hbar(t)$ and so we have $\text{INF}\,\widehat{\zeta}(t) = 1 - \text{INF}\,\hbar(t) \geq 1 - \text{INF}\,\hbar(utv) = \text{INF}\,\widehat{\zeta}(utv)$.

(5) ⇒ (3). Let $t, u, v \in S$. By assumption (5), we have $\text{INF}\,\hbar^\star(utv) \leq \text{INF}\,\hbar^\star(t)$ and then $\text{INF}\,\hbar(utv) = 1 - \text{INF}\,\hbar^\star(utv) \geq 1 - \text{INF}\,\hbar^\star(t) = \text{INF}\,\hbar(t)$.

(2) ⇒ (3) and (4) ⇒ (5). They are clear. □

Theorem 3.6. *If an IvFS $\tilde{\imath}$ of S is an IvFIId of S, then $\text{F}_{\tilde{\imath}}$ is a FIId of S.*

Proof. It follows from Lemmas 3.2 and 3.5. □

Theorem 3.7. *If $\hbar \in HFS^*(S)$ and $\hbar$ is a HFIId of S, then $\mathrm{F}_{\widehat{\zeta}}$ is a FIId of S for all $\widehat{\zeta} \in IC(\hbar)$.*

Proof. The proof follows from Lemmas 3.4 and 3.5. ☐

For a HFS $\hbar$ on T and an element γ of $[0,1]$, the sets

$U_{\mathrm{INF}}(\hbar;\gamma) = \{t \in T \mid \mathrm{INF}\,\hbar(t) \geq \gamma\}$ and $L_{\mathrm{INF}}(\hbar;\gamma) = \{t \in T \mid \mathrm{INF}\,\hbar(t) \leq \gamma\}$

are called an inf-**upper** γ**-level subset** and an inf-**lower** γ**-level subset** of $\hbar$, respectively.

Theorem 3.8. *Let $\hbar$ be a HFS on S. Then $\hbar$ is an* inf*-HFIId of S if and only if, for all $\gamma \in [0,1]$, if $U_{\mathrm{INF}}(\hbar;\gamma) \neq \emptyset$, then $U_{\mathrm{INF}}(\hbar;\gamma)$ is an IId of S.*

Proof. Assume that $\hbar$ is an inf-HFIId of S. Let $\gamma \in [0,1]$ and $U_{\mathrm{INF}}(\hbar;\gamma) \neq \emptyset$. Choose $\Bbbk \in \wp([0,1])$ such that $\mathrm{INF}\Bbbk = \gamma$, we get $\mathrm{INF}[\hbar;\Bbbk] = U_{\mathrm{INF}}(\hbar;\gamma)$. By the hypothesis, we get $U_{\mathrm{INF}}(\hbar;\gamma) = \mathrm{INF}[\hbar;\Bbbk]$ is an IId of S.

Conversely, suppose that the necessary condition holds. Let $\Bbbk \in \wp([0,1])$ and $\mathrm{INF}[\hbar;\Bbbk] \neq \emptyset$. Choose $\gamma := \mathrm{INF}\Bbbk$, we see that $\mathrm{INF}[\hbar;\Bbbk] = U_{\mathrm{INF}}(\hbar;\gamma)$ is an IId of S. Therefore, $\hbar$ is an inf-HFIId of S. ☐

Corollary 3.9. *Let $\tilde{\imath}$ be an IvFIId of S. Then, for all $\gamma \in [0,1]$, if $U_{\mathrm{INF}}(\tilde{\imath};\gamma) \neq \emptyset$, then $U_{\mathrm{INF}}(\tilde{\imath};\gamma)$ is an IId of S.*

Proof. The proof follows from Lemma 3.2 and Theorem 3.8. ☐

Theorem 3.10. *Let $\hbar$ be a HFS on S and $\widehat{\zeta} \in IC(\hbar)$. Then $\widehat{\zeta}$ is an* inf*-HFIId of S if and only if, for all $\gamma \in [0,1]$, if $L_{\mathrm{INF}}(\hbar;\gamma) \neq \emptyset$, then $L_{\mathrm{INF}}(\hbar;\gamma)$ is an IId of S.*

Proof. Assume that $\widehat{\zeta}$ is an inf-HFIId of S. Let $\gamma \in [0,1]$ and $L_{\mathrm{INF}}(\hbar;\gamma) \neq \emptyset$. Choose $\Bbbk := \{1-\gamma\}$, we get $\mathrm{INF}[\widehat{\zeta};\Bbbk] = L_{\mathrm{INF}}(\hbar;\gamma)$. Since $\widehat{\zeta}$ is an inf-HFIId of S, we get $L_{\mathrm{INF}}(\hbar;\gamma) = \mathrm{INF}[\widehat{\zeta};\Bbbk]$ is an IId of S.

Conversely, suppose that the necessary condition holds. Let $\Bbbk \in \wp([0,1])$ and $\mathrm{INF}[\widehat{\zeta};\Bbbk] \neq \emptyset$. Choose $\gamma := 1 - \mathrm{INF}\Bbbk$ and by the hypothesis, we have that $\mathrm{INF}[\widehat{\zeta};\Bbbk] = L_{\mathrm{INF}}(\hbar;\gamma)$ is an IId of S. Hence, $\widehat{\zeta}$ is an inf-HFIId of S. ☐

Corollary 3.11. *Let $\hbar$ be a HFIId of S and $\hbar \in HFS^*(S)$. Then, for all $\gamma \in [0,1]$, if $L_{\mathrm{INF}}(\hbar;\gamma) \neq \emptyset$, then $L_{\mathrm{INF}}(\hbar;\gamma)$ is an IId of S.*

Proof. The proof now follows from Lemma 3.4 and Theorem 3.10. ☐

For a HFS $\hbar$ on T and an element $\Bbbk$ of $\wp([0,1])$, we define the HFS $\mathcal{H}_{\mathrm{INF}}(\hbar;\Bbbk)$ on T by $\mathcal{H}_{\mathrm{INF}}(\hbar;\Bbbk)(t) = \{\gamma \in \Bbbk \,|\, \mathrm{INF}\,\hbar(t) \geq \gamma\}$ for all $t \in T$. We denote $\mathcal{H}_{\mathrm{INF}}(\hbar;[0,1])$ by $\mathcal{H}^{\hbar}_{\mathrm{INF}}$. Then we have that (1) $\mathcal{H}^{\hbar}_{\mathrm{INF}}$ is an IvFS on T, (2) $\mathcal{H}_{\mathrm{INF}}(\hbar;\Bbbk)(t) \subseteq \Bbbk$ for all $t \in T$, and (3) $0 = \inf \mathcal{H}^{\hbar}_{\mathrm{INF}}(t) \leq \mathrm{INF}\,\mathcal{H}_{\mathrm{INF}}(\hbar;\Bbbk)(t) \leq \mathrm{INF}\,\hbar(t) = \sup \mathcal{H}^{\hbar}_{\mathrm{INF}}(t)$ for all $t \in T$.

Theorem 3.12. *A HFS $\hbar$ on S is an* inf*-HFIId of S if and only if $\mathcal{H}_{\mathrm{INF}}(\hbar;\Bbbk)$ is a HFIId of S for all $\Bbbk \in \wp([0,1])$.*

Proof. Suppose that $\hbar$ is an inf-HFIId of S. Let $t,u,v \in S$, $\Bbbk \in \wp([0,1])$ and $\gamma \in \mathcal{H}_{\mathrm{INF}}(\hbar;\Bbbk)(t)$. Then $\gamma \in \Bbbk$ and $\mathrm{INF}\,\hbar(t) \geq \gamma$. By the hypothesis and Lemma 3.5, we get $\mathrm{INF}\,\hbar(utv) \geq \mathrm{INF}\,\hbar(t) \geq \gamma$ and so $\gamma \in \mathcal{H}_{\mathrm{INF}}(\hbar;\Bbbk)(utv)$. Thus $\mathcal{H}_{\mathrm{INF}}(\hbar;\Bbbk)(t) \subseteq \mathcal{H}_{\mathrm{INF}}(\hbar;\Bbbk)(utv)$. Hence, $\mathcal{H}_{\mathrm{INF}}(\hbar;\Bbbk)$ is a HFIId of S.

Conversely, suppose that the necessary condition holds. Let $t,u,v \in S$. Choose $\Bbbk = \{\mathrm{INF}\,\hbar(t)\}$, we obtain $\mathrm{INF}\,\Bbbk \in \mathcal{H}_{\mathrm{INF}}(\hbar;\Bbbk)(t)$. By the hypothesis, we get $\mathcal{H}_{\mathrm{INF}}(\hbar;\Bbbk)$ is a HFIId of S. Thus $\mathrm{INF}\,\hbar(t) = \mathrm{INF}\,\Bbbk \in \mathcal{H}_{\mathrm{INF}}(\hbar;\Bbbk)(utv)$ and so $\mathrm{INF}\,\hbar(utv) \geq \mathrm{INF}\,\hbar(t)$. From Lemma 3.5, it follows that $\hbar$ is an inf-HFIId of S. □

The following are immediate results of Lemma 3.2, Lemma 3.4 and Theorem 3.12

Corollary 3.13. *The following are true.*

(1) If $\tilde{\imath}$ is an IvFIId of S, then $\mathcal{H}_{\mathrm{INF}}(\tilde{\imath};\Bbbk)$ is a HFIId of S for all $\Bbbk \in \wp([0,1])$.

(2) If $\hbar$ is a HFIId of S and $\hbar \in HFS^*(S)$, then $\mathcal{H}_{\mathrm{INF}}(\widehat{\zeta};\Bbbk)$ is a HFIId of S for all $\Bbbk \in \wp([0,1])$ and $\widehat{\zeta} \in IC(\hbar)$.

Theorem 3.14. *For a HFS $\hbar$ on S, the following are equivalent:*

(1) $\hbar$ is an inf*-HFIId of S.*

(2) $\mathcal{H}^{\hbar}_{\mathrm{INF}}$ is a HFIId of S.

(3) $\mathcal{H}^{\hbar}_{\mathrm{INF}}$ is an IvFIId of S.

Proof. (1) ⇒ (2). This follows from Theorem 3.12.

(2) ⇒ (3). Suppose that $\mathcal{H}^{\hbar}_{\mathrm{INF}}$ is a HFIId of S. Let $t,u,v \in S$. Then $\mathrm{INF}\,\hbar(t) \in \mathcal{H}^{\hbar}_{\mathrm{INF}}(t)$ and by the hypothesis, we have $\mathrm{INF}\,\hbar(t) \in \mathcal{H}^{\hbar}_{\mathrm{INF}}(utv)$. Thus $\sup \mathcal{H}^{\hbar}_{\mathrm{INF}}(utv) = \mathrm{INF}\,\hbar(utv) \geq \mathrm{INF}\,\hbar(t) = \sup \mathcal{H}^{\hbar}_{\mathrm{INF}}(t)$ and then

$$\mathcal{H}^{\hbar}_{\mathrm{INF}}(t) = \left[0, \sup \mathcal{H}^{\hbar}_{\mathrm{INF}}(t)\right] \preceq \left[0, \sup \mathcal{H}^{\hbar}_{\mathrm{INF}}(utv)\right] = \mathcal{H}^{\hbar}_{\mathrm{INF}}(utv).$$

Hence, $\mathcal{H}^{\hbar}_{\mathrm{INF}}$ is an IvFIId of S.

(3) $\Rightarrow$ (1). Assume that $\mathcal{H}^{\hbar}_{\mathrm{INF}}$ is an IvFIId of S. Then $\mathcal{H}^{\hbar}_{\mathrm{INF}}(t) \preceq \mathcal{H}^{\hbar}_{\mathrm{INF}}(utv)$ for all $t, u, v \in S$. Thus for all $t, u, v \in S$, we obtain that

$$\mathrm{INF}\, \hbar(utv) = \sup \mathcal{H}^{\hbar}_{\mathrm{INF}}(utv) \geq \sup \mathcal{H}^{\hbar}_{\mathrm{INF}}(t) = \mathrm{INF}\, \hbar(t).$$

By Lemma 3.5, $\hbar$ is an inf-HFIId of S. This completes the proof. □

The following are immediate results of Lemma 3.2, Lemma 3.4 and Theorem 3.14.

Corollary 3.15. *The following are true.*

(1) If $\tilde{\imath}$ is an IvFIId of S, then $\mathcal{H}^{\tilde{\imath}}_{\mathrm{INF}}$ is both a HFIId and an IvFIId of S.

(2) If $\hbar$ is a HFIId of S and $\hbar \in HFS^*(S)$, then $\mathcal{H}^{\widehat{\zeta}}_{\mathrm{INF}}$ is both a HFIId and an IvFIId of S for all $\widehat{\zeta} \in IC(\hbar)$.

A semigroup S is called **intra-regular** if, for each $t \in S$, there exist $u, v \in S$ such that $t = ut^2v$.

Lemma 3.16. *[4] A semigroup S is intra-regular if and only if $f(t) = f(t^2)$, for every FIId f of S and for all $t \in S$.*

Theorem 3.17. *Let S be a semigroup. The following are equivalent:*

(1) S is intra-regular.

(2) $\mathrm{INF}\, \hbar(t) = \mathrm{INF}\, \hbar(t^2)$ *for every* inf-*HFIId $\hbar$ of S and $t \in S$.*

(3) $\mathrm{INF}\, \widehat{\zeta}(t) = \mathrm{INF}\, \widehat{\zeta}(t^2)$ *for every* inf-*HFIId $\hbar$ of S, $\widehat{\zeta} \in IC(\hbar)$ and $t \in S$.*

(4) $\mathcal{H}^{\hbar}_{\mathrm{INF}}(t) = \mathcal{H}^{\hbar}_{\mathrm{INF}}(t^2)$, *for every* inf-*HFIId $\hbar$ of S and $t \in S$.*

Proof. The proof follows from Lemmas 3.5 and 3.16. □

By an **ideal** of S we mean a nonempty subset T of S such that $ST \subseteq T$ and $TS \subseteq T$. A semigroup S is called **simple** if it contains no proper ideal. Note that every simple semigroup is intra-regular. A HFS $\hbar$ on T is called inf-**constant** if $\mathrm{INF}\, \hbar(u) = \mathrm{INF}\, \hbar(v)$ for all $u, v \in T$. It is easily seen that every constant hesitant fuzzy set is inf-constant but the converse is not true.

Lemma 3.18. *[4] A semigroup S is simple if and only if every fuzzy interior ideal of S is constant.*

Theorem 3.19. *For a semigroup S, the following statements are equivalent:*

(1) S is simple.

(2) Every inf*-HFIId of S is* inf*-constant.*

(3) Every infimum complement of inf*-HFIId of S is* inf*-constant.*

(4) $\mathcal{H}^{\hbar}_{\mathrm{INF}}$ is constant for every inf*-HFIId $\hbar$ of S.*

Proof. It follows from Lemmas 3.5 and 3.18. □

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