International Journal of Mathematics and
Computer Science, **17**(2022), no. 2, 685–693

Moment conditions for the periodic integer-valued autoregressive model

Bader S. Almohaimeed

Department of Mathematics
College of Science
Qassim University
Buraydah 51431, Saudi Arabia

email: bsmhiemied@qu.edu.sa

(Received October 8, 2021, Accepted November 15, 2021)

Abstract

We provide a necessary and sufficient condition for an integer-valued autoregression with periodic parameters and driven by an independent and periodically distributed innovation to have finite moments of any order.

1 Introduction

The integer-valued autoregression ($INAR$) studied by Al-Osh and Alzaid (1987) and by McKenzie (1985) is one of the best-known models for representing integer-valued time series that are frequently observed in real applications (economics, finance, insurance, signal processing, etc.). In recent decades, a growing literature on $INAR$ models has been built up, encompassing model structure, estimation methods and their statistical properties, various real applications, and many extensions (Latour 1998, Kedem and Fokianos 2002, McKenzie 2003 Davis et al. 2016, Ahmad and Francq 2016, Weiss 2018 and the reference therein).

Among the various extensions of the $INAR$ model, there is the periodic $INAR$ ($PINAR$) model introduced by Monteiro et al. (2010) in which the

Key words and phrases: Periodic INAR model, periodic ergodicity, periodic stationarity, finiteness of moment, binomial thinning.
AMS (MOS) Subject Classifications: 62M20, 60E05, 60J80, 60G10.
ISSN 1814-0432, 2022, http://ijmcs.future-in-tech.net

parameters are periodic over time. The $PINAR$ model aims at modeling integer-valued data that are characterized by seasonality, which appears in various applications (Bourguignon et al. 2016, Bracher and Held, 2017, Filho et al. 2021).

Building a time series model generally requires studying the existence of finite moments for the statistical procedures to have important statistical properties such as consistency and asymptotic normality. Within the framework of the first-order $PINAR(1)$ model, many estimation methods have been proposed such as conditional least squares estimation, conditional maximum likelihood method (Monteiro et al. 2010) and quasi-maximum likelihood estimation (Almohaimeed 2021). The consistency and asymptotic normality of these methods require the existence of moments up to the fourth order. When the process is driven by a Poisson distributed innovation, it is well known (Monteiro et al. 2010) that the marginal distributions of this process are also Poisson with finite moments of any order. For other distributions, conditions for the existence of moments are to be determined.

In this paper, we propose a necessary and sufficient condition for the first-order $PINAR(1)$ model to have higher order finite moments. We will show that apart from the conditions on the innovation process, the moment condition for any order is the same as the periodic stationarity condition proposed by Monteiro et al. (2010).

In the rest of this paper, we proceed as follows: In section 2, we present the $PINAR(1)$ model. The main result is given in section 3. We conclude this paper in section 4.

2 The periodic $INAR(1)$ model

Let $S \geq 1$ be a positive integer. A sequence of random variables $\{\varepsilon_t, t \in \mathbb{Z}\}$ is said to be independent and S-periodically distributed (ipd_S in short) if $\{\varepsilon_t, t \in \mathbb{Z}\}$ is independent and ε_t has the same distribution as ε_{kS+t} for all $t, k \in \mathbb{Z}$. Let $\{\varepsilon_t, t \in \mathbb{Z}\}$ be an ipd_S sequence with mean $\omega_t > 0$ and variance $\sigma_t^2 > 0$. Obviously, ω_t and σ_t^2 are S-periodic over time in the sense $\omega_t = \omega_{kS+t}$ and $\sigma_t^2 = \sigma_{kS+t}^2$ for all $t, k \in \mathbb{Z}$. An integer-valued stochastic process $\{X_t, t \in \mathbb{Z}\}$ is said to be a first-order periodic integer-valued autoregression ($PINAR(1)$ in short) if it is given by the following stochastic difference equation

$$X_t = \alpha_t \circ X_{t-1} + \varepsilon_t, \quad t \in \mathbb{Z}, \tag{1}$$

where α_t is S-periodic over t; i.e., $\alpha_t = \alpha_{kS+t}$ for all $t, k \in \mathbb{Z}$. The symbol $\circ$ denotes the binomial thinning operator (Steutel and Van Harn 1979) defined by

$$\alpha_t \circ X_{t-1} = \sum_{j=1}^{X_{t-1}} Y_{ti}$$

where $\{Y_{ti}, i \in \mathbb{N}\}_t$, called the counting series, is an ipd_S sequence of Bernoulli variables independent of X and satisfying

$$P(Y_{ti} = 1) = \alpha_t \in [0, 1].$$

Model (1) was proposed by Monteiro et al. (2010), generalizing the $INAR(1)$ model of Al-Osh and Alzaid (1987) to the case where the parameters are S-periodic time-varying.

3 Moment condition for the $PINAR(1)$ model

Monteiro et al. (2010) showed that equation (1) admits a unique strictly periodically stationary and periodically ergodic solution if and only if

$$\prod_{i=1}^{S} \alpha_i < 1. \tag{2}$$

For definitions of periodic stationarity and periodic ergodicity see, for example, Boyles and Gardner (1983), Aknouche (2015) and Aknouche et al. (2018). Naturally when $S = 1$, condition (2) reduces to the stationarity condition provided by Du and Li (1991) and Grunwald et al. (2000) for the $INAR(1)$.

When ε_t is Poisson distributed for all t, it is well-known (e.g., McKenzie, 2003, Monteiro et al. 2010) that the S marginal distributions of the periodically stationary solution $\{X_t, t \in \mathbb{Z}\}$ of (1) are also Poisson distributed and therefore they admit moments of any order. However, if the distributions of $\{\varepsilon_t, t \in \mathbb{Z}\}$ are not Poisson, in general the S marginal distributions of the process $\{X_t, t \in \mathbb{Z}\}$ are not necessarily the same as those of $\{\varepsilon_t, t \in \mathbb{Z}\}$. The moment conditions for the $PINAR(1)$ model with general distributions for the innovation sequence are thus of interest. In what follows, we will show that for any positive integer r, under the existence of the rth moments of the innovation, condition (2) is still necessary and sufficient for the existence of a strictly periodically stationary solution of (1) satisfying $E(X_t^r) < \infty$. Let

$\left\{ {r \atop i} \right\}$ denote the Stirling number of the second kind (Graham et al. 1988). First, consider the following three lemmas.

Lemma 1 (Knoblauch 2008, **Theorem 4.1)**
If X is binomially distributed, that is $X \sim B(N, \alpha)$, then

$$E(X^r) = \sum_{i=0}^{r} \left\{ {r \atop i} \right\} \alpha^i N^i. \tag{3}$$

Lemma 2 *For any random variable X with finite rth moment, we have*

$$E((\alpha \circ X)^r) = \sum_{i=0}^{r} \left\{ {r \atop i} \right\} \alpha^i E\left(X^i\right) = \alpha^r E(X^r) + P_{r-1}, \tag{4}$$

where

$$P_{r-1} = \sum_{i=0}^{r-1} \left\{ {r \atop i} \right\} \alpha^i E\left(X^i\right)$$

is a polynomial in α with degree $r-1$ and depends on $E(X^i)$ $(i = 0, ..., r-1)$.

Proof Since $(\alpha \circ X) | X$ is binomially distributed $\sim B(X, \alpha)$, the result follows from Lemma 1 and the properties of $\left\{ {r \atop i} \right\}$.

Lemma 3 (Bittanti et al. (2009)) *Let $\{a_t, t \in \mathbb{Z}\}$ and $\{b_t, t \in \mathbb{Z}\}$ be two S-periodic sequences of positives numbers; i.e., $a_t = a_{kS+t} > 0$ and* $b_t = b_{kS+t} > 0$ for all $t, k \in \mathbb{Z}$. *The ordinary difference equation*

$$x_t = a_t x_{t-1} + b_t, \quad t \in \mathbb{Z},$$

has a unique solution $\{x_t, t \in \mathbb{Z}\}$ if and only if

$$\prod_{v=1}^{S} a_v < 1.$$

We now state the main result of this paper.

Theorem 1 *For the $PINAR(1)$ model (1), assume that $E(\varepsilon_t^r) < \infty$ for all t $(r \geq 1)$. There exists a strictly periodically stationary solution $\{X_t, t \in \mathbb{Z}\}$ to (1) satisfying $E(X_v^r) < \infty$ for all $0 \leq v \leq S-1$, if and only if (2) holds.*

Proof i) *Sufficiency*:

If (2) is satisfied, then equation (1) admits a unique strictly periodically stationary and periodically ergodic solution given by (cf. Monteiro et al, 2010)

$$X_t = \sum_{j=0}^{\infty} \prod_{i=0}^{j-1} \alpha_{t-i} \circ \varepsilon_{t-j}, \ \ t \in \mathbb{Z}, \tag{5}$$

where the series in (5) converges *a.s.* Let $\left\{\overline{X}_t, t \in \mathbb{Z}\right\}$ be an integer-valued sequence defined by

$$\begin{cases} \overline{X}_t = \alpha_t \circ \overline{X}_{t-1} + \varepsilon_t & t \geq 1 \\ \overline{X}_t = 0 & t \leq 0, \end{cases} \tag{6}$$

and consider a random variable $X^{(v)}$ $(0 \leq v \leq S-1)$ having the same distribution as the term X_{nS+v} of the unique stationary solution given by (5). It is clear that for any $0 \leq v \leq S-1$, we have

$$\overline{X}_{nS+v} \underset{n\to\infty}{\overset{\mathcal{D}}{\rightarrow}} X^{(v)}$$

where $\underset{n\to\infty}{\overset{\mathcal{D}}{\rightarrow}}$ denotes the convergence in distribution as $n \to \infty$. Hence, from the weak convergence theory (cf. Billingsley, 1968), to show that $E\left(X_t^r\right)$ is finite for all $t \in \mathbb{Z}$ it suffices to show that

$$\lim_{t\to\infty} E\left(\overline{X}_t^r\right) < \infty$$

since the real function $x \mapsto x^r$ is continuous. We will use induction on r.

Assume that $\lim_{n\to\infty} E\left(\overline{X}_t^{r-1}\right) < \infty$ for all $t \in \mathbb{Z}$. From (6) one can write

$$\overline{X}_t = \sum_{j=0}^{t-1} \prod_{i=0}^{j-1} \alpha_{t-i} \circ \varepsilon_{t-j},\ t \geq 0$$

so $E\left(\overline{X}_t^r\right) < \infty$ for all $t \geq 0$. Therefore, by (6) and Lemma 2 we obtain the following S-periodic ordinary difference equation

$$\begin{aligned} E\left(\overline{X}_t^r\right) &= \sum_{i=0}^{r} \tbinom{r}{i} E\left(\alpha_t \circ \overline{X}_{t-1}\right)^i E\left(\varepsilon_t^{r-i}\right) \\ &= E\left(\alpha_t \circ \overline{X}_{t-1}\right)^r + \sum_{i=0}^{r-1} \tbinom{r}{i} E\left(\alpha_t \circ \overline{X}_{t-1}\right)^i E\left(\varepsilon_t^{r-i}\right) \\ &= \alpha_t^r E\left(\overline{X}_{t-1}^r\right) + P_{t,r-1} + \sum_{i=0}^{r-1} \tbinom{r}{i} E\left(\alpha_t \circ \overline{X}_{t-1}\right)^i E\left(\varepsilon_t^{r-i}\right) \\ &= \alpha_t^r E\left(\overline{X}_{t-1}^r\right) + \overline{P}_{t,r-1}, \end{aligned} \tag{7}$$

where from Lemma 2

$$P_{t,r-1} = \sum_{i=0}^{r-1} \begin{Bmatrix} r \\ i \end{Bmatrix} \alpha_t^i E\left(X^i\right))$$

and

$$\overline{P}_{t,r-1} = P_{t,r-1} + \sum_{i=0}^{r-1} \binom{r}{i} E\left(\alpha_t \circ \overline{X}_{t-1}\right)^i E\left(\varepsilon_t^{r-i}\right) \tag{8}$$

is finite by the induction hypothesis. Since equation (7) is an S-periodic ordinary difference equation, it is follows from Lemma 3 that under (2)

$$\lim_{t\to\infty} E\left(\overline{X}_t^r\right) < \infty,$$

establishing the result.

ii) Necessity:

Let $\{X_t, t \in \mathbb{Z}\}$ be the unique strictly periodically stationary solution of (1) such that $E\left(X_t^r\right) < \infty$ for all t. Again, using Lemma 2 we have

$$\begin{aligned} E\left(X_t^r\right) &= E\left(\alpha_t \circ X_{t-1}\right)^r + \sum_{i=0}^{r-1} \binom{r}{i} E\left(\alpha_t \circ X_{t-1}\right)^i E\left(\varepsilon_t^{r-i}\right) \\ &= \alpha_t^r E\left(X_{t-1}^r\right) + P_{t,r-1} + \sum_{i=0}^{r-1} \binom{r}{i} E\left(\alpha_t \circ X_{t-1}\right)^i E\left(\varepsilon_t^{r-i}\right) \\ &= \alpha_t^r E\left(X_{t-1}^r\right) + \widetilde{P}_{t,r-1} \end{aligned} \tag{9}$$

where, similar to (8), $\widetilde{P}_{t,r-1}$ is given by

$$\widetilde{P}_{t,r-1} = P_{t,r-1} + \sum_{i=0}^{r-1} \binom{r}{i} E\left(\alpha_t \circ X_{t-1}\right)^i E\left(\varepsilon_t^{r-i}\right).$$

Iterating the difference equation (9) S times, we obtain

$$E\left(X_t^r\right) = \left(\prod_{i=0}^{S-1} \alpha_{t-i}^r\right) E\left(X_{t-S}^r\right) + \sum_{j=1}^{S-1} \prod_{i=0}^{S-1} E\left(X_{t-i}^r\right) \widetilde{P}_{t-j,r-1}.$$

Using the periodic stationarity of the process $\{X_t, t \in \mathbb{Z}\}$, which implies that

$$E\left(X_t^r\right) = E\left(X_{t-S}^r\right),$$

it follows that

$$E\left(X_t^r\right) = \frac{\sum_{j=1}^{S-1} \prod\limits_{i=0}^{S-1} E\left(X_{t-i}^r\right) \widetilde{P}_{t-j,r-1}}{1 - \prod\limits_{i=0}^{S-1} \alpha_{t-i}^r} > 0. \tag{10}$$

In view of (10), we must have

$$\prod_{i=1}^{S} \alpha_i^r < 1,$$

which implies (2). $\square$

For the standard $INAR(1)$ model corresponding to $S = 1$ and $\alpha_t = \alpha$ for all t, the moment condition reduces to

$$\alpha_1 < 1$$

which is the stationarity and ergodicity condition provided by Du and Li (1991). See also Zheng et al. (2007).

4 Concluding remarks

In this paper, conditions for the existence of moments for the periodic $INAR(1)$ model have been proposed. The condition on the autoregressive parameters coincides with the periodic stationarity condition proposed by Monteiro et al. (2010). This is due to the linearity (thinning) in the operator of the $INAR(1)$ model, a fact that is related to periodic autoregression (cf. Aknouche 2015). Many extensions of the present result would be desirable. In particular, it would be important to find moment conditions for the pth-order $PINAR(p)$ model, the multivariate $PINAR(p)$, the random coefficient $PINAR$, among others.

References

[1] A. Ahmad, C. Francq, Poisson qmle of count time series models. *Journal of Time Series Analysis* **37**, (2014), 291–314.

[2] A. Aknouche, Explosive strong periodic autoregressions with multiplicity one, *Journal of Statistical Planning and Inference,* **161**, (2015), 50–72.

[3] A. Aknouche, E. Al-Eid, N. Demouche, Generalized quasi-maximum likelihood inference for periodic conditionally heteroskedastic models, *Statistical Inference for Stochastic Processes,* **21**, (2018), 485–511.

[4] M. A. Al-Osh, A. A. Alzaid, First-order integer-valued autoregressive (INAR(1)) process, *Journal of Time Series Analysis* **8,** (1987), 261–275.

[5] B. S. Almohaimeed, Asymptotic negative binomial quasi-likelihood inference for periodic integer-valued time series models, (2021), *Working paper.*

[6] P. Billingsley, Convergence of Probability Measure, Wiley, New York, 1968.

[7] S. Bittanti, P. Colaneri, *Periodic Systems - Filtering and control*, Springer-Verlag, 2009.

[8] J. Bracher, L. Held, Periodically stationary multivariate autoregressive models, *arXiv preprint, arXiv:1707.04635*, (2017)..

[9] M. Bourguignon, K. L. P. Vasconcellos, V. A. Reisen, M. Ispány, A Poisson INAR(1) process with a seasonal structure, *Journal of Statistical Computation and Simulation*, **86**, (2016), 373–387.

[10] R. A. Boyles, W. A. Gardner, Cycloergodic properties of discrete-parameter nonstationary stochastic processes, *IEEE, Trans. Infor. Theory* **29**, (1983), 105–114.

[11] R. A. Davis, S. H. Holan, R. Lund, N. Ravishanker, *Handbook of discrete-valued time series,* Chapman and Hall, 2016.

[12] J.-G. Du, Y. Li, The integer-valued autoregressive (INAR(p)) model, *Journal of Time Series Analysis*, **12**, (1991), 130–142.

[13] P. R. Filho, V. A. Reisen, P. Bondon, M. Ispany, M. Melo, F. Sarquis, A periodic and seasonal statistical model for non-negative integer-valued time series with an application to dispensed medications in respiratory diseases, *Applied Mathematical Modeling* **96**, (2021), 545–558.

[14] R. L. Graham, D. E. Knuth, O. Patashnik, *Concrete Mathematics*, Addison–Wesley, Reading MA, 1988.

[15] G. Grunwald, R. Hyndman, L. Tedesco, R. Tweedie, Non-Gaussian conditional linear AR(1) models, *Australian and New Zealand Journal of Statistics,* **42**, (2000), 479–495.

[16] B. Kedem, K. Fokianos, *Regression models for time series analysis*, Wiley, New York, 2002.

[17] A. Knoblauch, Closed-Form Expressions for the Moments of the Binomial Probability Distribution, *SIAM Journal on Applied Mathematics,* **69**, (2008), 197–204.

[18] A. Latour, Existence and stochastic structure of a non-negative integer valued autoregressive process, *Journal of Time Series Analysis,* **19**, (1998), 439–455.

[19] E. McKenzie, Some simple models for discrete variate time series, *Water Resources Bulletin.* **21**, (1985), 645–650.

[20] E. McKenzie, *Discrete variate time series*. In Handbook of statistics, Elsevier Science, Amsterdam, 2003.

[21] F. W. Steutel, K. Van Harn, Discrete analogues of self-decomposability and stability, *Annals of Probability* **5**, (1979), 893–899.

[22] C. H. Weiss, *An Introduction to Discrete-Valued Time Series*, John Wiley & Sons, 2018.

[23] H. T. Zheng, I. V. Basawa, S. Datta, The first order random coefficient integer-valued autoregressive processes, *Journal of Statistics, Planning and Inference,* **137**, (2007), 212–229.