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Applications on Fuzzy Translation and Fuzzy Multiplication of KU-algebras

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Abstract

In this paper, we introduce the concepts of fuzzy translation, fuzzy multiplication and fuzzy magnified translation of a KU-algebra. Moreover, We investigate fundamental properties of fuzzy KU-subalgebras and fuzzy KU-ideals regarding these concepts. Furthermore, we introduce Fuzzy extension KU-ideal and provide relations between this concept and fuzzy translation and fuzzy multiplication.

1 Introduction

The theory of fuzzy set was introduced in 1965 by Zadeh [11] as a generalization of classic set theory. Since then this new set has attracted the attention of many mathematicians as it has important applications in many fields such as decision making, control engineering, computer sciences, applied mathematics and information systems. This theory has been applied on many algebras; for instance, BCK-algebra [10] and BCI-algebra. Prabpayak and Leerawat introduced the algebraic structure KU-algebra and studied some of its properties [7], [8]. Many researchers applied fuzzy set properties on KU-algebra [3], [6], [5], [4], [2], [1], [9].

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In this paper, we study fuzzy translation, fuzzy multiplication and fuzzy magnified translation of a KU-algebra.

2 Preliminaries

Recall that a nonempty set X with a binary operation $*$ is called a KU-algebra if, for $r, s, t \in X$, the following conditions hold

(KU1) $(r * s) * [(s * t) * (r * t)] = 0$.

(KU2) $r * 0 = 0$.

(KU3) $0 * r = r$.

(KU4) $r * s = 0 = s * r$ implies that $r = s$.

A relation $\leq$ is defined on X as follows:

$$r \leq s \text{ if and only if } s * r = 0$$

Thus we can rewrite the above conditions as follows:

(KU1′) $(s * t) * (r * t) \leq r * s$.

(KU2′) $0 \leq r$.

(KU3′) $0 * r = r$.

(KU4′) $r \leq s$, $s \leq r$ implies that $r = s$.

Now, we can introduce some basic definitions.

Definition 2.1. *A nonempty subset N of a KU-algebra X is called a subalgebra of X if $r * s \in N$, for all $r, s \in N$.*

Definition 2.2. *Let X be a KU-algebra. A fuzzy set η in X is called a fuzzy subalgebra if it satisfies the following two conditions for $r, s \in X$:*

(1) $\eta(0) \geq \eta(r)$.

(2) $\eta(r) \geq \min\{\eta(r * s), \eta(s)\}$.

Definition 2.3. *A nonempty subset M of a KU-algebra X is called a KU-ideal of X if it satisfies the following conditions:*

(1) $0 \in M$.

(2) $r * (s * t) \in M, s \in M$ *implies that* $r * t \in M$, *or all* $r, s, t \in M$.

Throughout this paper, X is a KU-algebra unless stated otherwise.

3 Fuzzy Translation and Fuzzy Multiplication of KU-algebras

In this section, we present the concepts of fuzzy translation and fuzzy multiplication of a KU-algebra.

For a fuzzy subset η of$X$, we define

$$T = 1 - \sup\{\eta(r) : r \in X\}.$$

Now, we introduce the main definitions in this section.

Definition 3.1. *Let η be a fuzzy subset of X and $\delta \in [0, T]$. The map*

$$\eta_\delta^T : X \longrightarrow [0, 1]$$

is called a fuzzy δ-translation of η if it satisfies

$$\eta_\delta^T(r) = \eta(r) + \delta, \qquad \textit{for all } r \in X$$

Definition 3.2. *Let η be a fuzzy subset of X and $\gamma \in [0, 1]$. The map*

$$\eta_\gamma^M : X \longrightarrow [0, 1]$$

is called a fuzzy γ-multiplication of η if it satisfies

$$\eta_\gamma^M(r) = \gamma \cdot \eta(r), \qquad \textit{for all } r \in X$$

Example 1. *Consider X, the fuzzy KU-algebra represented in the following table,*

*	0	1	2	3
0	0	1	2	3
1	0	0	0	2
2	0	2	0	1
3	0	0	0	0

Define a fuzzy subset η of X by

$$\eta(0) = 0.7, \quad \eta(1) = \eta(2) = \eta(3) = 0.4.$$

Then η is a fuzzy Ku-subalgebra of X. Now, $T = 1 -$ sup $\{\eta(r) : r \in X\}$ which implies that $T = 0.5$. Choose $\delta = 0.2 \in [0.T]$ and $\gamma = 0.3 \in (0, 1]$ and define the map

$$\eta_{0.4}^T : X \longrightarrow [0.1]$$

by

$$\eta^T_{0.2}(r) = \begin{cases} 0.9 & \text{if } r = 0 \\ 0.6 & \text{otherwise} \end{cases}$$

Thus $\eta^T_{0.2}(r) = \eta(r) + 0.2$ *is a fuzzy 0.2-translation. Now, define the map*

$$\eta^M_{0.3} : X \longrightarrow [0.1]$$

by

$$\eta^M_{0.3}(r) = \begin{cases} 0.21 & \text{if } r = 0 \\ 0.12 & \text{otherwise} \end{cases}$$

Hence $\eta^M_{0.3}(r) = 0.3\eta(r)$ *is a fuzzy 0.3-multiplication.*

Theorem 3.3. *Let* η *be a fuzzy subset of* X*. Then, for* $\delta \in [0, T]$*, the fuzzy* δ*-translation* η^T_δ *of* η *is a fuzzy KU-subalgebra of* X *if and only if* η *is a fuzzy KU-subalgebra of* X*.*

Proof. Suppose that η is a fuzzy KU-subalgebra of X. Let $r, s \in X$ and $\delta \in [0, T]$. It follows that

$$\eta(r) \geq \min\{\eta(r * s), \eta(s)\}.$$

Now, we obtain

$$\begin{aligned} \eta^T_\delta(r) &= \eta(r) + \delta \\ &\geq \min\{\eta(r * s), \eta(s)\} + \delta \\ &= \min\{\eta(r * s) + \delta, \eta(s) + \delta\} \\ &= \min\{\eta^T_\delta(r * s), \eta^T_\delta(s)\} \end{aligned}$$

and hence η^T_δ is a fuzzy KU-subalgebra of X.

For the converse, assume that η^T_δ is a fuzzy KU-subalgebra of X. Let $r, s \in X$. Then

$$\begin{aligned} \eta(r) + \delta &= \eta^T_\delta(r) \\ &\geq \min\{\eta^T_\delta(r * s), \eta^T_\delta(s)\} \\ &= \min\{\eta(r * s) + \delta, \eta(s) + \delta\} \\ &\geq \min\{\eta(r * s), \eta(s)\} + \delta \end{aligned}$$

which implies that

$$\eta(r) \geq \min\{\eta(r * s), \eta(s)\}$$

and therefore, η is a fuzzy KU-subalgebra of X. □

Theorem 3.4. *Let η be a fuzzy subset of X. Then, for $\gamma \in (0, 1]$, the fuzzy γ-multiplication η_γ^M of η is a fuzzy KU-subalgebra of X if and only if η is a fuzzy KU-subalgebra of X.*

Proof. Suppose that η is a fuzzy KU-subalgebra of X. Let $r, s \in X$ and $\gamma \in (0, 1]$. It follows that

$$\eta(r) \geq \min\{\eta(r * s), \eta(s)\}.$$

Now, we obtain

$$\begin{aligned} \eta_\gamma^M(r) &= \gamma \cdot \eta(s) \\ &\geq \gamma \cdot \ \min\{\eta(r * s), \eta(s)\} \\ &= \min\{\gamma \cdot \eta(r * s), \gamma \cdot \eta(s)\} \\ &= \min\{\eta_\gamma^M(r * s), \eta_\gamma^M(s)\} \end{aligned}$$

and hence η_γ^M is a fuzzy KU-subalgebra of X.

For the converse, assume that η_γ^M is a fuzzy KU-subalgebra of X. Let $r, s \in X$. Then

$$\begin{aligned} \gamma \cdot \eta(r) &= \eta_\gamma^M(s) \\ &= \min\{\eta_\gamma^M(r * s), \eta_\gamma^M(s)\} \\ &= \min\{\gamma \cdot \eta(r * s), \gamma \cdot \eta(s)\} \\ &\geq \gamma \cdot \ \min\{\eta(r * s), \eta(s)\} \end{aligned}$$

which implies that

$$\eta(r) \geq \min\{\eta(r * s), \eta(s)\}$$

and therefore η is a fuzzy KU-subalgebra of X. □

Recall that a fuzzy set, η, of X is called a fuzzy KU-ideal of X if it satisfies the following conditions:
For $r, s, t \in X$:

(1) $\eta(0) \geq \eta(r)$.

(2) $\eta(r * t) \geq \min\{\eta(r * (s * t)), \eta(s)\}$.

Theorem 3.5. *Let η be a fuzzy subset of X. Then for $\delta \in [0, T]$, the fuzzy δ-translation η_δ^T of η is a fuzzy KU-ideal of X if and only if η is a fuzzy KU-ideal of X.*

Proof. Assume that η is a fuzzy KU-ideal of X. Let $r, s \in X$ and $\delta \in [0, T]$. Then, for $r, s, t \in X$, we have

$$\eta(0) \geq \eta(r)$$
$$\eta(r * t) \geq \min\{\eta(r * (s * t)), \eta(s)\}.$$

Now,

(1) $\eta_\delta^T(0) = \eta(0) + \delta \geq \eta(r) + \delta = \eta_\delta^T(r)$.

(2)

$$\begin{aligned}\eta_\delta^T(r * t) &= \eta(r * t) + \delta \\ &\geq \min\{\eta(r * (s * t)), \eta(s)\} + \delta \\ &= \min\{\eta(r * (s * t)) + \delta, \eta(s) + \delta\} \\ &= \min\{\eta_\delta^T(r * (s * t)), \eta_\delta^T(s)\}\end{aligned}$$

and thus η_δ^T is a fuzzy KU-ideal of X.

For the converse, suppose that η_δ^T is a fuzzy KU-ideal of X. Let $r, s \in X$. Then

(1) $\eta(0) + \delta = \eta_\delta^T(0) \geq \eta_\delta^T(r) = \eta(r) + \delta$ and thus $\eta(0) \geq \eta(r)$.

(2)

$$\begin{aligned}\eta(r * t) + \delta &= \eta_\delta^T(r * t) \\ &\geq \min\{\eta_\delta^T(r * (s * t)), \eta_\delta^T(s)\} \\ &= \min\{\eta(r * (s * t)) + \delta, \eta(s) + \delta\} \\ &\geq \min\{\eta(r * (s * t)), \eta(s)\} + \delta\end{aligned}$$

and hence $\eta(r * t) \geq \min\{\eta(r * (s * t)), \eta(s)\}$ which implies that η is a fuzzy KU-ideal of X as required. □

Theorem 3.6. *Let η be a fuzzy subset of X. Then, for $\gamma \in (0, 1]$, the fuzzy γ-multiplication η_γ^M of η is a fuzzy KU-ideal of X if and only if η is a fuzzy KU-ideal of X.*

Proof. Assume that η is a fuzzy KU-ideal of X. Let $r, s \in X$. Then, for $r, s, t \in X$, we have

$$\eta(0) \geq \eta(r)$$
$$\eta(r * t) \geq \min\{\eta(r * (s * t)), \eta(s)\}.$$

Now,

(1) $\eta_\gamma^M(0) = \gamma \cdot \eta(0) \geq \gamma \cdot \eta(r) = \eta_\gamma^M(r)$.

(2)

$$\begin{aligned}\eta_\gamma^M(r * t) &= \gamma \cdot \eta(r * t) \\ &\geq \gamma \cdot \ \min\{\eta(r * (s * t)), \eta(s)\} \\ &= \min\{\gamma \cdot \eta(r * (s * t)), \gamma \cdot \eta(s)\} \\ &= \min\{\eta_\gamma^M(r * (s * t)), \eta_\gamma^M(s)\}\end{aligned}$$

and thus η_γ^M is a fuzzy KU-ideal of X.

For the converse, suppose that η_γ^M is a fuzzy KU-ideal of X. Let $r, s \in X$. Then

(1) $\gamma \cdot \eta(0) = \eta_\gamma^M(0) \geq \eta_\gamma^M(r) = \gamma \cdot \eta(r)$ and thus $\eta(0) \geq \eta(r)$.

(2)

$$\begin{aligned}\gamma \cdot \eta(r * t) &= \eta_\gamma^M(r * t) \\ &\geq \min\{\eta_\gamma^M(r * (s * t)), \eta_\gamma^M(s)\} \\ &= \min\{\gamma \cdot \eta(r * (s * t)), \gamma \cdot \eta(s)\} \\ &\geq \gamma \cdot \ \min\{\eta(r * (s * t)), \eta(s)\}\end{aligned}$$

and hence $\eta(r * t) \geq \min\{\eta(r * (s * t)), \eta(s)\}$ which implies that η is a fuzzy KU-ideal of X as required. □

4 Fuzzy Magnified-$\gamma\delta$-Translation of KU-algebras

In this section, we introduce the concept of fuzzy magnified-$\gamma\delta$-translation of a KU-algebra. Then we study some properties regarding this concept.

Definition 4.1. *Let η be a fuzzy subset of X, $\delta \in [0, T]$ and $\gamma \in (0, 1]$. The map*

$$\eta_{\gamma\delta}^{MT} : X \longrightarrow [0, 1]$$

is called a fuzzy magnified-$\gamma\delta$-translation of η if it satisfies

$$\eta_{\gamma\delta}^{MT}(r) = \gamma \cdot \eta(r) + \delta, \qquad \text{for all } r \in X$$

Example 2. *Consider X, the fuzzy KU-algebra and the fuzzy KU-subalgebra, η, defined in example 1. Choose $\delta = 0.3$ and $\gamma = 0.5$ and define the map $\eta^{MT}_{(0.3)(0.5)} : X \longrightarrow [0.1]$ by:*

$$\eta^{MT}_{(0.5)(0.3)}(r) = \begin{cases} 0.65 & \textit{if } r = 0 \\ 0.5 & \textit{otherwise} \end{cases}$$

Then $\eta^{MT}_{(0.5)(0.3)}(r) = 0.5\eta(r) + 0.3$ is a fuzzy magnified-(0.5)(0.3)-translation.

Now, we prove the following theorems:

Theorem 4.2. *Suppose that η is a fuzzy subset of X, $\delta \in [0, T]$ and $\gamma \in (0, 1]$. Then η is a fuzzy KU-subalgebra of X if and only if the fuzzy magnified-$\gamma\delta$- translation of η, $\eta^{MT}_{(\gamma\delta)}$, is a fuzzy KU-subalgebra of X.*

Proof. Suppose that η is a fuzzy KU-subalgebra of X. Let $r, s \in X$. It follows that

$$\eta(r) \geq \min\{\eta(r * s), \eta(s)\}.$$

Now, we obtain

$$\begin{aligned} \eta^{MT}_{(\gamma\delta)}(r) &= \gamma \cdot \eta(s) + \delta \\ &\geq \min \gamma \cdot \{\eta(r * s), \eta(s)\} + \delta \\ &= \min\{\gamma \cdot \eta(r * s) + \delta, \gamma \cdot \eta(s) + \delta\} \\ &= \min\{\eta^{MT}_{(\gamma\delta)}(r * s), \eta^{MT}_{(\gamma\delta)}(s)\} \end{aligned}$$

and hence $\eta^{MT}_{(\gamma\delta)}$ is a fuzzy KU-subalgebra of X.

For the converse, assume that $\eta^{MT}_{(\gamma\delta)}$ is a fuzzy KU-subalgebra of X. Let $r, s \in X$. Then

$$\begin{aligned} \gamma \cdot \eta(r) + \delta &= \eta^{MT}_{(\gamma\delta)}(s) \\ &= \min\{\eta^{MT}_{(\gamma\delta)}(r * s), \eta^{MT}_{(\gamma\delta)}(s)\} \\ &= \min\{\gamma \cdot \eta(r * s) + \delta, \gamma \cdot \eta(s) + \delta\} \\ &\geq \gamma \cdot \ \min\{\eta(r * s), \eta(s)\} + \delta \end{aligned}$$

which implies that

$$\eta(r) \geq \min\{\eta(r * s), \eta(s)\}$$

and therefore η is a fuzzy KU-subalgebra of X. □

Theorem 4.3. *Let η be a fuzzy subset of X, $\delta \in [0,T]$ and $\gamma \in [0,1]$. Then η is a fuzzy KU-ideal of X if and only if the fuzzy magnified-$\gamma\delta$- translation of η, $\eta^{MT}_{(\gamma\delta)}$, is a fuzzy KU-ideal of X.*

Proof. Assume that η is a fuzzy KU-ideal of X. Then, for $r, s, t \in X$, we have

$$\eta(0) \geq \eta(r)$$

$$\eta(r * t) \geq \min\{\eta(r * (s * t)), \eta(s)\}.$$

Now,

(1) $\eta^{MT}_{(\gamma\delta)}(0) = \gamma \cdot \eta(0) + \delta \geq \gamma \cdot \eta(r) + \delta = \eta^{MT}_{(\gamma\delta)}(r)$.

(2)

$$\begin{aligned}\eta^{MT}_{(\gamma\delta)}(r * t) &= \gamma \cdot \eta(r * t) + \delta \\ &\geq \gamma \cdot \ \min\{\eta(r * (s * t)), \eta(s)\} + \delta \\ &= \min\{\gamma \cdot \eta(r * (s * t)) + \delta, \gamma \cdot \eta(s) + \delta\} \\ &= \min\{\eta^{MT}_{(\gamma\delta)}(r * (s * t)), \eta^{MT}_{(\gamma\delta)}(s)\}\end{aligned}$$

and thus $\eta^{MT}_{(\gamma\delta)}$ is a fuzzy KU-ideal of X.

For the converse, suppose that $\eta^{MT}_{(\gamma\delta)}$ is a fuzzy KU-ideal of X. Let $r, s \in X$. Then

(1) $\gamma \cdot \eta(0) + \delta = \eta^{MT}_{(\gamma\delta)}(0) \geq \eta^{MT}_{(\gamma\delta)}(r) = \gamma \cdot \eta(r) + \delta$ and thus $\eta(0) \geq \eta(r)$.

(2)

$$\begin{aligned}\gamma \cdot \eta(r * t) + \delta &= \eta^{MT}_{(\gamma\delta)}(r * t) \\ &\geq \min\{\eta^{MT}_{(\gamma\delta)}(r * (s * t)), \eta^{MT}_{(\gamma\delta)}(s)\} \\ &= \min\{\gamma \cdot \eta(r * (s * t)) + \delta, \gamma \cdot \eta(s) + \delta\} \\ &\geq \gamma \cdot \ \min\{\eta(r * (s * t)), \eta(s)\} + \delta\end{aligned}$$

and hence $\eta(r * t) \geq \min\{\eta(r * (s * t)), \eta(s)\}$ which implies that η is a fuzzy KU-ideal of X as required. ◻

5 Extension KU-Ideals

Let η_1, η_2 be two fuzzy sets of X in which η_2 is a fuzzy extension of η_1. If η_1 is a fuzzy KU-ideal of X implies that η_2 is a fuzzy KU-ideal of X, then η_2 is a fuzzy extension KU-ideal of η_1.

Theorem 5.1. *Let η be a fuzzy KU-ideal of X. Then for $\delta \in [0,1]$, the fuzzy δ-translation η^T_δ of η is a fuzzy extension KU-ideal of η.*

Proof. Since η is a fuzzy KU-ideal of X, theorem 3.5 implies that η^T_δ is a fuzzy KU-ideal of X. For $r \in X$, we have

$$\begin{aligned} \eta^T_\delta(r) &= \eta(r) + \delta \\ &\geq \eta(r) \end{aligned}$$

and therefore, η^T_δ of η is a fuzzy extension KU-ideal of η. □

Theorem 5.2. *Let η be a fuzzy KU-ideal of X. Then, for $\delta, \zeta \in [0,1]$ in which $\delta \geq \zeta$, the fuzzy δ-translation η^T_δ is a fuzzy extension KU-ideal of the fuzzy ζ-translation η^T_ζ.*

Proof. That η is a fuzzy KU-ideal of X, implies that η^T_δ and η^T_ζ are fuzzy KU-ideals of X by theorem 3.5. For $r \in X$, we have

$$\begin{aligned} \eta(r) + \delta &\geq \eta(r) + \zeta \\ \eta^T_\delta(r) &\geq \eta^T_\zeta(r) \end{aligned}$$

and thus η^T_δ is a fuzzy extension KU-ideal of η^T_ζ. □

Theorem 5.3. *Let η be a fuzzy set of X and $\delta, \gamma \in [0,1]$. If the fuzzy γ-multiplication η^M_γ of η is a fuzzy KU-ideal of X, then the fuzzy δ-translation η^T_δ is a fuzzy extension KU-ideal of η^M_γ.*

Proof. By theorem 3.6, η is a fuzzy KU-ideal of X. Then η^T_δ is a fuzzy KU-ideal of X by theorem 3.5. Now, for $r \in X$ we have

$$\begin{aligned} \eta^T_\delta(r) &= \eta(r) + \delta \\ &\geq \eta(r) \\ &\geq \gamma \cdot \eta(r) \\ &= \eta^M_\gamma(r) \end{aligned}$$

and hence η^T_δ is a fuzzy extension KU-ideal of η^M_γ. □

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