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Parametrization Approach for Solving Index-4 Linear Differential-Algebraic Control Systems

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Abstract

Based on the theory of variational formulation, we find an approximate solution to index-4 time-invariant linear differential-algebraic control equations. The critical points are subject to the differential-algebraic equations (DAEs) with consistent initial conditions class of the formulated variational functional. The points are the solution of the suggested index-4 DAEs with control problem and vice-versa. Moreover, using the generalized Ritz bases method, the formulated variational system is transferred from an indirect method to a direct one. Numerical examples are provided to evaluate the proposed approach to obtain simplicity, efficiency, and accuracy.

1 Introduction

The DAEs system is frequently used to model many real-life application problems. Therefore, determining novel findings, efficient and reliable methods

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to solve DAEs has emerged as a promising field for engineers and mathematicians. With and without controls, the studies [1]-[3] proposed numerical methods using different approaches for solving higher (greater than one) index DAEs. To determine an equivalent reduced ordinary differential equation, index refers to the minimum number of differentiation times of DAEs [4].

The homotopy analysis method is implemented to approximate an analytical solution in solving DAEs with different classes [5]. The researchers in [6], [7] proposed numerical methods for solving DAEs with index-2 and index-3. Usually, the index reduction method has been used in reducing the DAEs from a higher index to a lower one. The authors in [8] used the domain decomposition method and suggested a novel technique to solve nonlinear higher index DAEs.

In this paper, we use the variational formulation theory in the proposed method to obtain an appropriate equation. The determined equation's critical points are considered the solution of index-3 DAEs and vice versa. It is important to mention that the approach is connected with the calculus inverse variation problem. To approximate the analytical solution, the approach starts by implementing the direct method of the calculus of variation. Next, in the obtained variation formulation, a parameterization approach is followed to represent the solution as a linear combination of the basis elements of the given setting space (separable Banach space). Therefore, an infinite-dimensional optimization problem can be converted into an equivalent finite-dimensional problem via parameterization. Thus, after applying a direct method of calculus of variation, the variational formulation approximately (of unknown parameters) is solved and becomes a function of unknown independent parameters. To reach our objective differentiation of this function (with respect to its unknown parameters) would leads to a system of linear algebraic equations. Then, solving the corresponding linear algebraic system, the unknown parameters are determined. For low-index special structure DAEs, this approach is a generalization of the suggestion in [9], [10].

2 Method of Solution of Index-4 Linear Differential-Algebraic Control Systems

In this section, using a variational formulation approach, the solvability of index-4 time-invariant DAEs is discussed. Consider the following problem:

$$\dot{x_1} = A_{11}x_1(t) + A_{12}x_2(t) + B_1u(t) + f_1(t), \tag{2.1}$$

$$0 = A_{21}x_1(t) + B_2u(t) + f_2(t), \tag{2.2}$$

where $u(t) \in \triangle_u$ (defined according to the problem) refers to the admissible control class and represents a uniquely solvable problem.

Let $u \in c^3(I, R^r), f_1 \in c^2(I, R^{n_0}), f_2 \in c^3(I, R^{n-n_0})$ and $I = [t_0, t_f]$, with $A_{21}A_{12} = 0, A_{21}A_{11}A_{12} = 0$ and $A_{21}A_{11}^2A_{12}$ invertible. For the algebraic equation 2.2, the differentiation index (which is the minimum number of differentiations) is used to determine u and $x_2(t)$ as a continuous function of $x_1(t)$. Differentiating three times with respect to t using equation 2.2, we get

$$0 = A_{21}A_{11}^3x_1 + A_{21}A_{11}^2A_{12}x_2 + A_{21}A_{11}^2B_1u + A_{21}A_{11}B_1\dot{u} + A_{21}B_1\ddot{u} + B_2\dddot{u}$$
$$+A_{21}A_{11}^2f_1 + A_{21}A_{11}\dot{f}_1 + A_{21}\ddot{f}_1 + \dddot{f}_2$$

Set $c = A_{21}A_{11}^2A_{12}$, with c invertible:

$$x_2(t) = -c^{-1}(A_{21}A_{11}^3x_1 + \tilde{u} + \tilde{f}) = £(x_1(t)) + £(\tilde{u}, \tilde{f}), \tag{2.3}$$

where $\tilde{u} = A_{21}A_{11}^2B_1u + A_{21}A_{11}B_1\dot{u} + A_{21}B_1\ddot{u} + B_2\dddot{u}$

$$\tilde{f} = A_{21}A_{11}^2f_1 + A_{21}A_{11}\dot{f}_1 + A_{21}\ddot{f}_1 + \dddot{f}_2$$

$$£(x_1(t)) = -c^{-1}(A_{21}A_{11}^3x_1)$$
$$£(\tilde{u}, \tilde{f}) = -c^{-1}(\tilde{u} + \tilde{f})$$

Reducing the index-4 system in equations 2.1-2.2 leads to the following state-space system (defined on manifolds):

$$\dot{x}_1(t) + (A_{12}£(x_1(t)) - A_{11})x_1(t) = -A_{12}£(\tilde{u}, \tilde{f}) + B_1u(t) + f_1(t), \tag{2.4}$$

$$-A_{21}x_1(t) = B_2u(t) + f_2(t), \tag{2.5}$$

$$-A_{21}A_{11}x_1(t) = A_{21}B_1u(t) + B_2\dot{u}(t) + A_{21}f_1(t) + \dot{f}_2(t), \tag{2.6}$$

$$-A_{21}A_{11}^2x_1(t) = A_{21}A_{11}B_1u(t)+A_{21}B_2\dot{u}(t)+B_2\ddot{u}+A_{21}A_{11}f_1(t)+A_{21}\dot{f}_2(t)+\ddot{f}_2. \tag{2.7}$$

Defining the class of consistent initial condition by

$$\omega^0 = \left\{ \begin{array}{c} x_1(t_0) \in R^{rank(E)} | - A_{21}x_1(t_0) = B_2u(t_0) + f_2(t_0) \\ -A_{21}A_{11}x_1(t_0) = A_{21}B_1u(t_0) + B_2\dot{u}(t_0) + A_{21}f_1(t_0) + \dot{f}_2(t_0) \\ -A_{21}A_{11}^2x_1(t_0) = A_{21}A_{11}B_1u(t_0) + A_{21}B_1\dot{u}(t_0) + B_2\ddot{u}(t_0) \\ +A_{21}A_{11}f_1(t_0) + A_{21}\dot{f}_1(t_0) + \ddot{f}_2(t_0) \\ \text{for given} f_1(t_0), f_2(t_0), \dot{f}_1(t_0), \dot{f}_2(t_0), \ddot{f}_2(t_0), u(t_0), \dot{u}(t_0), \ddot{u}(t_0) \end{array} \right\}$$

one can redefine equations 2.4-2.7 as follows:

$$L_1x_1(t) = G_1(u, \dot{u}, \ddot{u}, \dddot{u}, f_1, \dot{f}_1, \ddot{f}_1, \dddot{f}_2)$$

$$L_2x_1(t) = G_2(u, f_2)$$

$$L_3x_1(t) = G_3(u, \dot{u}, f_1, \dot{f}_2)$$

$$L_4x_1(t) = G_4(u, \dot{u}, \ddot{u}, f_1, \dot{f}_1, \ddot{f}_2),$$

where

$$L_1x_1(t) = \frac{d}{dt}x_1(t) + (A_{12}\pounds(x_1(t)) - A_{11})x_1$$

$$G_1 = -A_{12}\pounds(\tilde{u}, \tilde{f}) + B_1u(t) + f_1(t)$$

$$L_2x_1(t) = -A_{21}x_1(t)$$

$$G_2 = B_2u(t) + f_2(t)$$

$$L_3x_1(t) = -A_{21}A_{11}x_1(t)$$

$$G_3 = A_{21}B_1u(t) + B_2\dot{u}(t) + A_{21}f_1(t) + \dot{f}_2(t)$$

$$L_4x_1(t) = -A_{21}A_{11}^2x_1(t)$$

$$G_4 = A_{21}A_{11}B_1u(t) + A_{21}B_2\dot{u}(t) + B_2\ddot{u} + A_{21}A_{11}f_1(t) + A_{21}\dot{f}_2(t) + \ddot{f}_2$$

Therefore, 2.4-2.7 become

$$L(x_1) = [L_1, L_2, L_3, L_4]^T(x_1) = G(u, \dot{u}, \ddot{u}, \dddot{u}, f_1, \dot{f}_1, \ddot{f}_1, \dddot{f}_2) = [G_1, G_2, G_3, G_4]^T,$$

where, in the optimal control problem, the pair $(x; u) \in D(L) \subset c(I, R^{n_0}), L$ is the linear operator and $D(L)$ is the domain of L. Thus, on the compact interval of solution I, the norm on $c(I, R^{n_0})$ is equipped with the sequential maximum norm.

Since the operator $\frac{d}{dt}$ appears in $L_1x_1(t)$, the linear operator Lx_1 is not symmetric with the given usual bilinear form $\langle x_1, x_2\rangle = \int_{t_0}^{t^f} x_1^T x_2 dt, x_1, x_2 \in$

$c(I, R^{n_0})$ (see [9],[10]). No variational formulation exists unless the linear operator is redefined or its bilinear form.

To create a functional (variational) equivalent to a linear problem $Lu = f$, where L is not symmetric with respect to the chosen bilinear form, one can define the functional $F_{\omega^0}[x_1]$ as follows:

$$F_{\omega^0}[x_1] = \int_{t_0}^{tf} \left\{ \begin{array}{c} \frac{1}{2}L_1^T x_1(t) L_1 x_1(t) + \frac{1}{2}L_2^T x_1(t) L_2 x_1(t) + \frac{1}{2}L_3^T x_1(t) L_3 x_1(t) \\ +\frac{1}{2}L_4^T x_1(t) L_4 x_1(t) + \frac{1}{2}L_5^T x_1(t_0) L_5 x_1(t_0) + \frac{1}{2}L_6^T x_1(t_0) L_6 x_1(t_0) \\ +\frac{1}{2}L_7^T x_1(t_0) L_7 x_1(t_0) - G_1^T(t) L_1 x_1(t) - G_2^T(t) L_2 x_1(t) \\ -G_3^T(t) L_3 x_1(t) - G_4^T(t) L_4 x_1(t) - G_5^T(t_0) L_5 x_1(t_0) \\ -G_6^T(t_0) L_6 x_1(t_0) - G_7^T(t_0) L_7 x_1(t_0) \end{array} \right\} dt \tag{2.8}$$

From a practical perspective, the functional $F_{\omega^0}[x_1]$ has to be evaluated to determine the critical points. To approximate the solution using a finite number of basis functions of separable Banach (Hilbert) space with a supremum norm, a direct method of the variational problem is adapted. This is done because finding the critical points of a functional is equivalent to solving the necessary Euler equation corresponding to the given problem. Consequently, we first set the following parameterisation:

$$x_1^j = x_{10} + \sum_{i=1}^{m_j} a_i^j H_i^j(t), j = 1, 2, ..., n_0, m_j \text{arbitrary} \tag{2.9}$$

$$x_2^l = \pounds(x_1^j, u, f), l = 1, 2, ..., n - n_0, j = 1, ..., n_0, \tag{2.10}$$

where $x_{10} \in \omega^0, H_i^j$ is a linearly independent basis function of time t.

Secondly, the equations 2.9 and 2.10 are substituted into 2.8

$$F[x_1] = F(a_0^1, a_1^1, a_2^1, ..., a_{m_1}^1, a_0^2, a_1^2, a_2^2, ..., a_{m_2}^2, ..., a_0^{n_0}, a_1^{n_0}, a_2^{n_0}, ..., a_{m_{n_0}}^{n_0}), \tag{2.11}$$

where the total number of unknown variables is $n = n_0 + n - n_0$.

Then, with respect to $a_i^j, i = 0, ..., m_j, j = 1, 2, ..., n_0$

$$i.e., \frac{\partial F}{\partial a_i^j} = 0, \forall i = 0, ..., m_j, j = 1, 2, ..., n_0 \tag{2.12}$$

the critical point of variational formulation 2.8 is equivalent to finding the derivative of the functional 2.11. The linear system of algebraic equations has been obtained from equation 2.12 with the consistency of initial conditions class because the variational formulation is of quadratic type. The functions $u(t), \dot{u}(t), \ddot{u}(t), \dddot{u}(t), f(t), \dot{f}(t), \ddot{f}(t), \dddot{f}(t)$ are chosen from the class of admissible functions. An approximate solution $x_1(t), x_2(t)$ is determined according to 2.9, 2.10 once this system 2.12 is solved for a^i_j. Thus, the original solution of 2.1, 2.2 is determined approximately.

3 Illustration

For the linear semi-explicit DAE problem:

$$\dot{x_1} = A_{11}x_1 + A_{12}x_2 + B_1u + f_1$$

$$0 = A_{21}x_1 + u + f_2,$$

where $x_1 = \begin{bmatrix} x_{11} \\ x_{12} \\ x_{13} \end{bmatrix}$, $x_2 = x_{21}$, $A_{11} = \begin{bmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & 0 \end{bmatrix}$, $A_{12} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$,

$B_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $f_1 = \begin{bmatrix} f_{11} \\ f_{12} \\ f_{13} \end{bmatrix}$, $A_{21} = \begin{bmatrix} 1 & -1 & 1 \end{bmatrix}$,$f_2 = f_{21}, t \in [t_0, t_f]$,

with $\begin{bmatrix} f_{11} \\ f_{12} \\ f_{13} \\ f_{21} \end{bmatrix} = \begin{bmatrix} t^5 + t^3 + 2t^2 + t - cost \\ t^5 + t^3 + t^2 + 3t + 1 \\ 5t^4 + 4t^2 + t + 1 \\ t^2 - cost - t^3 - t^5 + 1, \end{bmatrix}$, where, in the algebraic constraint, we note that the variable x_{21} does not occur. Differentiating three times with respect to t, we obtain:

$$x_2(t) = -c^{-1}(A_{21}A_{11}^3x_1 + A_{21}A_{11}^2u + A_{21}A_{11}B_1\dot{u} + A_{21}B_1\ddot{u} + B_2\dddot{u} + A_{21}A_{11}^2f_1 + A_{21}A_{11}\dot{f_1} + A_{21}\ddot{f_1} + \dddot{f}_2)$$

$$= -2x_{11} - 2x_{12} - 2x_{13} + 2t^5 + 2t^3 + 3t^2 + 2t,$$

where $c^{-1} = (A_{21}A_{11}^2A_{12})^{-1} = 1$, and the reduced system will be

$$\dot{x}_1(t) + (A_{12}A_{21}A_{11}^3 - A_{11})x_1(t) = A_{12}(A_{21}A_{11}^2u + A_{21}A_{11}B_1\dot{u} + A_{21}B_1\ddot{u} + B_2\dddot{u} + A_{21}A_{11}^2f_1 + A_{21}A_{11}\dot{f_1} + A_{21}\ddot{f_1} + \dddot{f}_2) + B_1u + f_1$$
$$-A_{21}x_1(t) = B_2u + f_2$$
$$-A_{21}A_{11}x_1(t) = A_{21}B_1u + B_2\dot{u} + A_{21}f_1 + \dot{f_2}$$
$$-A_{21}A_{11}^2x_1(t) = A_{21}A_{11}B_1u + A_{21}B_1\dot{u} + B_2\ddot{u} + A_{21}A_{11}f_1 + A_{21}\dot{f_1} + \ddot{f_2}$$

The class of consistency initial condition is

$$\omega^0 = \left\{ \begin{array}{c} x_1(t_0) \in R^{rank(E)}|(x_{12} - x_{13} = -t^5 - t^3 + t^2 + 1)|_{t=t_0} \\ (-x_{13} = -t^5 - t^3)|_{t=t_0} \\ (x_{11} + x_{12} = t^2 + t + 1)|_{t=t_0} \end{array} \right\}$$

The variational formulation with the class of consistent initial condition is defined as follows:

$$F[x_1]=\frac{1}{2}\int_{t_0}^{t^f}\left\{\begin{array}{c}\dot{x}_1(t)+(A_{12}A_{21}A_{11}^3-A_{11})x_1(t)]^T[\dot{x}_1(t)\\ +(A_{12}A_{21}A_{11}^3-A_{11})x_1(t)]-2G_1(t)[\dot{x}_1(t)\\ +(A_{12}A_{21}A_{11}^3-A_{11})x_1(t)]+[A_{21}x_1(t)]^T[A_{21}x_1(t)]\\ -2[B_2u+f_2]^T[-A_{21}x_1(t)]+[A_{21}A_{11}x_1(t)]^T[A_{21}A_{11}x_1(t)]\\ -2[A_{21}B_1u(t)+B_2\dot{u}(t)+A_{21}f_1+\dot{f}_2(t)]^T[-A_{21}A_{11}x_1]\\ +[A_{21}A_{11}^2x_1(t)]^T[A_{21}A_{11}^2x_1(t)]\\ -2[A_{21}A_{11}B_1u(t)+A_{21}B_1\dot{u}+B_2\ddot{u}\\ +A_{21}A_{11}f_1(t)+A_{21}\dot{f}_1(t)+\ddot{f}_2(t)]^T[-A_{21}A_{11}^2x_1(t)]\\ +[A_{21}x_1(t_0)]^T[A_{21}x_1(t_0)]-2[B_2u(t_0)+f_2(t_0)]^T[-A_{21}x_1(t_0)]\\ +[A_{21}A_{11}x_1(t_0)]^T[A_{21}A_{11}x_1(t_0)]\\ -2[A_{21}B_1u(t_0)+B_2\dot{u}(t_0)+A_{21}f_1+\dot{f}_2]^T[-A_{21}A_{11}x_1(t_0)]\\ +[A_{21}A_{11}^2x_1(t_0)]^T[A_{21}A_{11}^2x_1(t_0)]\\ -2[A_{21}A_{11}B_1u(t_0)+A_{21}B_1\dot{u}(t_0)\\ +B_2\ddot{u}(t_0)+A_{21}A_{11}f_1(t_0)+A_{21}\dot{f}_1(t_0)+\ddot{f}_2(t_0)]^T[-A_{21}A_{11}^2x_1(t_0)]\end{array}\right\} \tag{3.13}$$

Now, set $x_{11}(t)=a_0^1+\sum_{i=1}^5 a_i^1t^i, x_{12}(t)=a_0^2+\sum_{i=1}^5 a_i^2t^i, x_{13}(t)=a_0^3+\sum_{i=1}^5 a_i^3t^i$.

Taking $\frac{\partial F}{\partial \overrightarrow{a^1}}=0$, $\frac{\partial F}{\partial \overrightarrow{a^2}}=0$, $\frac{\partial F}{\partial \overrightarrow{a^3}}=0$, leads to the linear algebraic equation $A\begin{pmatrix}\overrightarrow{a^1}\\ \overrightarrow{a^2}\\ \overrightarrow{a^3}\end{pmatrix}=B$, which is solvable directly for $\begin{pmatrix}\overrightarrow{a^1}\\ \overrightarrow{a^2}\\ \overrightarrow{a^3}\end{pmatrix}=A^{-1}B$. As a result, we obtain the approximate solution $(x_{11}(t),x_{12}(t),x_{13}(t),x_{21}(t))$. The values of the unknown coefficients $\overrightarrow{a^1},\overrightarrow{a^2},\overrightarrow{a^3}$ are listed in table 1.

Table 1: Unknown coefficients of the representation of $x_{11}(t),x_{12}(t),x_{13}(t)$

Coef. of x_{11} states	Numerical values	Coef. of x_{12} states	Numerical values	Coef. of x_{13} states	Numerical values
a_0^1	0	a_0^2	1	a_0^3	0
a_1^1	1	a_1^2	-7×10^{-40}	a_1^3	6×10^{-40}
a_2^1	7×10^{-40}	a_2^2	1	a_2^3	-6×10^{-39}
a_3^1	-1×10^{-39}	a_3^2	-1×10^{-39}	a_3^3	1
a_4^1	-7×10^{-40}	a_4^2	2×10^{-38}	a_4^3	-2×10^{-38}
a_5^1	1×10^{-39}	a_5^2	-8×10^{-39}	a_5^3	1

Table 2: The absolute error between the exact and approximate solution in the suggested method $x_{11}, x_{12}, x_{13}, x_{21}$ at $t \in [0, 1]$

t	Abs. error of x_{11}	Abs. error of x_{12}	Abs. error of x_{13}	Abs. error of x_{21}
0	0	0	0	0
0.1	2×10^{-41}	3×10^{-40}	2×10^{-41}	0
0.2	0	0	0	0
0.3	9×10^{-41}	3×10^{-40}	2×10^{-41}	0
0.4	0	3×10^{-40}	2×10^{-41}	1×10^{-39}
0.5	0	0	0	0
0.6	1×10^{-40}	0	0	3×10^{-40}
0.7	0	0	0	5×10^{-17}
0.8	0	3×10^{-40}	0	1×10^{-39}
0.9	1×10^{-40}	0	5×10^{-17}	3×10^{-17}
1	0	0	0	0

The following:

$$x_1 = \begin{bmatrix} t \\ 1 + t^2 \\ t^3 + t^5 \end{bmatrix},$$

$$x_2 = t^2 - 2,$$

satisfy equations 2.1, 2.2 for a given $u(t) = \cos t \in c^3(I, R)$.

The absolute error between the exact and approximate solution in the suggested method $x_{11}(t), x_{12}(t), x_{13}(t), x_{21}(t)$ is stated in table 2.

The method's accuracy is demonstrated in the table even for simple polynomial basis functions.

In Figure 1, we plot and compare the exact solution with the numerical solution listed in Table 2.

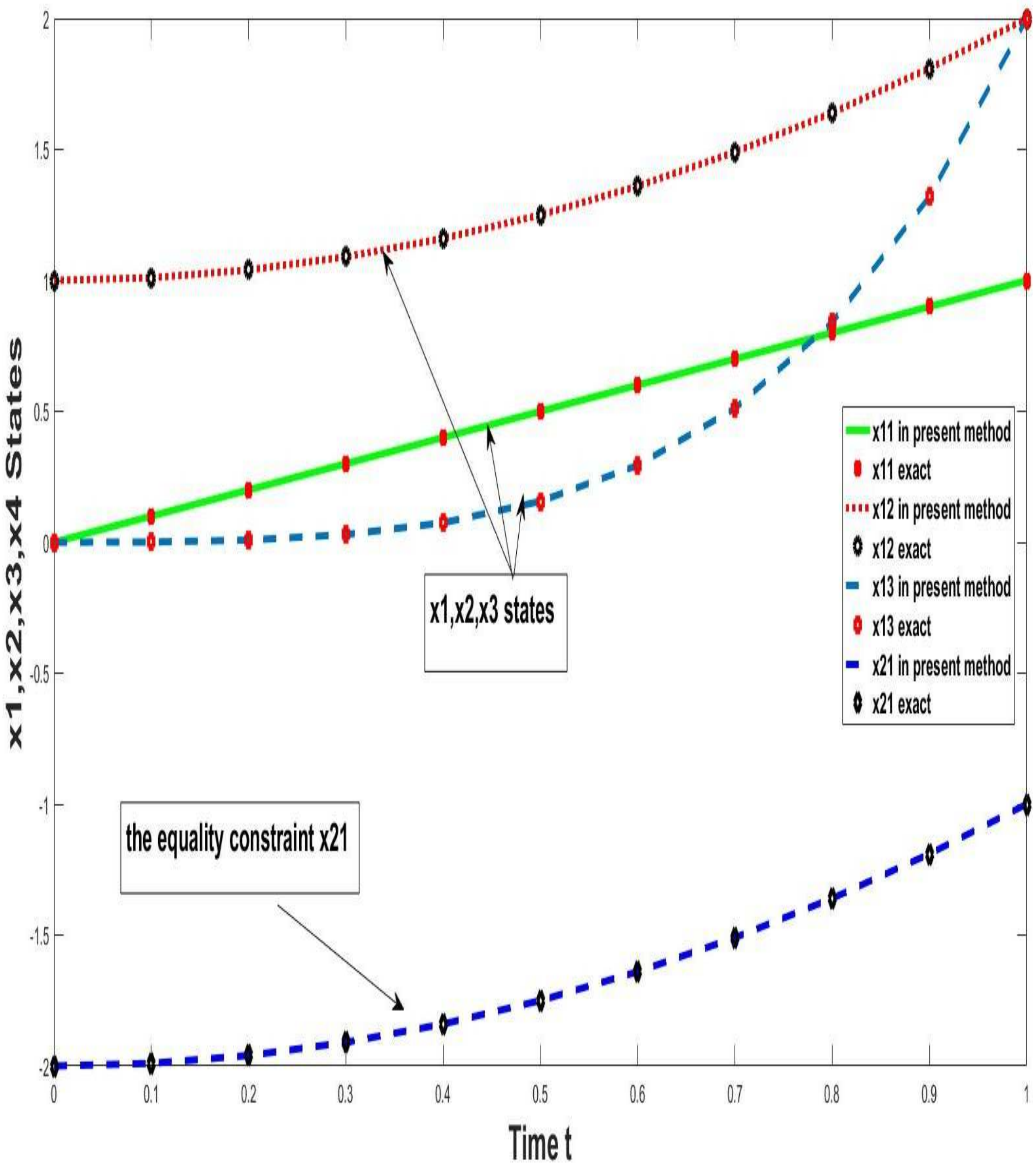

Figure 1: The differential-equality states in the present method coincide with the exact solution with respect to an open-loop control $u(t) = \cos t$.

4 Discussion and Conclusions

As a separable Banach space, the solvability's setting space is assumed to guarantee that the countable linear independent set of functions exist. In this paper, both an implicit function theorem and the differentiation index have been used to be the base of the proposed approach. In the beginning, the approach repeats the differentiation index until the required conditions (of using implicit function theorem) are met. Accordingly, the algebraic equations for the unknown state-space parameters are solved as reduced-order ordinary differential equations. With consistent initial conditions, these equations are defined on a class of algebraic constraints. Then, the operator form has been used to write the system. In this case, the variational formulation wss developed. Next, as elements of a linear combination of elements of the setting space basis, the solution has been formulated. Generally, as a perturbation in the non-homogenous functions, this system was treated as a perturbation. An example of index-4 illustrated our result. Even with a very simple basis function, the required accuracy of the proposed approach has been demonstrated. To include splines, an orthonormal basis can be modified easily. According to the theoretical justification and the numerical results, the proposed method is recommended for application problems (optimal linear regulator DAEs are included).

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