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The length of terms and their measurement

Khwancheewa Wattanatripop, Thawhat Changphas

Department of Mathematics
Faculty of Science
Khon Kaen University
Khon Kaen 40002, Thailand

email: khwancheewa12@gmail.com, thacha@kku.ac.th

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Abstract

One of natural measures of the complexity of a term is the length of term, the total number of both operation symbols and variables that occur in the term. Based on the usual superposition operation and the generalized superposition operation of terms, two significant formulas for counting the length of terms in these operations are established. We also define a novel class of terms which is called terms of a fixed length. The partial superposition for such terms is proposed under a suitable condition and the partial algebra of terms of a fixed length is constructed.

1 Introduction and Preliminaries

Term is an important and fundamental concept in universal algebra and related topics. Moreover, it is an elementary knowledge to study in computer science (see [8]). Recently, descriptions and applications of terms for classifying algebras were proposed in [9]. By the set $\{f_i \mid i \in I\}$ we denote an indexed set of operation symbols of type $\tau = (n_i)_{i\in I}$ where f_i is an n_i-ary operation symbol for $n \in \mathbb{N}^+ := \{1, 2, \ldots\}$. For $n \in \mathbb{N}^+$, let $X_n = \{x_1, \ldots, x_n\}$ be an n-elements alphabet of variables and let $X = \{x_1, \ldots, x_n, \ldots\}$ be countably infinite. An n-ary term of type τ is defined as follows:

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(i) Every $x_j \in X_n$ is an n-ary term of type τ.

(ii) If $t_1, \ldots, t_{n_i}$ are n-ary terms of type τ and f_i is an n_i-ary operation symbol, then $f_i(t_1, \ldots, t_{n_i})$ is an n-ary term of type τ.

(iii) The set $W_\tau(X_n)$ of all n-ary terms of type τ is the smallest set which contains $x_1, \ldots, x_n$ and is closed under finite applications of (ii).

Let $W_\tau(X_n)$ be the set of all n-ary terms of type τ and $W_\tau(X)$ be the many-sorted set of all terms of type τ, i.e., $W_\tau(X) := (W_\tau(X_n))_{n\in\mathbb{N}^+}$. Nowadays, there are several applications of terms in different ways, see [2, 3, 5, 6, 15]. For further contributions of terms in studying algebraic structures and constraint satisfaction problem, we refer to [4, 12, 13].

For $m, n \in \mathbb{N}^+$, a many-sorted superposition operation

$$S^n_m : W_\tau(X_n) \times (W_\tau(X_m))^n \to W_\tau(X_m)$$

is defined by

(i) $S^n_m(t, t_1, \ldots, t_n) := t_i$ if $t = x_i \in X_n$ and

(ii) $S^n_m(t, t_1, \ldots, t_n) := f_i(S^n_m(t'_1, t_1, \ldots, t_n), \ldots, S^n_m(t'_{n_i}, t_1, \ldots, t_n))$ if $t = f_i(t'_1, \ldots, t'_{n_i}) \in W_\tau(X_n)$.

Hence the many-sorted algebra

$$clone\tau := ((W_\tau(X_n))_{n\in\mathbb{N}^+}, (S^n_m)_{m,n\in\mathbb{N}^+}, (x_i)_{1\leq i\leq n, n\in\mathbb{N}^+})$$

can be formed. Moreover, the algebra $clone\tau$ satisfies the identities (C1), (C2), (C3):

(C1) $\tilde{S}^n_m(\tilde{S}^p_n(\tilde{Z}, \tilde{Y}_1, \ldots, \tilde{Y}_p), \tilde{X}_1, \ldots, \tilde{X}_n) \approx \tilde{S}^p_m(\tilde{Z}, \tilde{S}^n_m(\tilde{Y}_1, \tilde{X}_1, \ldots, \tilde{X}_n), \ldots, \tilde{S}^n_m(\tilde{Y}_p, \tilde{X}_1, \ldots, \tilde{X}_n)), m, n, p \in \mathbb{N}^+$;

(C2) $\tilde{S}^n_m(\lambda_j, \tilde{X}_1, \ldots, \tilde{X}_n) \approx \tilde{X}_j, n, m \in \mathbb{N}^+, 1 \leq j \leq n$;

(C3) $\tilde{S}^n_n(\tilde{Y}, \lambda_1, \ldots, \lambda_n) \approx \tilde{Y}, n \in \mathbb{N}^+$

where $\tilde{S}^n_m$, $\tilde{S}^p_n$, $\tilde{S}^p_m$, $\tilde{S}^n_n$ are operation symbols, $\tilde{Z}, \tilde{Y}_1, \ldots, \tilde{Y}_p, \tilde{X}_1, \ldots, \tilde{X}_n, \tilde{Y}$ are variables for terms, and λ_j are symbol for variables.

For a mapping which maps every n_i-ary operation symbol f_i to an n-ary term of type τ is said to be a *hypersubstitution of type* τ and denoted by σ. Any hypersubstitution σ can be uniquely extended to a mapping

$$\hat{\sigma} : W_\tau(X) \to W_\tau(X)$$

which defined by the following ways:

(i) $\hat{\sigma}[x] := x$ for any variable $x \in X$; and

(ii) $\hat{\sigma}[f_i(t_1, \ldots, t_n)] := S^n_m(\sigma(f_i), \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_n])$ where m is the maximum of the arities of the terms $\hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_n]$ and assume that $\hat{\sigma}[t_j]$, for all $1 \leq j \leq n$ are already defined.

The set of all hypersubstitutions of type τ is denoted by $Hyp(\tau)$. The associative binary operation $\circ_h$ is defined by the following way:

$$\sigma_1 \circ_h \sigma_2 := \hat{\sigma}_1 \circ \sigma_2$$

where $\circ$ is the usual composition of functions. The identity $\sigma_{id} : \{f_i \mid i \in I\} \to W_\tau(X)$ such that $\sigma_{id}(f_i) := f_i(x_1, \ldots, x_n)$. Hence, the structure $(Hyp(\tau); \circ_h, \sigma_{id})$ forms a monoid [8].

Generally, a generalization of superposition S^n_m which is called the generalized superposition S^m, for a fixed $m \geq 1$, was proposed by S. Leeratanavalee [11] in 2000. Namely, an $(m+1)$-ary operation $S^m : (W_\tau(X))^{m+1} \to W_\tau(X)$ defined inductively by the following steps: For $t, t_1, \ldots, t_m \in W_\tau(X)$,

(i) If $t = x_i$; $1 \leq i \leq m$, then $S^m(x_i, t_1, \ldots, t_m) := t_i$.

(ii) If $t = x_i$; $m < i$, then $S^m(x_i, t_1, \ldots, t_m) := x_i$.

(iii) If $t = f_i(s_1, \ldots, s_{n_i})$, then
$S^m(t, t_1, \ldots, t_m) := f_i(S^m(s_1, t_1, \ldots, t_m), \ldots, S^m(s_{n_i}, t_1, \ldots, t_m))$.

Then we can form the structure, $(m+1)$-ary algebra $(W_\tau(X), S^m, (x_i)_{i\geq 1})$ consisting the universe $W_\tau(X)$ together with one $(m+1)$-ary operation and infinitely many nullary operations.

Recall from [10] that a *generalized hypersubstitution of type* τ is an arbitrary mapping σ from the set of n_i-ary operation symbols to the set $W_\tau(X)$ which does not necessarily preserve the arity. Any generalized hypersubstitution σ can be uniquely extended to a mapping $\widehat{\sigma} : W_\tau(X) \to W_\tau(X)$, which is defined by

(i) $\widehat{\sigma}[x_i] := x_i \in X$,

(ii) $\widehat{\sigma}[f_i(t_1, \ldots, t_{n_i})] := S^{n_i}(\sigma(f_i), \widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_{n_i}])$ where $\widehat{\sigma}[t_j]$, $1 \leq j \leq n_i$ are already defined.

In [7], Denecke and Wismath used some functional measures of a complexity of a term to construct a formula for the complexity of the composition

$S_m^n(s, t_1, \ldots, t_n)$ and the formula for the complexity of $\hat{\sigma}[t]$ where t is a compound term. Two measures which the authors used are the variable count and the operation symbol count.

The variable count of a term t is the total number of occurrences of variables in t, and denoted by $vb(t)$. If t is a variable, then $vb(t) = 1$ and if $t = f_i(t_1, \ldots, t_n)$, then $vb(t) = \sum_{j=1}^{n} vb(t_j)$. For the operation symbol count of a term t is denoted by $op(t)$ which is the total number of occurrences of operation symbols in t. If t is a variable, then $op(t) = 0$ and if

$t = f_i(t_1, \ldots, t_n)$, then $op(t) = 1 + \sum_{j=1}^{n} op(t_j)$.

For example, let $\tau = (3)$ be a type with one ternary operation symbol g. Consider a term $t = g(g(x_1, x_3, x_2), x_1, g(x_3, x_2, x_2))$. Then

$$\begin{aligned} vb(t) &= \sum_{j=1}^{3} vb(t_j) \\ &= vb(g(x_1, x_3, x_2)) + vb(x_1) + vb(g(x_3, x_2, x_2)) \\ &= 3 + 1 + 3 \\ &= 7. \end{aligned}$$

Furthermore,

$$\begin{aligned} op(t) &= 1 + \sum_{j=1}^{3} op(t_j) \\ &= 1 + op(g(x_1, x_3, x_2)) + op(x_1) + op(g(x_3, x_2, x_2)) \\ &= 1 + 1 + 0 + 1 \\ &= 3. \end{aligned}$$

In [14], W. Puninagool and S. Leeratanavalee generalized the concept of complexity of compositions and hypersubstitutions which were studied by K. Denecke and S. L. Wismath [7] to complexity of generalized superpositions and generalized hypersubstitutions. Moreover, the authors obtained theory of M-strongly solid varieties, examining the k-normalization chains of a variety with respect to these complexity measurements.

In 2018, E. Aichinger, N. Mudrinski and J. Opršal defined the concept of the total number of occurrences of both variables and operation symbols contained in a term t which is called the *length of the term* t, and denoted by $\text{len}(t)$. Bounds for the length of the terms expressing these functions was considered. In conclusion, they showed that this bound is often robust against the change of the basic operations of the structure. For more details, see [1].

In this paper, we first use the above concepts to construct a formula for counting the length of terms in the usual superposition operation. Moreover, the formula for the complexity of $\hat{\sigma}[t]$ in terms of a compound term t and a hypersubstitution σ is obtained. Analogously, applying the generalized superposition operation and generalized hypersubstitution is also studied. Furthermore, we define a new term by using the idea in [1] which is called a *term of a fixed length*. Based on the superposition operations, the many-sorted partial algebra of terms of a fixed length, denoted by $clone_{fl}\tau_n$, is obtained. In conclusion, it is proved that $clone_{fl}\tau_n$ satisfies the identities of the clone of terms type τ_n as weak identities.

2 The length count of terms under superposition operation and generalized superposition operation

In this section, the formulas for counting the length of terms in the usual superposition operation and the generalized superposition operation are established. Here, we first recall the definition of length of term.

Definition 2.1. *([1]) Let $t \in W_\tau(X_n)$ be any n-ary term of type τ. The length of a term t, denoted by* $\text{len}(t)$*, is the total number of occurrences of both operation symbols and variables that occur in t. This can be defined inductively by*

(i) $\text{len}(t) = 1$ *if t is a variable;*

(ii) $\text{len}(t) = \sum_{j=1}^{n_i} \text{len}(t_j) + 1$ *if $t = f_i(t_1, \ldots, t_{n_i})$.*

Example 2.2. *Let $\tau = (3)$ be a type with one ternary operation symbol g. Consider a term $t = g(x_1, x_3, g(x_2, x_2, x_4))$. Then*

$$\begin{aligned} \text{len}(t) &= \text{len}(x_1) + \text{len}(x_3) + \text{len}(g(x_2, x_2, x_4)) + 1 \\ &= 1 + 1 + (\text{len}(x_2) + \text{len}(x_2) + \text{len}(x_4) + 1) + 1 \\ &= 1 + 1 + 4 + 1 \\ &= 7. \end{aligned}$$

To define the formula for counting the length of terms in the superposition operation, we also need a function counting the number of occurrences of a specific variable x_k in a term t. Let t be an n-ary term of type τ. For each variable x_k, then the x_k-variable count $vb_k(t)$ of t is defined by inductively as follows:

(i) $vb_k(x_k) = 1$;

(ii) if x_k does not occur in t, then $vb_k(t) = 0$;

(iii) if $t = f_i(t_1, \ldots, t_n)$ and $x_k \in var(t)$, then $vb_k(t) = \sum_{j=1}^{n} vb_k(t_j)$.

Now, the complexity theorem under the usual superposition can be shown as follows:

Theorem 2.3. *Let $s \in W_\tau(X_n)$ and $t_1, \ldots, t_n \in W_\tau(X_m)$. Then*

$$\mathrm{len}(S^n_m(s, t_1, \ldots, t_n)) = \sum_{j=1}^{n} vb_j(s)\mathrm{len}(t_j) + op(s).$$

Proof. Let $s \in W_\tau(X_n)$ and $t_1, \ldots, t_n \in W_\tau(X_m)$. We will proof by induction on the complexity of the term s. If $s = x_k \in X_n$, then $\mathrm{len}(S^n_m(x_k, t_1, \ldots, t_n)) = \mathrm{len}(t_k)$. Since $vb_j(x_k) = 0$ for all $1 \leq j \neq k \leq n$ and $op(x_k) = 0$,

$$\sum_{j=1}^{n} vb_j(x_k)\mathrm{len}(t_j) + op(x_k) = 1 \cdot \mathrm{len}(t_k) + 0 = \mathrm{len}(t_k) = \mathrm{len}(S^n_m(x_k, t_1, \ldots, t_n)).$$

Let $s = f_i(s_1, \ldots, s_{n_i})$ and assume that the formula is satisfied for $s_1, \ldots, s_{n_i}$. Then

$$\begin{aligned}
\mathrm{len}(S^n_m(s, t_1, \ldots, t_n)) &= \mathrm{len}(S^n_m(f_i(s_1, \ldots, s_{n_i}), t_1, \ldots, t_n)) \\
&= \mathrm{len}(f_i(S^n_m(s_1, t_1, \ldots, t_n), \ldots, S^n_m(s_{n_i}, t_1, \ldots, t_n))) \\
&= \sum_{k=1}^{n_i} \mathrm{len}(S^n_m(s_k, t_1, \ldots, t_n)) + 1 \\
&= \sum_{k=1}^{n_i} \Big(\Big(\sum_{j=1}^{n} vb_j(s_k)\mathrm{len}(t_j) \Big) + op(s_k) \Big) + 1 \\
&= \sum_{k=1}^{n_i} \Big(\sum_{j=1}^{n} vb_j(s_k)\mathrm{len}(t_j) \Big) + \sum_{k=1}^{n_i} op(s_k) + 1
\end{aligned}$$

$$= \sum_{j=1}^{n}\Big(\sum_{k=1}^{n_i} vb_j(s_k)\mathrm{len}(t_j)\Big) + op(s)$$
$$= \sum_{j=1}^{n} vb_j(s)\mathrm{len}(t_j) + op(s).$$

□

Remark 2.4. *By the fact that the length of term is the total number of occurrences of both variables and operation symbols, we also obtain another proof of Theorem 2.3 using Theorem 3.1 in [7].*

Example 2.5. *Let $\tau = (2)$ be a type with a binary operation symbol f. We would like to count the length of term $S_4^3(f(x_1, x_2), f(x_3, x_2), x_1, f(x_1, x_4))$. According to the definition of superposition operation, then by direct calculation we have $S_4^3(f(x_1, x_2), f(x_3, x_2), x_1, f(x_1, x_4)) = f(f(x_3, x_2), x_1)$. Hence, $\mathrm{len}(S_4^3(f(x_1, x_2), f(x_3, x_2), x_1, f(x_1, x_4))) = \mathrm{len}(f(f(x_3, x_2), x_1)) = 5$. However, by applying Theorem 2.3, we then have*

$$\begin{aligned} & \mathrm{len}(S_4^3(f(x_1, x_2), f(x_3, x_2), x_1, f(x_1, x_4))) \\ = & \sum_{j=1}^{3} vb_j(f(x_1, x_2))\mathrm{len}(t_j) + op(f(x_1, x_2)) \\ = & (1)(3) + (1)(1) + (0)(3) + 1 \\ = & 5. \end{aligned}$$

It turns out that the results are also equal. We see that the counting a length of superposition using our formula is more convenient than using the usual definition.

Example 2.6. *Let $\tau = (2, 3)$ be a type of algebras with one binary operation symbol f and one ternary operation symbol g. We observe that $S_3^2(f(x_2, x_1), g(x_2, x_3, x_1), g(x_1, x_2, x_1)) = f(g(x_1, x_2, x_1), g(x_2, x_3, x_1))$. This means that*

$$\mathrm{len}(S_3^2(f(x_2, x_1), g(x_2, x_3, x_1), g(x_1, x_2, x_1))) = 9.$$

Using Theorem 2.3,

$$\begin{aligned} & \mathrm{len}(S_3^2(f(x_2, x_1), g(x_2, x_3, x_1), g(x_1, x_2, x_1))) \\ = & \sum_{j=1}^{2} vb_j(f(x_2, x_1))\mathrm{len}(t_j) + op(f(x_2, x_1)) \\ = & (1)(4) + (1)(4) + 1 \\ = & 9. \end{aligned}$$

By the definition of hypersubstitution $\hat{\sigma}[t]$ where t is a composite term, then the following corollary is obtained.

Corollary 2.7. *Let $t = f_i(t_1, \ldots, t_{n_i}) \in W_\tau(X_n)$ and σ be a hypersubstitution of type τ. Then*

$$\mathrm{len}(\hat{\sigma}[f_i(t_1, \ldots, t_{n_i})]) = \sum_{j=1}^{n_i} vb_j(\sigma(f_i))\mathrm{len}(\hat{\sigma}[t_j]) + op(\sigma(f_i)).$$

The following theorem shows a formula for counting the terms in generalized superposition operation.

Theorem 2.8. *Let $s, t_1 \ldots, t_m \in W_\tau(X)$. Then*

$$\mathrm{len}(S^m(s, t_1, \ldots, t_m)) = \sum_{j=1}^{m} vb_j(s)\mathrm{len}(t_j) + op(s) + \sum_{j>m} vb_j(s).$$

Proof. Let $s, t_1 \ldots, t_m \in W_\tau(X)$. We will give a proof by induction on the complexity of the term s. If $s = x_k \in X$ for some $1 \le k \le m$, then $\mathrm{len}(S^m(s, t_1, \ldots, t_n)) = \mathrm{len}(t_k)$. Since $vb_j(s) = 0$ for all $j \neq k$ and $op(s) = 0$,

$$\sum_{j=1}^{m} vb_j(s)\mathrm{len}(t_j) + op(s) + \sum_{j>m} vb_j(s) = \mathrm{len}(t_k) = \mathrm{len}(S^m(s, t_1, \ldots, t_n)).$$

If $s = x_k \in X$ for some $m < k$, then $\mathrm{len}(S^m(s, t_1, \ldots, t_n)) = \mathrm{len}(x_k)$. Since $vb_j(s) = 0$ for all $1 \le j \neq k$, $op(s) = 0$ and $\sum_{j>m} vb_j(s) = 1$,

$$\sum_{j=1}^{m} vb_j(s)\mathrm{len}(t_j) + op(s) + \sum_{j>m} vb_j(s) = 1 = \mathrm{len}(x_k) = \mathrm{len}(S^m(s, t_1, \ldots, t_n)).$$

Let $s = f_i(s_1, \ldots, s_{n_i})$ and assume that this formula is satisfied for $s_1, \ldots, s_{n_i}$. Then

$$\begin{aligned} \mathrm{len}(S^m(s, t_1, \ldots, t_m)) &= \mathrm{len}(S^m(f_i(s_1, \ldots, s_{n_i}), t_1, \ldots, t_m)) \\ &= \mathrm{len}(f_i(S^m(s_1, t_1, \ldots, t_m), \ldots, S^m(s_{n_i}, t_1, \ldots, t_m))) \\ &= \sum_{k=1}^{n_i} \mathrm{len}(S^m(s_k, t_1, \ldots, t_m)) + 1 \end{aligned}$$

$$\begin{aligned}
&= \sum_{k=1}^{n_i}\Big(\sum_{j=1}^{m} vb_j(s_k)\mathrm{len}(t_j) + op(s_k) + \sum_{j>m} vb_j(s_k)\Big) + 1 \\
&= \sum_{k=1}^{n_i}\Big(\sum_{j=1}^{m} vb_j(s_k)\mathrm{len}(t_j)\Big) + \sum_{k=1}^{n_i} op(s_k) + \sum_{k=1}^{n_i}\Big(\sum_{j>m} vb_j(s_k)\Big) \\
&\quad +1 \\
&= \sum_{j=1}^{m}\Big(\sum_{k=1}^{n_i} vb_j(s_k)\mathrm{len}(t_j)\Big) + op(s) + \sum_{j>m}\Big(\sum_{k=1}^{n_i} vb_j(s_k)\Big) \\
&= \sum_{j=1}^{m} vb_j(s)\mathrm{len}(t_j) + op(s) + \sum_{j>m} vb_j(s).
\end{aligned}$$

□

Remark 2.9. *The proof in Theorem 2.8 can be shown in another way by Proposition 3.1 in [14].*

Example 2.10. *Let $\tau = (2,1)$ be a type with a binary operation symbol f and a unary operation symbol g. Consider the terms $S^2(f(x_1,x_3), g(x_4), f(x_3,x_2))$. We obtain directly that $S^2(f(x_1,x_3), g(x_4), f(x_3,x_2)) = f(g(x_4),x_3)$. This implies that $\mathrm{len}(S^2(f(x_1,x_3), g(x_4), f(x_3,x_2))) = \mathrm{len}(f(g(x_4),x_3)) = 4$. In other ways, using the formula in Theorem 2.8, we have*

$$\begin{aligned}
&\mathrm{len}(S^2(f(x_1,x_3), g(x_4), f(x_3,x_2))) \\
&= \sum_{j=1}^{2} vb_j(f(x_1,x_3))\mathrm{len}(t_j) + op(f(x_1,x_3)) + \sum_{j>2} vb_j(f(x_1,x_3)) \\
&= (1)(2) + (0)(3) + 1 + 1 \\
&= 4.
\end{aligned}$$

By the notion of an extension of a generalized hypersubstitution $\hat{\sigma}[t]$ where t is a composite term, then the following corollary is obtained.

Corollary 2.11. *Let $t = f_i(t_1,\ldots,t_{n_i}) \in W_\tau(X)$ and σ be a generalized hypersubstitution of type τ. Then*

$$\mathrm{len}(\hat{\sigma}[f_i(t_1,\ldots,t_{n_i})]) = \sum_{j=1}^{n_i} vb_j(\sigma(f_i))\mathrm{len}(\hat{\sigma}[t_j]) + op(\sigma(f_i)) + \sum_{j>m} vb_j(\sigma(f_i)).$$

3 Terms of a fixed length and their corresponding partial algebras

Let $n \geq 1$ be a fixed positive integer. Throughout this section, by $\tau_n := (n_i)_{i \in I}$ we denote the sequence of arity of n-ary operation symbol having the same arity n, i.e., $n_i = n$ for every $i \in I$. Based on the concept of length of terms, we introduce the following specific kind of term and study their properties.

Definition 3.1. *An n-ary term of a fixed length of type τ_n is inductively defined by:*

(i) Each variable $x_i \in X_n$ is an n-ary term of a fixed length of type τ_n.

(ii) If $t_1, \ldots, t_n$ are n-ary terms of a fixed length of type τ_n and $\operatorname{len}(t_j) = \operatorname{len}(t_k)$ for all $1 \leq j, k \leq n$, then $f_i(t_1, \ldots, t_n)$ is a term of a fixed length of type τ_n.

(iii) The set $W^{fl}_{\tau_n}(X_n)$ of all n-ary terms of a fixed length of type τ_n is the least set containing $x_1, \ldots, x_n$ and closed under finite applications of (ii).

Let $W^{fl}_{\tau_n}(X)$ be the set of all terms of a fixed length of type τ_n, i.e.,

$$W^{fl}_{\tau_n}(X) := (W^{fl}_{\tau_n}(X_n))_{n \in \mathbb{N}^+}.$$

Example 3.2. *We consider a type $\tau_2 = (2)$ with a binary operation symbol f. Then $x_1, x_2, f(x_1, x_1), f(x_1, x_2), f(x_2, x_1), f(f(x_1, x_1), f(x_2, x_1))$, $f(f(x_2, x_2), f(x_1, x_2))$ are examples of binary terms of a fixed length of type (2). But, the following terms are not binary terms of a fixed length of type (2): $f(x_1, f(x_2, x_1)), f(f(x_1, x_2), x_2)$.*

Consider a type $\tau_3 = (3)$ with a ternary operation symbol f. Then $f(x_1, x_3, x_2) \in W^{fl}_{(3)}(X_3)$ and $f(x_2, x_1, x_3), x_2, x_4 \in W^{fl}_{(3)}(X_4)$ whereas $S^3_4(f(x_1, x_3, x_2), f(x_2, x_1, x_3), x_2, x_4) = f(f(x_2, x_1, x_3), x_4, x_2) \notin W^{fl}_{(3)}(X_4)$. This shows, in general, that $W^{fl}_{\tau_n}(X) = (W^{fl}_{\tau_n}(X_n))_{n \in \mathbb{N}^+}$ is not closed under the superposition operation S^n_n. In Lemma 3.3, we construct some condition which build this term is closed under the operation S^n_n.

Lemma 3.3. *If $t = f_i(t_1, \ldots, t_n), s_1, \ldots, s_n \in W^{fl}_{\tau_n}(X_n)$ and $\operatorname{len}(s_j) = \operatorname{len}(s_k)$ for all $1 \leq j, k \leq n$, then $S^n_n(f_i(t_1, \ldots, t_n), s_1, \ldots, s_n) \in W^{fl}_{\tau_n}(X_n)$.*

Proof. Assume that $t = f_i(t_1, \ldots, t_n), s_1, \ldots, s_n \in W^{fl}_{\tau_n}(X_n)$ and $\text{len}(s_j) = \text{len}(s_k)$ for all $1 \le j, k \le n$. Note that

$$S^n_n(f_i(t_1, \ldots, t_n), s_1, \ldots, s_n) = f_i(S^n_n(t_1, s_1, \ldots, s_n), \ldots, S^n_n(t_n, s_1, \ldots, s_n)).$$

We have to show that

(i) $S^n_n(t_j, s_1, \ldots, s_n) \in W^{fl}_{\tau_n}(X_n)$ for all $1 \le j \le n$.

(ii) $\text{len}(S^n_n(t_j, s_1, \ldots, s_n)) = \text{len}(S^n_n(t_k, s_1, \ldots, s_n))$ for all $1 \le j, k \le n$.

Since t_j is a term of a fixed length, t_j is a variable or a composite term $f_i(t'_1, \ldots, t'_n)$ which $\text{len}(t'_j) = \text{len}(t'_k)$ for all $1 \le j, k \le n$. We substitute the terms from $\{s_1, \ldots, s_n\}$ for variables in t_j. By the assumption that $\text{len}(s_j) = \text{len}(s_k)$ for all $1 \le j, k \le n$, i.e., the length count of $s_1, \ldots, s_n$ are equal. Then we have $S^n_n(t_j, s_1, \ldots, s_n) \in W^{fl}_{\tau_n}(X_n)$ for all $1 \le j \le n$.

Next, we show that (ii) holds. Since $t = f_i(t_1, \ldots, t_n) \in W^{fl}_{\tau_n}(X_n)$, $\text{len}(t_j) = \text{len}(t_k)$ for all $1 \le j < k \le n$. Since $\text{len}(s_j) = \text{len}(s_k)$ for all $1 \le j, k \le n$, according to the definition of superposition operation of S^n_n, we have $\text{len}(S^n_n(t_j, s_1, \ldots, s_n)) = \text{len}(S^n_n(t_k, s_1, \ldots, s_n))$ for all $1 \le j, k \le n$. □

Lemma 3.3 leads us to define partial maps on $(W^{fl}_{\tau_n}(X_n))_{n \in \mathbb{N}^+}$.
For $n \in \mathbb{N}^+$, let

$$\overline{S}^n_n : (W^{fl}_{\tau_n}(X_n))^{n+1} \multimap W^{fl}_{\tau_n}(X_n)$$

be such that

$$\overline{S}^n_n(t, s_1, \ldots, s_n) = \begin{cases} S^n_n(t, s_1, \ldots, s_n) & \text{if} \quad \text{len}(s_j) = \text{len}(s_k) \text{ for all} \\ \quad 1 \le j, k \le n; & \\ \quad \text{not defined} & \text{otherwise.} \end{cases}$$

Now, we obtain the many-sorted partial algebra

$$clone_{fl}\ \tau_n = ((W^{fl}_{\tau_n}(X_n))_{n \in \mathbb{N}^+}, (\overline{S}^n_n)_{n \in \mathbb{N}^+}, (x_i)_{1 \le i \le n \in \mathbb{N}^+}).$$

Moreover, we have the following theorem.

Theorem 3.4. *$clone_{fl}\ \tau_n$ satisfies $(C1) - (C3)$ as weak identity.*

Proof. For $(C1)$, let $n \in \mathbb{N}^+$. By replacing the variables in $(C1)$ by $t_1, \ldots, t_n$, $s_1, \ldots, s_n, u \in W^{fl}_{\tau_n}(X_n)$, and the operation symbols by the partial fundamental operations of $clone_{fl}\tau_n$, we have

$$\bar{S}_n^n(\bar{S}_n^n(u, s_1, \ldots, s_n), t_1, \ldots, t_n) \approx$$
$$\bar{S}_n^n(u, \bar{S}_n^n(s_1, t_1, \ldots, t_n), \ldots, \bar{S}_n^n(s_n, t_1, \ldots, t_n)).$$

If $\text{len}(s_j) = \text{len}(s_k)$ for all $1 \leq j, k \leq n$ and $\text{len}(t_q) = \text{len}(t_r)$ for all $1 \leq q, r \leq n$, then the left-hand side is defined, and

$$\bar{S}_n^n(\bar{S}_n^n(u, s_1, \ldots, s_n), t_1, \ldots, t_n) = S_n^n(S_n^n(u, s_1, \ldots, s_n), t_1, \ldots, t_n).$$

Moreover, $\bar{S}_n^n(s_1, t_1, \ldots, t_n), \ldots, \bar{S}_n^n(s_n, t_1, \ldots, t_n)$ are defined. Then

$$\bar{S}_n^n(s_1, t_1, \ldots, t_n) = S_n^n(s_1, t_1, \ldots, t_n), \ldots, \bar{S}_n^n(s_n, t_1, \ldots, t_n) = S_n^n(s_n, t_1, \ldots, t_n).$$

Since $\text{len}(s_j) = \text{len}(s_k)$ for all $1 \leq j, k \leq n$ and $\text{len}(t_q) = \text{len}(t_r)$ for all $1 \leq q, r \leq n$. It follows that the right-hand side is defined. Then

$$\begin{aligned} & \bar{S}_n^n(u, \bar{S}_n^n(s_1, t_1, \ldots, t_n), \ldots, \bar{S}_n^n(s_n, t_1, \ldots, t_n)) \\ = \; & S_n^n(u, S_n^n(s_1, t_1, \ldots, t_n), \ldots, S_n^n(s_n, t_1, \ldots, t_n)). \end{aligned}$$

Since S_n^n satisfies the superassociative law, $clone_{fl}\tau_n$ satisfies $(C1)$ as a weak identity.

For (C2), let $n \in \mathbb{N}^+$, and $1 \leq j \leq n$. By replacing $\tilde{S}_n^n$ by $\bar{S}_n^n$, λ_i by $x_j \in X_n$, and $\tilde{X}_1, \ldots, \tilde{X}_n$ by $t_1, \ldots, t_n \in W_{\tau_n}^{fl}(X_n)$, we then have

$$\bar{S}_n^n(x_j, t_1, \ldots, t_n).$$

If $\text{len}(t_q) = \text{len}(t_r)$ for all $1 \leq q, r \leq n$, then $\bar{S}_n^n(x_j, t_1, \ldots, t_n)$ is defined, and

$$\bar{S}_n^n(x_j, t_1, \ldots, t_n) = S_n^n(x_j, t_1, \ldots, t_n) = t_j.$$

Hence $clone_{fl}\tau_n$ satisfies (C2) as a weak identity.

Finally, for (C3), let $n \in \mathbb{N}^+$. By replacing $\tilde{S}_n^n$ by $\bar{S}_n^n$, $\tilde{Y}$ by $s \in W_{\tau_n}^{fl}(X_n)$, then

$$\bar{S}_n^n(s, x_1, \ldots, x_n).$$

Since $\text{len}(x_j) = \text{len}(x_k)$ for all $1 \leq j, k \leq n$, then $\bar{S}_n^n(s, x_1, \ldots, x_n)$ is defined, and

$$\bar{S}_n^n(s, x_1, \ldots, x_n) = S_n^n(s, x_1, \ldots, x_n) = s.$$

Thus $clone_{fl}\tau_n$ satisfies (C3) as a weak identity. □

4 Conclusions

The primary aim of this present paper is to introduce a novel measurement of terms, called the length count of terms. Two formulas for counting the length of terms under the many-sorted usual superposition operation and the generalized superposition operation were successfully established. Based on the theory of clone of term, the notion of terms of a fixed length is introduced. Using the superposition of terms, the many-sorted partial algebra of terms of a fixed length, denoted by $clone_{fl}\tau_n$ was completely obtained. It turned out that $clone_{fl}\tau_n$ satisfies (C1), (C2), (C3) as weak identities. For the future research direction, applications of terms of a fixed length in hypersubstitution theory are still interesting topics.

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