International Journal of Mathematics and
Computer Science, **16**(2021), no. 4, 1647–1652

Elementary Solution to the Damped Pendulum

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(Received May 11, 2021, Accepted June 15, 2021)

Abstract

In this work, we obtain an approximate elementary trigonometric solution to a damped pendulum equation. We also compare both graphically and numerically this solution with Runge-Kutta numerical solution and with the Johannessen's solution.

1 Introduction

Since the time of Galileo, the pendulum has constituted a physical object fascinating physicists and becoming one of the paradigms in the study of physics and natural phenomena. In the framework of nonlinear dynamics, there is no doubt that the pendulum is one of the objects that have deserved more attention in modeling all kinds of phenomena related to oscillations, bifurcations and chaos. The simple pendulum has been used as a physical model to solve problems such as: non-linear plasma oscillations, Duffing oscillators, motion of spacecraft over slowly rotating asteroids, etc. In this work, we consider the damped pendulum equation and give an elementary approximate trigonometric solution to it.

Key words and phrases: Damped pendulum, nonlinear odes, Duffing equation, Jacobian elliptic functions.
AMS (MOS) Subject Classifications: 34Axx, 34A05,34A34, 34A25, 33E05.
ISSN 1814-0432, 2021, http://ijmcs.future-in-tech.net

The equation of undamped motion of a simple pendulum with damping reads

$$\ddot{\theta}+2\varepsilon\dot{\theta}\ +\omega^2\sin\theta=0,\ \theta(0)=0 \text{ and } \theta'(0)=\dot{\theta}_0, \tag{1.1}$$

where $\theta\equiv\theta\,(t)$ gives the angular position of the pendulum with respect to the vertical, $\omega=\sqrt{g/l}$ represents the angular frequency in unit of rad $(1/s)$, with $g=9.81m/s^2$ denoting the acceleration of gravity and l is the length of the massless pendulum arm.

2 Approximate Analytical solution

Let

$$\theta(t)=2\arctan(a\exp(-\varepsilon t)\sin(f(t))), \tag{2.2}$$

where a is a constant that is determined from the initial condition $\theta'(0)=\dot{\theta}_0$ and $f=f(t)$ is some function to be determined. Plugging equation (2.2) into (1.1) yields

$$\begin{array}{l}\ddot{\theta}+2\varepsilon\dot{\theta}\ +\omega^2\sin\theta=-\frac{2a}{\left(a^2e^{-2t\varepsilon}(1-\cos(2f(t)))+2\right)^2}\\ \left[\begin{array}{c}\left\{a^2\left(5f'(t)^2+9\varepsilon^2-3\omega^2\right)+4e^{2t\varepsilon}(f'(t)^2+\varepsilon^2-\omega^2)\right\}\sin(\eta)+\\ -\left\{\left(a^2+4e^{2t\varepsilon}\right)f''(t)+4a^2\varepsilon f'(t)\right\}\cos(\eta)\\ a^2(f''(t)+4f'(t)\varepsilon)\cos(3\eta)+a^2\left(f'(t)^2-3\varepsilon^2+\omega^2\right)\sin(3\eta)\end{array}\right]\end{array} \tag{2.3}$$

Now we require that

$$\begin{array}{l}a^2\left(5f'(t)^2+9\varepsilon^2-3\omega^2\right)+4e^{2t\varepsilon}(f'(t)^2+\varepsilon^2-\omega^2)=0.\\ \left(a^2+4e^{2t\varepsilon}\right)f''(t)+4a^2\varepsilon f'(t)=0.\end{array} \tag{2.4}$$

Eliminating $f''(t)$ from the last system gives the ode

$$f'(t)=\sqrt{\omega^2-\varepsilon^2-\frac{2a^2\left(2\varepsilon^2+\omega^2\right)}{5a^2+4e^{2t\varepsilon}}}. \tag{2.5}$$

Choose the function $f=f(t)$ so that $f(0)=0$. Then

$$f(t)=\int_0^t\sqrt{\omega^2-\varepsilon^2-\frac{2a^2\left(2\varepsilon^2+\omega^2\right)}{5a^2+4e^{2\tau\varepsilon}}}d\tau \tag{2.6}$$

The function $f=f(t)$ is almost linear. For practical purposes, we may use the approximation

$$f(t)\approx f'(0)t=\sqrt{\omega^2-\varepsilon^2-\frac{2a^2\left(2\varepsilon^2+\omega^2\right)}{5a^2+4}}t.$$

The value for the constant a is obtained by solving the equation $\theta'(0) = \dot{\theta}_0$. From this, we get the following:

$$12\left(3\varepsilon^2 - \omega^2\right)a^4 - \left(16\left(\omega^2 - \varepsilon^2\right) - 5\dot{\theta}_0^2\right)a^2 + 4\dot{\theta}_0^2 = 0. \tag{2.7}$$

We choose the following value:

$$a = \frac{1}{2}\operatorname{sign}(\dot{\theta}_0)\sqrt{\frac{\sqrt{32\dot{\theta}_0^2\left(\omega^2 - 13\varepsilon^2\right) + 256\left(\varepsilon^2 - \omega^2\right)^2 + 25\dot{\theta}_0^4} - 5\dot{\theta}_0^2 + 16\omega^2 - 16\varepsilon^2}{6(3\varepsilon^2 - \omega^2)}}. \tag{2.8}$$

3 Analysis and Discussion

We have obtained an elementary approximate analytical solution to the damped pendulum equation. Johannessen [2] got the following approximate analytical solution:

$$\theta(t) = 2\sin^{-1}\left(\frac{\dot{\theta}_0 e^{-\varepsilon t}}{2\sqrt{\omega^2 - \varepsilon^2}} \times \operatorname{sn}\left(\xi(t), \frac{e^{-2t\varepsilon}\dot{\theta}_0^2}{4\left(\omega^2 - \varepsilon^2\right)}\right)\right), \tag{3.9}$$

where

$$\xi(t) = \sqrt{\omega^2 - \varepsilon^2}\,t - \frac{9\dot{\theta}_0^2\left(1 - (4\varepsilon t + 1)e^{-4\varepsilon t}\right)}{4096\varepsilon\left(\omega^2 - \varepsilon^2\right)^{3/2}} - \frac{\left(1 - (2\varepsilon t + 1)e^{-2\varepsilon t}\right)\dot{\theta}_0^2}{32\varepsilon\sqrt{\omega^2 - \varepsilon^2}}.$$

We will compare Johannessen's formula with the elementary formulas we have obtained in the previous section.

Example. Let

$$\varepsilon = 0.2,\ \theta_0 = 0,\ \dot{\theta}_0 = 0.25,\ \omega = 1 \text{ and } 0 \leq t \leq T = 20. \tag{3.10}$$

The i.v.p. to be solved is

$$x''(t) + 0.4x'(t) + \sin(x(t)) = 0 \wedge x(0) = 0 \wedge x'(0) = 0.25 \text{ for } 0 \leq t \leq 20. \tag{3.11}$$

Figure 1 shows the graph of three solutions: Runge-Kutta (dashed), Johannessen's solution (black) and our proposed solution (blue).

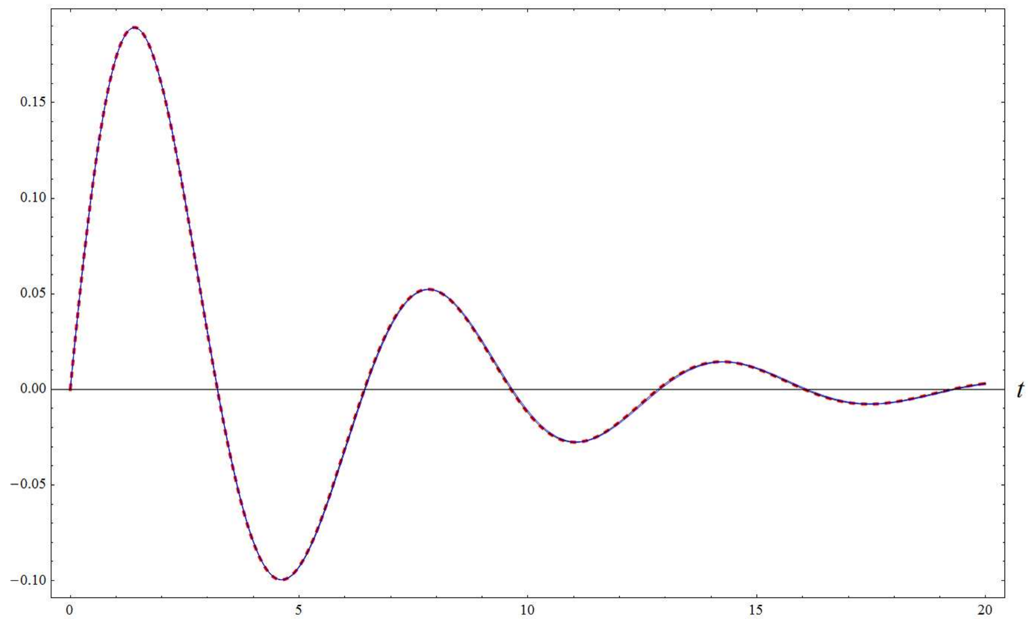

Figure 1

2ε	$\text{Error}_{\text{RungeKutta}}$	$\text{Error}_{\text{Johannessen}}$	$\text{Error}_{\text{Our Solution}}$
0.02	4.01209×10^{-6}	0.0000195881	0.00187958
0.058	4.01478×10^{-6}	0.0000571511	0.00181357
0.096	3.83307×10^{-6}	0.000095238	0.00178244
0.134	3.57606×10^{-6}	0.000133938	0.00175131
0.172	3.31052×10^{-6}	0.000173344	0.00172123
0.21	3.06186×10^{-6}	0.000213556	0.00169301
0.248	2.82045×10^{-6}	0.000254682	0.00166726
0.286	2.55263×10^{-6}	0.000296834	0.00164445
0.324	2.2159×10^{-6}	0.000340138	0.00162493
0.362	1.77794×10^{-6}	0.000384725	0.00160896
0.4	1.40213×10^{-6}	0.000430743	0.00159671

$\dot{\theta}_0$	$\text{Error}_{\text{RungeKutta}}$	$\text{Error}_{\text{Johannessen}}$	$\text{Error}_{\text{Our Solution}}$
0.1	5.66901×10^{-6}	0.000012985	0.000110576
0.19	1.42741×10^{-6}	0.0000890639	0.000752433
0.28	1.81618×10^{-6}	0.000285046	0.00237698
0.37	4.2885×10^{-6}	0.000657727	0.00538544
0.46	4.62252×10^{-6}	0.0012639	0.010106
0.55	2.22462×10^{-6}	0.00216037	0.0167728
0.64	6.95462×10^{-6}	0.00340393	0.0255084
0.73	4.3753×10^{-6}	0.00505137	0.0363128
0.82	9.46827×10^{-6}	0.00715949	0.049061
0.91	0.0000114051	0.00978509	0.0651477
1.	6.58771×10^{-6}	0.012985	0.0867195

Table 1.

Table 1 shows errors for different choices of the damping coefficient and initial speed with $\omega = 1$ for $0 \leq t \leq 20$.

We may use the following approximation to get an elementary solution from Johannessen's solution ($-1 \leq m < 1$):

$$\operatorname{sn}(t,m) \approx \sin_m(t) := \frac{\sin\left(\sqrt{1+\frac{1}{14}\,(A-m-12)}t\right)}{\sqrt{1+\frac{1}{14}\,(A-m-12)\cos^2\left(\sqrt{\frac{1}{14}\,(A-m-12)}t\right)}}. \tag{3.12}$$

$$A = \sqrt{m^2 - 144m + 144} \tag{3.13}$$

The Jacobian fucntion $\operatorname{cn}(t,m)$ may also be approximated as follows:

$$\operatorname{cn}(t,m) \approx \cos_m(t) := \frac{\sqrt{A-m+2}\cos\left(\frac{\sqrt{A-m+2}}{\sqrt{14}}t\right)}{\sqrt{14+(A-m-12)\cos^2\left(\frac{\sqrt{A-m+2}}{\sqrt{14}}t\right)}}. \tag{3.14}$$

Table 3 shows the error obtained by approximating the Jacobian functions by means of the trigonometric functions using formulas (3.12)
- (3.14) for $T = 4K(m)$.

$2\,\varepsilon$	$Error_{RungeKutta}$	$Error_{Johannessen}$	$Error_{Our\ Solution}$
0.02	4.01209×10^{-6}	0.0000195881	0.00187958
0.058	4.01478×10^{-6}	0.0000571511	0.00181357
0.096	3.83307×10^{-6}	0.000095238	0.00178244
0.134	3.57606×10^{-6}	0.000133938	0.00175131
0.172	3.31052×10^{-6}	0.000173344	0.00172123
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0.362	1.77794×10^{-6}	0.000384725	0.00160896
0.4	1.40213×10^{-6}	0.000430743	0.00159671

$\dot{\theta}_0$	$Error_{RungeKutta}$	$Error_{Johannessen}$	$Error_{Our\ Solution}$
0.1	5.66901×10^{-6}	0.000012985	0.000110576
0.19	1.42741×10^{-6}	0.0000890639	0.000752433
0.28	1.81618×10^{-6}	0.000285046	0.00237698
0.37	4.2885×10^{-6}	0.000657727	0.00538544
0.46	4.62252×10^{-6}	0.0012639	0.010106
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0.73	4.3753×10^{-6}	0.00505137	0.0363128
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0.91	0.0000114051	0.00978509	0.0651477
1.	6.58771×10^{-6}	0.012985	0.0867195

Table 1.

We have obtained an elementary solution to the damped pendulum equation. Letting $\varepsilon = 0$, we obtain an approximate trigonometric solution to the undamped pendulum [1]. We conclude that Johannessen's solution gives a more accurate solution. However, Johannessen's solution is not elementary. Despite this fact, we may obtain an elementary solution from Johannessen's formula. Finally, notice that if a problem may be solved using the Jacobian functions, such problem may also be solved in an elementary way using trigonometric functions and the elementary solution will be reasonably good.

References

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[2] K. Johannessen, An analytical solution to the equation of motion for the damped nonlinear pendulum, 2014 Eur. J. Phys. 35 035014.