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Extending Fibonacci's Method for Computing Pythagorean Triples

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Abstract

In this paper, we extend Fibonacci's method for computing Pythagorean triples of the form $(x, y, y + 1)$ to compute Pythagorean triples of the form $(x, y, y + \ell)$ in three ways. Each of these three constructions exploits a certain relationship between x and ℓ which must be true of any Pythagorean triple. Constructions 1 and 2 provide new proofs of already known formulae, whereas Construction 3 provides a novel method of computing Pythagorean triples.

1 Introduction

A *Pythagorean triple* is a triple (x, y, z) of natural numbers satisfying

$$x^2 + y^2 = z^2.$$

Fibonacci's method [7] of computing Pythagorean triples is based on the series expansion of a perfect square and is briefly summarized: the square of y is the sum of the first y odd numbers ($y^2 = \sum_{i=1}^{y}(2i-1)$). Therefore, addition

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of the next odd number $2y+1$ results in the next perfect square $(y+1)^2 = \sum_{i=1}^{y+1}(2i-1)$. Hence, it is easy to generate triples of the form $(x, y, y+1)$ since $x^2 = (y+1)^2 - y^2 = 2y+1$, leading to $y = \frac{x^2-1}{2}$. Fibonacci's method was recently studied by Amato [1] in the special case that y is odd. Herein, we extend Fibonacci's method to the general triple $(x, y, z) = (x, y, y+\ell)$ (Fibonacci's method corresponds to the case $\ell = 1$). We can rewrite the Pythagorean equation as

$$x^2 = z^2 - y^2 = (z+y)(z-y) = ((y+\ell)+y)((y+\ell)-y) = (2y+\ell)\ell.$$

From $x^2 = (2y+\ell)\ell$, we immediately derive a proposition.

Proposition 1.1. *If $(x, y, y+\ell)$ is a Pythagorean triple, then*

1. $x > \ell$,
2. $x \equiv \ell \pmod 2$ – *i.e.*, $x = \ell + 2p$ *for some* $p \in \mathbb{N}$,
3. $\frac{x^2}{\ell} > \ell$, *and*
4. $\frac{x^2}{\ell} \equiv \ell \pmod 2$.

$\square$

This proposition is used to construct natural solutions to the Pythagorean equation of the form $(x, y, y+\ell)$ via three constructions:

1. The (a, b, c) method depends on factoring ℓ.
2. The (ℓ, p) method depends on x having the same parity as ℓ.
3. The (ℓ, q) method depends on x^2/ℓ having the same parity as ℓ.

Constructions 1 and 2 are shown to provide new proofs of techniques known already to be complete, whereas Construction 3 provides a novel approach to computing Pythagorean triples.

2 Construction 1: The (a, b, c) Method

Suppose that $(x, y, z) = (x, y, y+\ell)$ is a Pythagorean triple. The (a, b, c) method generates (x, y, z) by writing $\ell = a^2 b$ as a product of paired and unpaired factors. Since x^2 is a perfect square and

$$x^2 = (2y+\ell)\ell = (2y+\ell)a^2 b,$$

$2y+\ell$ must have b as a factor and $\frac{2y+\ell}{b} = c^2$ is a perfect square; that is, $2y+\ell = bc^2$. This factorization immediately gives

$$x = \sqrt{(2y+\ell)\ell} = \sqrt{(bc^2)(a^2b)} = abc$$

and

$$y = \frac{bc^2 - \ell}{2} = \frac{bc^2 - a^2b}{2} = \frac{(c^2-a^2)b}{2}$$

Finally, we complete the Pythagorean triple by computing

$$z = y + \ell = \frac{(c^2-a^2)b}{2} + a^2b = \frac{(c^2+a^2)b}{2}$$

The Pythagorean triples are expressed as

$$(x,y,z) = \left(abc, \frac{(c^2-a^2)b}{2}, \frac{(c^2+a^2)b}{2}\right)$$

This argument culminates in our first construction.

Construction 1. *Suppose $\ell = a^2b$ and (b,c) satisfies $c > a$ and either*

1. *b is even, or*

2. *$a \equiv c \pmod 2$.*

Then

$$(x,y,z) = \left(abc, \frac{(c^2-a^2)b}{2}, \frac{(c^2+a^2)b}{2}\right)$$

is a Pythagorean triple.

Construction 1 generates any multiple of the primitive triangles given by Euclid's formula [4, Book X, Lemma 1], which we would call the (a,c) method. Euclid's (a,c) method is known to be complete for primitives and is non-repeating (see [5, Appendix B] or [9, Theorem 2]). Our (a,b,c) method is complete as it generates all b-multiples of the primitives. This provides a new proof that Euclid's formula can be extended to all triples, which is already known in the literature [6, 3].

3 Construction 2: The (ℓ, p) Method

Suppose that $(x, y, z) = (x, y, y + \ell)$ is a Pythagorean triple. The (ℓ, p) method generates (x, y, z) by explicitly exploiting the fact that x must have the same parity as ℓ. That is, $x = \ell + 2p$ for some natural p and $x^2 = \ell^2 + 4\ell p + 4p^2$. Solving the equation

$$(2y + \ell)\ell = x^2 = (\ell + 2p)^2 = \ell^2 + 4p\ell + 4p^2$$

for y gives

$$y = 2p + \frac{2p^2}{\ell}$$

and

$$z = y + \ell = \ell + 2p + \frac{2p^2}{\ell}.$$

This argument culminates in our second construction.

Construction 2. *Given ℓ, suppose that $p \in \mathbb{N}$ satisfies $\frac{2p^2}{\ell} \in \mathbb{N}$. Then*

$$(x, y, z) = \left(\ell + 2p, 2p + \frac{2p^2}{\ell}, \ell + 2p + \frac{2p^2}{\ell}\right)$$

is a Pythagorean triple.

Proposition 3.1. *Construction 2 is complete.*

Proof. Suppose that $(x, y, z) = (x, y, y+\ell)$ is any Pythagorean triple and set $p = \frac{x-\ell}{2}$, which is positive by Proposition 1.1(1). By Proposition 1.1(2), x and ℓ have the same parity; that is, the difference $x-\ell$ is even and $p = \frac{x-\ell}{2}$ is natural. This choice of p therefore generates the Pythagorean triple (x, y, z) as described in Construction 2 since $x = \ell + 2p$. □

Relationship to Dickson's Solution Dickson [2] generated Pythagorean triples for any pair $(\ell, m) \in \mathbb{N}$ by the formula

$$(x, y, z) = \left(l + \sqrt{2m\ell}, m + \sqrt{2m\ell}, l + m + \sqrt{2m\ell}\right)$$

When substituting $m = \frac{2p^2}{\ell}$, our natural solutions then coincide with Dickson's. Pythagorean triples using Dickson's formula are irrational except when $2ml$ is a perfect square. Our equation provides only rational solutions; natural solutions are obtained when $\frac{2p^2}{\ell}$ is a natural number (i.e., $2p^2$ is a natural multiple of ℓ). This provides, then, an alternative proof that Dickson's method is complete than to what is currently found in the literature [8].

4 Construction 3: The (ℓ, q) Method

Suppose that (x, y, z) is a Pythagorean triple. The (ℓ, q) method depends on explicitly constraining x^2/ℓ to the same parity as ℓ. By Proposition 1.1, x and ℓ have the same parity. Therefore, $x = \ell + 2p$ for some $p \in \mathbb{N}$ and $x^2 = (\ell + 2p)^2$. Further, x^2/ℓ and ℓ have the same parity. This means that for odd ℓ, $x^2 = (2q - 1)\ell$ for some q, and for even ℓ, $x^2 = 2q\ell$ for some q.

Therefore, for odd ℓ, the equation $x^2 - x^2 = 0$ becomes

$$(\ell + 2p)^2 - (2q - 1)\ell = 4p^2 + 4p\ell + \ell^2 - (2q - 1)\ell = 0$$

and has solutions

$$\begin{aligned} p &= \frac{-4\ell + \sqrt{16\ell^2 - 16(\ell^2 - (2q-1)\ell)}}{8} \\ &= \frac{-\ell + \sqrt{\ell^2 - (\ell^2 - (2q-1)\ell)}}{2} \\ &= \frac{-\ell + \sqrt{(2q-1)\ell}}{2}. \end{aligned}$$

All $q > \frac{\ell+1}{2}$ such that $(2q - 1)\ell$ is a perfect square yield triples

$$(x, y, z) = \left(\sqrt{(2q-1)\ell}, q - \frac{\ell+1}{2}, q + \frac{\ell-1}{2}\right)$$

Analogously, for even ℓ, the equation

$$(\ell + 2p)^2 - 2q\ell = 4p^2 + 4p\ell + \ell^2 - 2q\ell = 0$$

has solutions

$$p = \frac{-4\ell + \sqrt{16\ell^2 - 16(\ell^2 - 2q\ell)}}{8} = \frac{-\ell + \sqrt{\ell^2 - (\ell^2 - 2q\ell)}}{2} = \frac{-\ell + \sqrt{2q\ell}}{2}$$

All $q > \frac{\ell}{2}$ such that $2q\ell$ is a perfect square yield triples

$$(x, y, z) = \left(\sqrt{2q\ell}, q - \frac{\ell}{2}, q + \frac{\ell}{2}\right)$$

This argument culminates in our third construction.

Construction 3. *Given ℓ, suppose that $q \in \mathbb{N}$ satisfies $q > \frac{\ell+1}{2}$ and either*

1. *$x^2 = (2q-1)\ell$ if ℓ is odd, or*
2. *$x^2 = 2q\ell$ if ℓ is even.*

Then

$$(x, y, z) = \left(\sqrt{(2q-1)\ell}, q - \frac{\ell+1}{2}, q + \frac{\ell-1}{2}\right)$$

is a Pythagorean triple if ℓ is odd and

$$(x, y, z) = \left(\sqrt{2q\ell}, q - \frac{\ell}{2}, q + \frac{\ell}{2}\right)$$

is a Pythagorean triple if ℓ is even.

Corollary 4.1. *Given ℓ, there is a family of Pythagorean triples (x, y, z) indexed by $m > \ell$ satisfying $m \equiv \ell \pmod 2$.*

Proof. Let

$$m = \begin{cases} 2q & \ell \equiv 0 \pmod 2 \\ 2q-1 & \ell \equiv 1 \pmod 2 \end{cases}$$

Then, since $m = \ell + 2y$ and by Construction 3, there is a Pythagorean triple

$$(x, y, z) = \left(\sqrt{m\ell}, \frac{m-\ell}{2}, \frac{m+\ell}{2}\right)$$

□

Note When m and ℓ are both perfect squares, the technique outlined in the argument of Construction 3 yields Euclid's solution [4].

Proposition 4.2. *Construction 3 is complete.*

Proof. Suppose that $(x, y, z) = (x, y, y + \ell)$ is any Pythagorean triple. If ℓ is even, set

$$q = \frac{x^2}{2\ell} = \frac{(2y+\ell)\ell}{2\ell} = \frac{2y+\ell}{2}$$

which is natural since $2y + \ell$ is even. If ℓ is odd, set

$$q = \frac{x^2+\ell}{2\ell} = \frac{(2y+\ell)\ell+\ell}{2\ell} = \frac{2y+\ell+1}{2}$$

In both cases, $q > \frac{\ell+1}{2}$, q satisfies the parity-appropriate condition (1) or (2) in Construction 3. Therefore, the Pythagorean triple (x, y, z) can be constructed from q. □

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