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Trigonometric Solution to the Pendulum Equation

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Abstract

In this work, an approximate analytic solution which is called a semi-analytical solution to the pendulum equation is obtained. Moreover, the semi-analytical solution is compared to the approximate numerical solution obtained with the aid of Runge-Kutta method.

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1 Introduction

We consider the simple pendulum equation which has an exact solution that is expressed in terms of the Jacobian elliptic functions. The solution may also be expressed using the Weierstrass elliptic function. However, elliptic functions are hard to evaluate. For this reason, we will give an approximate trigonometric solution to it.

The equation of undamped motion of simple pendulum with no friction or dissipation reads

$$\ddot{\theta} + \omega^2 \sin\theta = 0, \theta(0) = \theta_0 \text{ and } \theta'(0) = 0, \tag{1.1}$$

where $\theta \equiv \theta(t)$ gives the angular position of the pendulum with respect to the vertical, $\omega = \sqrt{g/l}$ represents the angular frequency in unit of rad $(1/s)$, $g = 9.81 m/s^2$ denotes the acceleration of gravity and l gives the length of massless pendulum arm. In this case, the simple pendulum moves with a simple harmonic motion indefinitely without decay because the only effect on the pendulum motion is the conservative force, so the mechanical energy will remain constant during the movement of the pendulum. But this behavior does not mimic the actual reality of the pendulum motion due to the presence of several forces that work to impede its motion such as air resistance, friction, viscosity, etc. These forces are called dissipative force which leads to cause a loss of pendulum energy and the mechanical energy does not remain constant and the pendulum amplitude decreases during the motion. Therefore, the dissipative force must be included in the equation of motion.

2 Approximate Analytical solution

Let $\theta(t) = 2\arctan\left(\tan(\theta_0/2)\frac{\sqrt{1+\lambda}\cos(\sqrt{w}t)}{\sqrt{1+\lambda\cos^2(\sqrt{w}t)}}\right)$. Define the residual $R(t) = \theta''(t) + \omega^2 \sin\theta(t)$. We have

$$R(0) = \frac{\sin(\theta_0)}{\lambda+1}(\lambda\omega - w + \omega). \tag{2.2}$$

We define $w = \omega^2(1+\lambda)$. Let $c_0 = \tan(\theta_0/2)$. Then

$$R(t) = \frac{1}{\cos(\sqrt{w}t)} \left(\begin{array}{c} c_0\sqrt{\lambda+1}\lambda^2\omega\left(c_0^2\lambda + c_0^2 + \lambda\right)\cos\left(6t\sqrt{w}\right) + \\ 2c_0\sqrt{\lambda+1}\lambda\cos\left(4t\sqrt{w}\right) \\ \left(\begin{array}{c} 3c_0^2\lambda^2\omega + 7c_0^2\lambda\omega + \\ 4c_0^2\lambda w + 4c_0^2 w + 4c_0^2\omega + 3\lambda^2\omega + 6\lambda\omega + 4\lambda w \end{array} \right) + \\ c_0\sqrt{\lambda+1}\cos\left(2t\sqrt{w}\right) \\ \left(\begin{array}{c} 15c_0^2\lambda^3\omega + 47c_0^2\lambda^2\omega + \\ 48c_0^2\lambda\omega - 16c_0^2\lambda^2 w + 16c_0^2 w + \\ 16c_0^2\omega + 15\lambda^3\omega + 48\lambda^2\omega + 48\lambda\omega - 16\lambda^2 w + 16\lambda w \end{array} \right) + \\ 2c_0\sqrt{\lambda+1} \left(\begin{array}{c} 5c_0^2\lambda^3\omega + 17c_0^2\lambda^2\omega + 20c_0^2\lambda\omega - 12c_0^2\lambda^2 w - \\ 36c_0^2\lambda w - 24c_0^2 w + 8c_0^2\omega + 5\lambda^3\omega + 18\lambda^2\omega + \\ 24\lambda\omega - 12\lambda^2 w - 40\lambda w - 16w + 16\omega \end{array} \right) \end{array} \right) \tag{2.3}$$

We will choose the value of λ so that

$$\begin{array}{c} 5c_0^2\lambda^3\omega + 17c_0^2\lambda^2\omega + 20c_0^2\lambda\omega - 12c_0^2\lambda^2 w - \\ 36c_0^2\lambda w - 24c_0^2 w + 8c_0^2\omega + 5\lambda^3\omega + 18\lambda^2\omega + \\ 24\lambda\omega - 12\lambda^2 w - 40\lambda w - 16w + 16\omega = 0. \end{array} \tag{2.4}$$

This gives the following cubic in λ :

$$16\tan^2\left(\frac{\theta_0}{2}\right) + 8\left(4 + 5\tan^2\left(\frac{\theta_0}{2}\right)\right)\lambda + \left(3 + \frac{62}{1+\cos(\theta_0)}\right)\lambda^2 + 7\sec^2\left(\frac{\theta_0}{2}\right)\lambda^3 = 0 \tag{2.5}$$

We choose the least in magnitude real root of (2.5). A good approximate solution for λ is

$$\lambda = \frac{4\sin^2\left(\frac{\theta_0}{2}\right)\left(20\cos(\theta_0) - 7\left(\cos(2\theta_0) + 75\right)\right)}{421\cos(\theta_0) + 66\cos(2\theta_0) - 5\cos(3\theta_0) + 3614}. \tag{2.6}$$

The solution is periodic and its period is approximately $T = 2\pi/(\omega\sqrt{1+\lambda})$.

3 Analysis and Discussion

We obtained approximate analytical solution to the pendulum equation. Let us compare it with the exact solution [1]- [2]. The exact solution to the i.v.p (1.1) reads

$$\theta_{\text{exact}}(t) = 2\tan^{-1}\left(\tan\left(\frac{\theta_0}{2}\right)\text{cn}\left(\omega t \,\middle|\, \sin^2\left(\frac{\theta_0}{2}\right)\right)\right). \tag{3.7}$$

Let $\theta_0 = \pi/2$. The cubic (2.5) is $14\ \lambda^3 + 65\lambda^2 + 72\lambda + 16 = 0$ whose approximate roots are $\lambda_2 = -3.10515, \lambda_2 = -1.24117, \lambda_3 = -0.296537$. We choose the least in magnitude root: $\lambda = -0.296537$. If we evaluate the value of λ by means of formula (2.6), then we get $\lambda = -0.291995$ which differs from the exact value by 0.0045. The approximate trigonometric solution is

$$\theta_{\text{approx}}(t) = 2\tan^{-1}\left(\frac{0.838727\cos(0.419364t)}{\sqrt{1. - 0.296537\cos^2(0.419364t)}}\right). \tag{3.8}$$

The approximate period is $T_{\text{approx}} = 14.98267$ which differs from the exact period $T_{\text{exact}} = 8K(1/2)$ by $|T_{\text{approx}} - T_{\text{exact}}| = 0.15$. The error for the approximate trigonometric solution reads $\max_{-T/2\leq t\leq T/2} |\theta_{\text{exact}}(t) - \theta_{\text{approx}}(t)| = 0.0282873$. Figure 1 shows a comparison between the exact and the approximate trigonometric solutions.

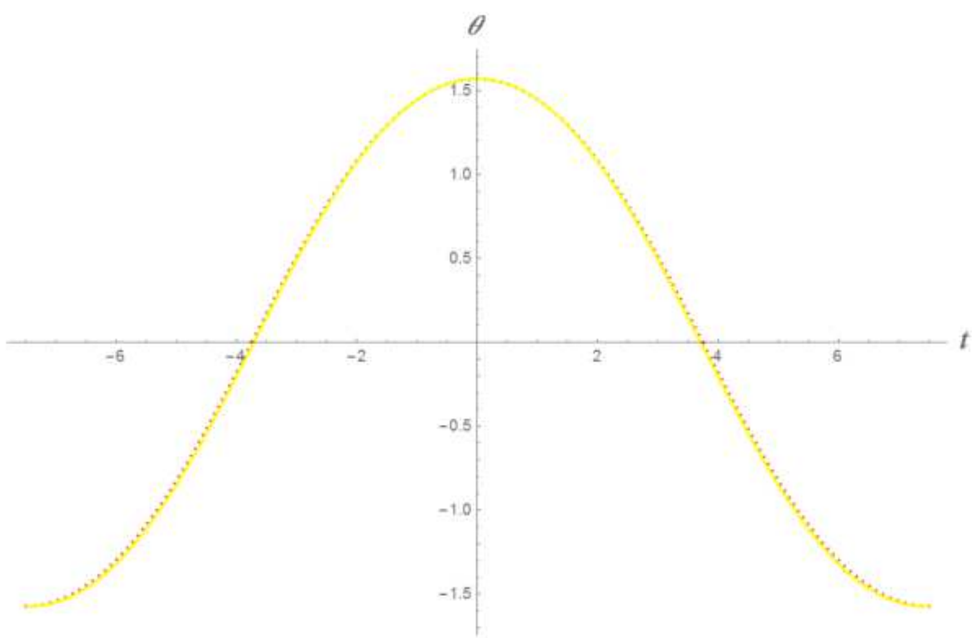

Figure 1. Comparison between exact and approximate solution to pendulum for $\theta_0 = \pi/2$.

Table 1 shows the error of the approximate solution compared with the exact solution for different initial amplitudes and $\omega = 1$.

θ_0	$\max_{-\frac{T}{2}\le\theta\le\frac{T}{2}} \left\|\theta_{exact}(\theta_0)-\theta_{approx}(\theta_0)\right\|$	θ_0	$\max_{-\frac{T}{2}\le\theta\le\frac{T}{2}} \left\|\theta_{exact}(\theta_0)-\theta_{approx}(\theta_0)\right\|$
5	1.35373×10^{-8}	50	0.00139653
10	4.33451×10^{-7}	55	0.00226233
15	3.29815×10^{-6}	60	0.00352213
20	0.0000139394	65	0.00529369
25	0.0000426718	70	0.00773478
30	0.000106482	75	0.0110188
35	0.000231071	80	0.0153616
40	0.000452715	85	0.0210188
45	0.000820039	90	0.0282803

Table 1.

We also may estimate the error using the following polynomial ($\omega = 1$):

$$\begin{aligned} P(\theta_0) \;=\;& 0.0000419225\theta_0^{10} + 2.18\times10^{-6}\theta_0^9 - 0.00032109\theta_0^8 - 0.0000103024\theta_0^7 + \\ & 0.00195378\theta_0^6 + 0.0000154305\theta_0^5 + 0.001178320\theta_0^4 - 8.05\times10^{-6}\theta_0^3 - \\ & 0.00006955450\theta_0^2 + 1.06\times10^{-6}\theta_0 \end{aligned} \quad (3.9)$$

For example, if $\theta_0 = 30°$, the approximate value for the error using formula (3.9) will be $P(\pi/6) = 0.000107909$ which differs from the error in Table 1 by 1.42656×10^{-6}. In Table 2, we compare the exact value of λ with the value computed using formula (2.6) for different values of the initial amplitude and $\omega = 1$.

θ	$\lambda_{exact}(\theta)$	$\lambda_{approx}(\theta)$	$\left\|\lambda_{exact}(\theta)-\lambda_{approx}(\theta)\right\|$	θ_0	$\lambda_{exact}(\theta_0)$	$\lambda_{approx}(\theta_0)$	$\left\|\lambda_{exact}(\theta_0)-\lambda_{approx}(\theta_0)\right\|$
5	−0.000951835	−0.000951835	6.02615×10^{-13}	50	−0.0941927	−0.0941384	0.0000542278
10	−0.0038062	−0.0038062	1.53799×10^{-10}	55	−0.113701	−0.113588	0.000113525
15	−0.00855968	−0.00855967	3.92162×10^{-9}	60	−0.134951	−0.134729	0.000221784
20	−0.0152065	−0.0152064	3.88912×10^{-8}	65	−0.157905	−0.157497	0.000408598
25	−0.0237384	−0.0237381	2.29659×10^{-7}	70	−0.182522	−0.181806	0.000715687
30	−0.0341445	−0.0341435	9.76201×10^{-7}	75	−0.208752	−0.207552	0.00119948
35	−0.0464113	−0.046408	3.30478×10^{-6}	80	−0.236541	−0.234607	0.00193348
40	−0.0605222	−0.0605127	9.46451×10^{-6}	85	−0.265826	−0.262816	0.00301006
45	−0.0764571	−0.0764333	0.000023839	90	−0.296537	−0.291995	0.00454146

Table 2.

Remark 1. Given that

$$\ddot{\theta} + \omega^2 \sin\theta = 0, \theta(0) = 0 \text{ and } \theta'(0) = \dot{\theta}_0,\ \dot{\theta}_0 \neq 0, \tag{3.10}$$

an approximate trigonometric solution is given by

$$\theta_{\text{approx}}(t) = 2\tan^{-1}\left(\frac{\sqrt{\mu+1}\dot{\theta}_0 \sin\left(t\sqrt{\frac{(\mu+1)\left(\dot{\theta}_0^2-2\right)}{4\mu-2}}\right)}{\sqrt{2}\sqrt{\frac{(\mu+1)\left(\dot{\theta}_0^2-2\right)}{2\mu-1}}\sqrt{\mu\cos^2\left(t\sqrt{\frac{(\mu+1)\left(\dot{\theta}_0^2-2\right)}{4\mu-2}}\right)+1}}\right), \tag{3.11}$$

where μ is the least in magnitude real root to the quartic

$$\begin{gathered} 4\dot{\theta}_0^2\left(2\omega - \dot{\theta}_0\right)\left(\dot{\theta}_0 + 2\omega\right) - 2\left(-36\dot{\theta}_0^2\omega^2 + 9\dot{\theta}_0^4 + 64\omega^4\right)\mu + \\ (101\dot{\theta}_0^2\omega^2 - 24\dot{\theta}_0^4 - 72\omega^4)\mu^2 + 2\left(15\dot{\theta}_0^2\omega^2 - 5\dot{\theta}_0^4 + 2\omega^4\right)\mu^3 + 8\omega^4\mu^4 = 0. \end{gathered} . \tag{3.12}$$

The default value is

$$\mu = \frac{2\dot{\theta}_0^2\left(4\omega^2 - \dot{\theta}_0^2\right)\left(-4320\dot{\theta}_0^2\omega^6 + 1972\dot{\theta}_0^4\omega^4 - 451\dot{\theta}_0^6\omega^2 + 57\dot{\theta}_0^8 + 4096\omega^8\right)}{\begin{gathered}-405504\dot{\theta}_0^2\omega^{10} + 277632\dot{\theta}_0^4\omega^8 - 107296\dot{\theta}_0^6\omega^6 + \\ 25512\dot{\theta}_0^8\omega^4 - 3694\dot{\theta}_0^{10}\omega^2 + 317\dot{\theta}_0^{12} + 262144\omega^{12}\end{gathered}}. \tag{3.13}$$

The exact solution is given by

$$\theta_{\text{exact}}(t) = 2\tan^{-1}\left(\frac{\dot{\theta}_0 \text{sn}\left(\frac{1}{2}t\sqrt{4-\dot{\theta}_0^2}\middle|\frac{\dot{\theta}_0^2}{\dot{\theta}_0^2-4}\right)}{\sqrt{4-\dot{\theta}_0^2}}\right) \tag{3.14}$$

Remark 2. Let

$$\ddot{\theta} + \omega^2 \sin\theta = 0, \theta(0) = \theta_0 \text{ and } \theta'(0) = \dot{\theta}_0,\ \theta_0\dot{\theta}_0 \neq 0 \tag{3.15}$$

The exact solution is

$$\theta_{\text{exact}}(t) = 2\tan^{-1}\left(c_0 \text{cn}\left(\omega t + \text{cn}^{-1}\left(\frac{\tan\left(\frac{\theta_0}{2}\right)}{c_0}, \frac{c_0^2}{c_0^2+1}\right), \frac{c_0^2}{c_0^2+1}\right)\right), \tag{3.16}$$

where $c_0^2 = \frac{4\omega^2}{2\omega^2(\cos(\theta_0)+1)-\dot{\theta}_0^2} - 1$.An approximate trigonometric solution is given by

$$\theta_{\text{approx}}(t) = 2\arctan\left(c_0 \frac{\sqrt{1+\lambda}\cos(\sqrt{w}t + c_1)}{\sqrt{1+\lambda\cos^2(\sqrt{w}t + c_1)}}\right), \tag{3.17}$$

where

$$c_1 = \cos^{-1}\left(\frac{\tan\left(\frac{\theta_0}{2}\right)}{\sqrt{c_0^2(\lambda+1) - \lambda\tan^2\left(\frac{\theta_0}{2}\right)}}\right). \tag{3.18}$$

The number c_0 is a solution to the sextic

$$\begin{aligned} &4(1+\lambda)^2\omega^2 c_0^6 - \left(\sec^4\left(\tfrac{\theta_0}{2}\right)\left((1+\lambda)(1+3\lambda)\omega^2\sin^2(\theta_0) + \dot{\theta}_0^2\right)\right)c_0^4 + \\ &4\lambda(2+3\lambda)\omega^2\tan^4\left(\tfrac{\theta_0}{2}\right)c_0^2 - 4\lambda^2\omega^2\tan^6\left(\tfrac{\theta_0}{2}\right) = 0. \end{aligned} \tag{3.19}$$

The number λ is the least in magnitude real root to the quintic

$$\begin{aligned} &54\left(\omega^2\left(5\sin\left(\tfrac{\theta_0}{2}\right) + \sin\left(\tfrac{3\theta_0}{2}\right)\right)^2 + 8\left(1+\cos(\theta_0)\right)\dot{\theta}_0^2\right) + \\ &\tfrac{9}{2}\left(\omega^2\left(1374 + 247\cos(\theta_0) - 78\cos(2\theta_0) - 7\cos(3\theta_0)\right) + 224\left(1+\cos(\theta_0)\right)\dot{\theta}_0^2\right)\lambda + \\ &\tfrac{3}{2}\left(\omega^2\left(9814 + 2859\cos(\theta_0) + 2\cos(2\theta_0) - 3\cos(3\theta_0)\right) + 490\left(1+\cos(\theta_0)\right)\dot{\theta}_0^2\right)\lambda^2 + \\ &\tfrac{7}{2}\left(4\omega^2\left(1087 + 275\cos(\theta_0) + 6\cos(2\theta_0)\right) + 49\left(1+\cos(\theta_0)\right)\dot{\theta}_0^2\right)\lambda^3 + \\ &98\omega^2\left(75 + 11\cos(\theta_0)\right)\lambda^4 + 1372\omega^2\lambda^5 = 0. \end{aligned} \tag{3.20}$$

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