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On the Diophantine Equation $a^x + (a+2)^y = z^2$, where $a \equiv 3 \pmod{20}$

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Abstract

In this paper, we show that $(x, y, z) = (1, 0, \sqrt{a+1})$, where $a = (10k-2)^2 - 1$ such that $k \in \mathbb{Z}$, is a non-negative integer solution for the Diophantine equation $a^x + (a+2)^y = z^2$, where $a \equiv 3 \pmod{20}$ and $a \in \mathbb{N}$.

1 Introduction

In 1844, Catalan [1] conjectured that $(3, 2, 2, 3)$ is the unique solution (a, b, x, y) for the Diophantine equation

$$a^x - b^y = 1,$$

where a, b, x and y are integers such that $\min\{a, b, x, y\} > 1$.

In 2004, the Catalan's conjecture was proved by Mihailescu [2].

In 2012, Sroysang [3] showed that $(1, 0, 2)$ is the only solution (x, y, z) for the Diophantine equation

$$3^x + 5^y = z^2,$$

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where x, y and z are non-negative integers.

In 2013, Chotchaisthit [4] studied the Diophantine equation

$$p^x + (p+1)^y = z^2,$$

where p is a mersenne prime.

In 2014, Sroysang [5] proved that $(1, 0, 12)$ is a unique non-negative integer solution (x, y, z) for the Diophantive equation

$$143^x + 145^y = z^2,$$

where x, y and z are non-negative integers.

In 2020, Dokchan and Pakapongpun [6] showed that the Diophantine equations

$$a^x + (a+2)^y = z^2,$$

where $a \equiv 5 \pmod{42}$ and $a \in \mathbb{N}$ has solutions in non-negative integers. Inspired by [3, 5] and [6], we study the Diophantine equation

$$a^x + (a+2)^y = z^2,$$

where $a \equiv 3 \pmod{20}$ and $a \in \mathbb{N}$.

2 Main results

Lemma 2.1. *If $a \equiv 3 \pmod{20}$ and x is and odd, then $a^x \equiv 2 \pmod{5}$ or $a^x \equiv 3 \pmod{5}$.*

Proof. Since $a \equiv 3 \pmod{20}$, $a \equiv 3 \pmod{5}$. So we can write $x = 4k-1$ or $x = 4k-3$, where $k \in \mathbb{Z}$.

If $x = 4k-1$, then $a^4 \equiv 1 \pmod{5}$ and thus $a^{4k-4} \equiv 1 \pmod{5}$. Hence,

$$a^x = a^{4k-1} = a^{4k-4} \cdot a^3 \equiv 2 \pmod{5}.$$

If $x = 4k-3$, and since $a^{4k-4} \equiv 1 \pmod{5}$, we have

$$a^x = a^{4k-3} = a^{4k-4} \cdot 3 \equiv a \equiv 3 \pmod{5}.$$

Therefore, $a^x \equiv 2 \pmod{5}$ or $a^x \equiv 3 \pmod{5}$.

□

Theorem 2.2. $(x, y, z) = (1, 0, \sqrt{a+1})$, *where* $a = (10k-2)^2 - 1$ *such that* $k \in \mathbb{Z}$ *is a non-negative integer solution for the Diophantine equation* $a^x + (a+2)^y = z^2$, *where* $a \equiv 3 \pmod{20}$ *and* $a \in \mathbb{N}$.

Proof. We consider the Diophantine equation $a^x + (a+2)^y = z^2$ in 3 cases as following.

Case 1: Suppose that $y = 0$. We obtain the Diophantine equation

$$a^x + 1 = z^2.$$

If $x = 0$, then $z^2 = 2$ which is impossible.

If $x = 1$, then $z^2 = a+1$. Thus $z = \sqrt{a+1}$. Since $a \equiv 3 \pmod{20}$, there is $m \in \mathbb{Z}$ such that $a = 3 + 20m$. Then $z = \sqrt{3+20m+1} = 2\sqrt{5m+1}$. It follows that $5m+1 = k^2$ for some $k \in \mathbb{Z}$ and so $k^2 \equiv 1 \pmod 5$. Hence

$$k \equiv 1 \pmod 5 \text{ or } k \equiv -1 \pmod{5.})$$

If $k \equiv 1 \pmod 5$, there is $t_1 \in \mathbb{Z}$ such that $k = 5t_1 + 1$. Since $5m+1 = k^2 = (5t_1+1)^2$, $m = 5t_1^2 + 2t_1$. Thus,

$$a = 3 + 20m = 3 + 20(5t_1^2 + 2t_1) = (10t_1 + 2)^2 - 1.$$

If $k \equiv -1 \pmod 5$, there is $t_2 \in \mathbb{Z}$ such that $k = 5t_2 - 1$. Since $5m+1 = k^2 = (5t_2 - 1)^2$, $m = 5t_2^2 - 2t_2$. Thus

$$a = 3 + 20m = 3 + 20(5t_2^2 - 2t_2) = (10t_2 - 2)^2 - 1.$$

But $(10t_1 + 2)^2 - 1 = (10t_2 - 2)^2 - 1$, when $t_1 = -t_2$. Therefore,

$$a = (10k - 2)^2 - 1,$$

for some $k \in \mathbb{Z}$. Thus $(x, y, z) = (1, 0, \sqrt{a+1})$ where $a = (10k-2)^2 - 1$ such that $k \in \mathbb{Z}$ is the solution of the Diophantine equation $a^x + (a+2)^y = z^2$ in this case.

If $x > 1$, then the Diophantine equation $z^2 - a^x = 1$ has no non-negative solution by the Catalan's conjecture.

Case 2: Suppose that $x = 0$. We obtain the Diophantine equation

$$1 + (a+2)^y = z^2.$$

If $y = 0$, then $z^2 = 2$, which is impossible.

If $y = 1$, then $z^2 = a + 3$. Since $z^2 = a + 3 \equiv 6 \pmod{20}$, which is impossible.

If $y > 1$, then the Diophantine equation $z^2 - (a+2)^y = 1$ has no non-negative solution by Catalan's conjecture.

Case 3: Suppose that $x \geq 1$ and $y \geq 1$. Since $a^x + (a+2)^y = z^2$ and $a \equiv 3 \pmod{20}$, a^x and $(a+2)^y$ are odd and so $a^x + (a+2)^y$ is even. Then z is even, so $z^2 \equiv 0 \pmod 4$. Since $a \equiv 3 \pmod{20}$, $a \equiv 3 \pmod 4$ and $a + 2 \equiv 3 + 2 \equiv 1 \pmod 4$. Thu,

$$(a+2)^y \equiv 1 \pmod 4.$$

It follows that

$$a^x + 1 \equiv a^x + (a+2)^y \equiv z^2 \equiv 0 \pmod 4.$$

Hence

$$a^x \equiv -1 \equiv 3 \pmod 4.$$

Thus x is odd. By Lemma 2.1,

$$a^x \equiv 2 \pmod 5 \text{ or } a^x \equiv 3 \pmod 5.$$

Since $a \equiv 3 \pmod{20}$, $(a+2)^y \equiv 0 \pmod{20}$ and so $(a+2)^y \equiv 0 \pmod 5$. It follows that

$$z^2 \equiv 2 \pmod 5 \text{ or } z^2 \equiv 3 \pmod 5.$$

This is a contradiction since for any integer z, $z^2 \equiv 0 \pmod 5$ or $z^2 \equiv 1 \pmod 5$ or $z^2 \equiv 4 \pmod 5$.

$\square$

Note: There are infinitely many integers a's which are of the form $a \equiv 3 \pmod{20}$ such that the exponential Diophantine equation $a^x + (a+2)^y = z^2$, where x, y and z are non-negative integers, has non-negative integer solutions. The following table shows some Exponential Diophantine equations for some particular values of a between 1 to 350.

$a^x + (a+2)^y = z^2$	Solution of the equation
$3^x + 5^y = z^2$	$(x, y, z) = (1, 0, 2)$
$23^x + 25^y = z^2$	No solution
$43^x + 45^y = z^2$	No solution
$63^x + 65^y = z^2$	$(x, y, z) = (1, 0, 8)$
$83^x + 85^y = z^2$	No solution
$103^x + 105^y = z^2$	No solution
$123^x + 125^y = z^2$	No solution
$143^x + 145^y = z^2$	$(x, y, z) = (1, 0, 12)$
$163^x + 165^y = z^2$	No solution
$183^x + 185^y = z^2$	No solution
$203^x + 205^y = z^2$	No solution
$223^x + 225^y = z^2$	No solution
$243^x + 245^y = z^2$	No solution
$263^x + 265^y = z^2$	No solution
$283^x + 285^y = z^2$	No solution
$303^x + 305^y = z^2$	No solution
$323^x + 325^y = z^2$	$(x, y, z) = (1, 0, 19)$
$343^x + 345^y = z^2$	No solution

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