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Interpretations of Kronecker product and ordinal product of poset matrices

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Abstract

We recall the notion of poset matrix, a square matrix with entries 0s and 1s, to represent posets. We also recall the frequently studied Kronecker product of matrices and introduce the notion of the ordinal product of matrices. We give some interpretations of these products in the case of poset matrices. We show that the Kronecker product of poset matrices represents the direct product of posets and the ordinal product of poset matrices represents the ordinal product of posets. Finally, we show that these results give matrix recognition of factorable posets and composite posets.

1 Introduction

Various operations on matrices are being considered in the literature due to their classical applications in science and engineering fields. Among these, we recall the Kronecker product of matrices. We introduce the notion of the ordinal product of matrices. According to Van Loan [9], the application areas where Kronecker products abound are all thriving that include, particularly, the areas of signal processing, image processing, semidefinite programming,

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and quantum computing. Therefore, the Kronecker product of various matrices have been considered by numerous authors who showed their applications to the related fields [9, 10].

On the other hand, due to the computational tractability property of posets, they provide the basic structures of several applied and theoretical problems in many fields of science and engineering [5]. Therefore, different methods for recognition of various classes of posets are considered in the literature [2]. Due to many computational aspects of incidence matrices, they have classical applications in recognizing different classes of posets and graphs [3, 6, 7, 8]. Khamis [2] recalled the notion of prime posets and decomposable posets and described an algorithmic method by using an incidence matrix to determine if a finite poset is a prime poset. These intuitions give us the idea of defining the class of factorable posets and the class of composite posets and giving their matrix recognition.

In Section 2, we recall some basic terminologies related to the direct product and the ordinal product of posets. We recall also some important definitions and common notations related to the poset matrix and its interpretations in posets. In Section 3, we recall the Kronecker product of matrices and show that the Kronecker product of poset matrices is also a poset matrix and it represents the direct product of posets. In Section 4, we define the ordinal product of matrices and show that the ordinal product of poset matrices is also a poset matrix and it represents the ordinal product of posets. In Section 5, we define the property of transitive blocks of poset matrices in a block poset matrix and give recognition of the class of factorable posets. In Section 6, we define the property of transitive blocks of 1s in a block poset matrix and give recognition of the class of composite posets.

2 Preliminaries

A *partially ordered set* or *poset* is a structure $\mathbf{A} = \langle A, \leqslant \rangle$ consisting of the nonempty set A with the order relation $\leqslant$ on A; that is, $\leqslant$ is reflexive, antisymmetric and transitive on A. The set A is called the *underlying set* or *ground set* of the poset $\mathbf{A}$. A poset $\mathbf{A}$ is called *finite* if the underlying set A is finite. Through this paper, we assume that every poset is finite and nonempty. We use the notations $\mathbf{1}$ for the singleton poset, $\mathbf{C}_n (n \geq 1)$ for n-element chain poset, $\mathbf{I}_n (n \geq 1)$ for n-element antichain poset, $\mathbf{L}_n (n \geq 2)$ for ladder poset with $2n$ elements, $\mathbf{B}_{m,n} (m \geq 1, n \geq 1)$ for the complete bipartite poset with m minimal elements and n maximal elements. For further details on posets, please refer to the classical book by Davey and Priestley [1].

Let $\mathbf{A} = \langle A, \leqslant_A \rangle$ and $\mathbf{B} = \langle B, \leqslant_B \rangle$ be two posets. A bijective map $\phi : A \to B$ is called an *order isomorphism* if for all $x, y \in A$, $x \leqslant_A y$ if and only if $\phi(x) \leqslant_B \phi(y)$. We write $\mathbf{A} \cong \mathbf{B}$ whenever $\mathbf{A}$ and $\mathbf{B}$ are *order isomorphic*.

We use the notations $\mathbf{A} + \mathbf{B}$ and $\mathbf{A} \oplus \mathbf{B}$ to denote the direct and the ordinal sums of the posets $\mathbf{A}$ and $\mathbf{B}$, respectively. From now on, we will briefly write $n\mathbf{A}$ for the direct sum $\mathbf{A} + \mathbf{A} + \cdots + \mathbf{A}$ and $\oplus^n \mathbf{B}$ for the ordinal sum $\mathbf{B} \oplus \mathbf{B} \oplus \cdots \oplus \mathbf{B}$. We now recall the definitions of the direct and ordinal products of posets. The *direct product* of the posets $\mathbf{A}$ and $\mathbf{B}$, denoted by $\mathbf{A} \times \mathbf{B}$, is defined as the poset $\langle A \times B, \leqslant_\times \rangle$ such that for all $(x, y), (x', y') \in A \times B$, $(x, y) \leqslant_\times (x', y')$ if $x \leqslant_A x'$ and $y \leqslant_B y'$. Here the posets $\mathbf{A}$ and $\mathbf{B}$ are called *direct factors* of $\mathbf{A} \times \mathbf{B}$. In Figure 1, the direct products $\mathbf{B_{1,2}} \times \mathbf{B_{2,1}}$ and $\mathbf{B_{2,1}} \times \mathbf{B_{1,2}}$ are shown by using the Hasse diagrams. For any posets $\mathbf{A}$ and $\mathbf{B}$, it is easy to show that $\mathbf{A} \times \mathbf{B} \cong \mathbf{B} \times \mathbf{A}$ and $\mathbf{I}_n \times \mathbf{B} \cong n\mathbf{B}$.

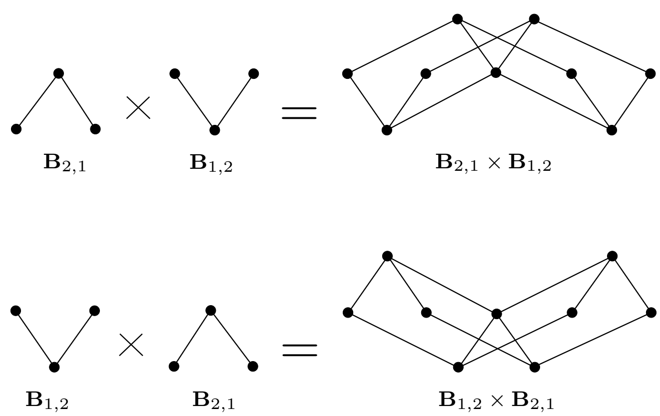

Figure 1: Hasse diagrams of posets giving $\mathbf{B}_{2,1} \times \mathbf{B}_{1,2}$ and $\mathbf{B}_{1,2} \times \mathbf{B}_{2,1}$.

The *ordinal product* of the posets $\mathbf{A}$ and $\mathbf{B}$, denoted by $\mathbf{A} \otimes \mathbf{B}$, is defined as the poset $\langle A \times B, \leqslant_\otimes \rangle$ such that for all $(x, y), (x', y') \in A \times B$, $(x, y) \leqslant_\otimes (x', y')$ if either (i) $x \leqslant_A x'$ or (ii) $x = x'$ and $y \leqslant_B y'$. Here the posets $\mathbf{A}$ and $\mathbf{B}$ are called *ordinal factors* of $\mathbf{A} \otimes \mathbf{B}$. In Figure 2, the ordinal products $\mathbf{B_{1,2}} \otimes \mathbf{B_{2,1}}$ and $\mathbf{B_{2,1}} \otimes \mathbf{B_{1,2}}$ are shown by using the Hasse diagrams. For any posets $\mathbf{A}$ and $\mathbf{B}$, it is easy to check that $\mathbf{A} \otimes \mathbf{B} \ncong \mathbf{B} \otimes \mathbf{A}$ and $\mathbf{I}_n \otimes \mathbf{B} \cong n\mathbf{B}$. We will show also by using the poset matrix that $\mathbf{C}_n \otimes \mathbf{B} \cong \oplus^n \mathbf{B}$.

From now on, we use the notations $M_{m,n}$ for an m-by-n matrix and M_m for a square matrix of order m. In particular, we use the notations I_n, O_n

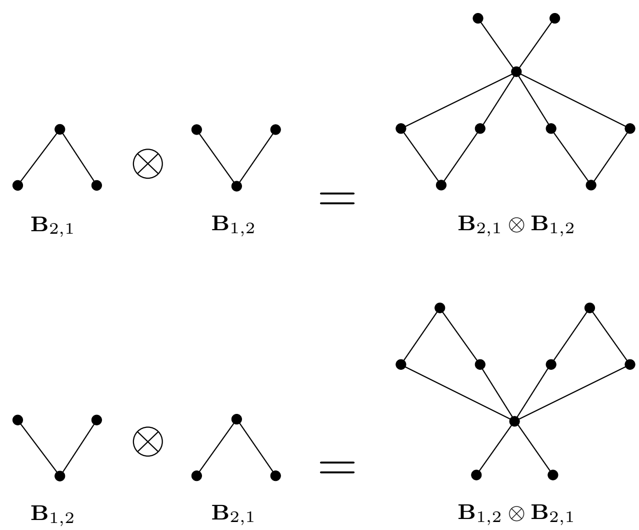

Figure 2: Hasse diagrams of posets giving $\mathbf{B}_{2,1} \otimes \mathbf{B}_{1,2}$ and $\mathbf{B}_{1,2} \otimes \mathbf{B}_{2,1}$.

and Z_n respectively for the identity matrix, the matrix with entries 1s only and the matrix with entries 0s only of order n. We also use the notation C_n for the matrix $[c_{ij}], 1 \leq i, j \leq n$ defined as $c_{ij} = 1$ for all $i \leq j$ and $c_{ij} = 0$ otherwise.

The notion of poset matrix was introduced by Mohammad and Talukder [4] who gave some recognition of series-parallel posets by using the poset matrix. A square $(0,1)$-matrix $M = [a_{ij}], 1 \leq i, j \leq m$ is called a *poset matrix* if and only if the following conditions hold:

1. $a_{ii} = 1$ for all $1 \leq i \leq m$ i.e. M is reflexive,
2. $a_{ij} = 1$ and $a_{ji} = 1$ imply $i = j$ i.e. M is antisymmetric and
3. $a_{ij} = 1$ and $a_{jk} = 1$ imply $a_{ik} = 1$ i.e. M is transitive.

An upper (or lower) triangular $(0,1)$-matrix with entries 1s in the main diagonal is clearly reflexive and antisymmetric. Therefore, an upper (or lower) triangular $(0,1)$-matrix with entries 1s in the main diagonal is a poset matrix if it is transitive. For example, both I_n and C_n, as defined above, are poset matrices for all $n \geq 1$ because these are upper triangular and clearly transitive. Some non-trivial examples of poset matrices are given as follows:

Example 2.1.

$$L = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \qquad L^{'} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

To each poset matrix $M_m = [a_{ij}]$, $1 \leq i, j \leq m$, a poset $\mathbf{A} = \langle A, \leqslant \rangle$, where $A = \{x_1, x_2, \ldots, x_m\}$ and x_i corresponds the i-th row (or column) of M_m, is associated by defining the order relation $\leqslant$ on A such that for all $1 \leq i, j \leq m$,

$$x_i \leqslant x_j \text{ if and only if } a_{ij} = 1.$$

Then it is said that the poset matrix M_m represents the poset $\mathbf{A}$ and vice versa. For example, the poset matrix I_n represents the n-element antichain poset $\mathbf{I}_n$ and the poset matrix C_n represents the n-element chain poset $\mathbf{C}_n$. Also the poset matrices L and $L^{'}$, given in Example 2.1, represent respectively the posets $\mathbf{B}_{2,1}$ and $\mathbf{B}_{1,2}$.

Let M_m be a poset matrix. Then, for some $1 \leq i, j \leq m$, interchanges of i-th and j-th rows along with interchanges of i-th and j-th columns in M_m is called *(i,j)-relabeling* of M_m. The following results were obtained in [4] where the authors gave interpretations of some operations in the poset matrix.

Theorem 2.1. *Any relabeling of a poset matrix is a poset matrix and it represents the same poset up to isomorphism.*

Theorem 2.2. *Every poset matrix can be relabeled to an upper (or lower) triangular matrix with 1s in the main diagonal by a finite number of relabeling.*

From now on, by a poset matrix we mean a poset matrix in upper triangular form.

3 Kronecker product of poset matrices

We now recall the Kronecker product of matrices. The *Kronecker product* (*tensor product* or *direct product*) of the matrices $M_{m,n} = [a_{ij}], 1 \leq i \leq m, 1 \leq j \leq n$ and $N_{p,q}$, denoted by $M_{m,n} \otimes N_{p,q}$, is an $(m \times p)$-by-$(n \times q)$ block matrix defined as follows:

$$M_{m,n} \otimes N_{p,q} = \begin{bmatrix} a_{11}N_{p,q} & a_{12}N_{p,q} & \cdots & a_{1n}N_{p,q} \\ a_{21}N_{p,q} & a_{22}N_{p,q} & \cdots & a_{2n}N_{p,q} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}N_{p,q} & a_{m1}N_{p,q} & \cdots & a_{mn}N_{p,q} \end{bmatrix}$$

Let $M_m = [a_{ij}], 1 \leq i, j \leq m$ and N_n be poset matrices. Since M_m is a (0,1)-matrix, the (i, j)-th block P_{ij} of the matrix $P_{m \times n} = M_m \otimes N_n =$

$[P_{ij}], 1 \leq i, j \leq m$ can be expressed as follows:

$$P_{ij} = \begin{cases} N_n \text{ if } a_{ij} = 1, \\ Z_n \text{ otherwise.} \end{cases} \tag{3.1}$$

The following example shows the Kronecker product of the poset matrices L and L' given in Example 2.1.

Example 3.1.

$$L \otimes L' = \left[\begin{array}{ccc|ccc|ccc} 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

We observe that the block matrix $L \otimes L'$ is a poset matrix and represents the poset $\mathbf{B}_{2,1} \times \mathbf{B}_{1,2}$ that can be checked immediately from the Hasse diagram of $\mathbf{B}_{2,1} \times \mathbf{B}_{1,2}$ shown in Figure 3. We establish this result in the following theorem, which gives an interpretation of the Kronecker product of poset matrices in posets.

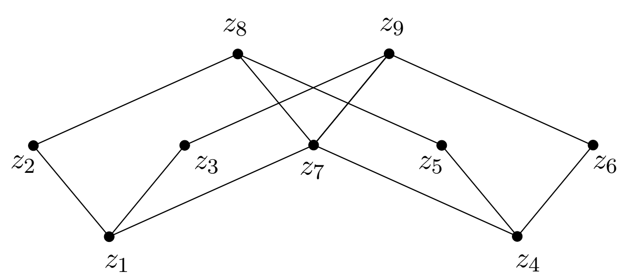

Figure 3: Hasse diagram of $\mathbf{B}_{2,1} \times \mathbf{B}_{1,2}$ with labeling.

Theorem 3.1. *Let the poset matrix M_m represent the poset $\mathbf{A}$ and the poset matrix N_n represent the poset $\mathbf{B}$. Then the matrix $M_m \otimes N_n$ is a poset matrix and it represents the poset $\mathbf{A} \times \mathbf{B}$.*

Proof. Let $M_m = [a_{ij}], 1 \leq i, j \leq m$, $N_n = [b_{ij}], 1 \leq i, j \leq n$, $\mathbf{A} = \langle A; \leqslant_A \rangle$, where $A = \{x_1, x_2, \ldots, x_m\}$ and $\mathbf{B} = \langle B; \leqslant_B \rangle$ where $B = \{y_1, y_2, \ldots, y_n\}$. Also, let $M_m \otimes N_n = P_{m\times n} = [p_{ij}], 1 \leq i, j \leq m \times n$ with block representation $[P_{ij}], 1 \leq i, j \leq m$. Since both M_m and N_n are upper triangular matrices, $P_{ij} = Z_n$ for all $i > j$. Thus $P_{m\times n}$ is upper triangular with 1s in the main diagonal and hence $P_{m\times n}$ is clearly reflexive and antisymmetric. For transitivity of $P_{m\times n}$, let $p_{ij} = p_{jk} = 1$ for some $1 \leq i \leq j \leq k \leq m \times n$. Then we have the following three cases:

1. $p_{ij}, p_{jk} \in P_{rr} = N_n$ for some $1 \leq r \leq m$. Then there exist $b_{i'j'}, b_{j'k'}, b_{i'k'} \in N_n$ such that $b_{i'j'} = q_{ij} = 1$, $b_{j'k'} = q_{jk} = 1$ and $b_{i'k'} = q_{ik}$. Since N_n is transitive, $q_{ik} = b_{i'k'} = 1$.

2. $p_{ij} \in P_{rs} = N_n$ and $p_{jk} \in P_{ss} = N_n$ for some $1 \leq r < s \leq m$. Then $p_{ik} \in P_{rs} = N_n$ and hence $p_{ik} = 1$.

3. $p_{ij} \in P_{rs} = N_n$ and $p_{jk} \in P_{st} = N_n$ for some $1 \leq r < s < t \leq m$. Then $p_{ik} \in P_{rt}$. Then, by the definition of Kronecker product of poset matrices, $a_{rs}, a_{st} \in M_m$; and $a_{rs} = a_{st} = 1$. Since M_m is transitive, $a_{rt} = 1$. Therefore $P_{rt} = N_n$ and, clearly, $p_{ik} = 1$.

Thus $P_{m\times n}$ is transitive and hence is a poset matrix.
We now show that $P_{m\times n}$ represents $\mathbf{A} \times \mathbf{B} = \langle A \times B; \leqslant_\times \rangle$, where $A \times B = \{(x_k, y_r) : 1 \leq k \leq m, 1 \leq r \leq n\}$. Then $A \times B \cong \{z_i : 1 \leq i \leq m \times n\} = Z$, because the mapping $(x_k, y_r) \mapsto z_i$ such that $n(k-1) + r = i$ gives an one-to-one correspondence between $A \times B$ and Z. Let $p_{ij} = 1$ in $P_{m\times n}$ for some $1 \leq i \leq j \leq m \times n$. Assign $r = i \mod n$, $s = j \mod n$, $k = \frac{i-r}{n} + 1$ and $l = \frac{j-s}{n} + 1$. Then $z_i \mapsto (x_k, y_r)$, $z_j \mapsto (x_l, y_s)$ and $p_{ij} = b_{kl} \in Q_{rs} = N_n$. Thus $b_{kl} = 1$ in N_n and, by the definition of Kronecker product of poset matrices, $a_{rs} = 1$ in M_m. Since N_n represents $\mathbf{A}$ and M_m represents $\mathbf{B}$, $x_k \leqslant_A x_l$ and $y_r \leqslant_B y_s$. Then, by the definition of direct product of posets, $(x_k, y_r) \leqslant_\times (x_l, y_s)$ i.e. $z_i \leqslant_\times z_j$.

For the converse, we can similarly show that $z_i \leqslant_\times z_j$ for some $1 \leq i, j \leq m \times n$ implies $p_{ij} = 1$ in $P_{m\times n}$. Hence $P_{m\times n}$ represents $\mathbf{A} \times \mathbf{B}$. □

4 Ordinal product of poset matrices

The ordinal sum of matrices was introduced in [4]. The ordinal sum of the matrices $M_{m,p}$ and $N_{n,q}$, denoted by $M_{m,p} \boxplus N_{n,q}$, is an $(m+n)$-by-$(p+q)$

block matrix defined as follows:

$$M_{m,p} \boxplus N_{n,q} = \left[\begin{array}{ccc} M_{m,p} & | & O_{m,q} \\ -- & . & -- \\ Z_{n,p} & | & N_{n,q} \end{array} \right]$$

The authors then gave a generalization of the ordinal sum of m poset matrices. They constructed the (i,j)-th block T_{ij} of the matrix $\boxplus_{k=1}^{m} N_{n_k} = T_t = [T_{ij}], 1 \leq i,j \leq m$, where $t = \sum_{k=1}^{m} n_k$, and gave its interpretation in posets as follows:

Theorem 4.1. *Let the poset matrix N_{n_i} represent the poset $\mathbf{B}_i$, where $1 \leq i \leq m$. Then the matrix $\boxplus_{k=1}^{m} N_{n_k}$ is a poset matrix and it represents the poset $\bigoplus_{k=1}^{m} \mathbf{B}_k$.*

Note that we write briefly $\boxplus^n N_n$ for the ordinal sum $N_n \boxplus N_n \boxplus \cdots \boxplus N_n$. We now define the ordinal product of matrices.

Definition 4.2. *The* ordinal product *of the matrices $M_{m,n} = [a_{ij}], 1 \leq i \leq m, 1 \leq j \leq n$ and $N_{p,q}$, denoted by $M_{m,n} \boxtimes N_{p,q}$, is an $(m \times p)$-by-$(n \times q)$ block matrix defined as follows:*

$$M_{m,n} \boxtimes N_{p,q} = \begin{bmatrix} a_{11}N_{p,q} & a_{12}O_{p,q} & \cdots & a_{1n}O_{p,q} \\ a_{21}O_{p,q} & a_{22}N_{p,q} & \cdots & a_{2n}O_{p,q} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}O_{p,q} & a_{m1}O_{p,q} & \cdots & a_{mn}N_{p,q} \end{bmatrix}$$

Let $M_m = [a_{ij}], 1 \leq i,j \leq m$ and N_n be poset matrices. Since M_m is a (0,1)-matrix, the (i,j)-th block Q_{ij} of the matrix $Q_{m \times n} = M_m \boxtimes N_n = [Q_{ij}], 1 \leq i,j \leq m$ can be expressed as follows:

$$Q_{ij} = \begin{cases} N_n \text{ if } i = j, \\ O_n \text{ if } i \neq j \text{ and } a_{ij} = 1, \\ Z_n \text{ otherwise.} \end{cases} \qquad (4.2)$$

The following example shows the ordinal product of the poset matrices L and L' given in Example 2.1.

Example 4.1.

$$L \boxtimes L' = \left[\begin{array}{ccc|ccc|ccc} 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ \hline 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

We observe that the block matrix $L \boxtimes L'$ is a poset matrix and it represents the poset $\mathbf{B}_{2,1} \otimes \mathbf{B}_{1,2}$ that can be checked immediately from the Hasse diagram of $\mathbf{B}_{2,1} \otimes \mathbf{B}_{1,2}$ shown in Figure 4. We establish the following result which gives an interpretation of the ordinal product of poset matrices in posets.

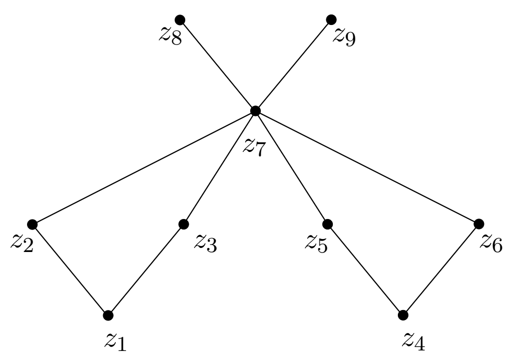

Figure 4: Hasse diagram of $\mathbf{B}_{2,1} \otimes \mathbf{B}_{1,2}$ with labeling.

Theorem 4.3. *Let the poset matrix M_m represent the poset $\mathbf{A}$ and let the poset matrix N_n represent the poset $\mathbf{B}$. Then the matrix $M_m \boxtimes N_n$ is a poset matrix which represents the poset $\mathbf{A} \otimes \mathbf{B}$.*

Proof. Let $M_m = [a_{ij}], 1 \leq i, j \leq m$, $N_n = [b_{ij}], 1 \leq i, j \leq n$, $\mathbf{A} = \langle A; \leqslant_A \rangle$ where $A = \{x_1, x_2, \ldots, x_m\}$ and $\mathbf{B} = \langle B; \leqslant_B \rangle$ where $B = \{y_1, y_2, \ldots, y_n\}$. Also, let $M_m \boxtimes N_n = Q_{m \times n} = [q_{ij}], 1 \leq i, j \leq m \times n$ with block representation $[Q_{ij}], 1 \leq i, j \leq m$. Since both M_m and N_n are upper triangular matrices, $Q_{ij} = Z_n$ for all $i > j$. Thus $Q_{m \times n}$ is upper triangular with elements 1s in the main diagonal and hence $Q_{m \times n}$ is clearly reflexive and antisymmetric. For transitivity of $Q_{m \times n}$, let $q_{ij} = q_{jk} = 1$ for some $1 \leq i \leq j \leq k \leq m \times n$. Then we have the following three cases:

1. $q_{ij}, q_{jk} \in Q_{rr} = N_n$ for some $1 \le r \le m$. Then there exist $b_{i'j'}, b_{j'k'}, b_{i'k'} \in N_n$ such that $b_{i'j'} = q_{ij} = 1$, $b_{j'k'} = q_{jk} = 1$ and $b_{i'k'} = q_{ik}$. Since N_n is transitive, $q_{ik} = b_{i'k'} = 1$.

2. $q_{ij} \in Q_{rs} = O_n$ and $q_{jk} \in Q_{ss} = N_n$ for some $1 \le r < s \le m$. Then $q_{ik} \in Q_{rs} = O_n$ and clearly $q_{ik} = 1$.

3. $q_{ij} \in Q_{rs} = O_n$ and $q_{jk} \in Q_{st} = O_n$ for some $1 \le r < s < t \le m$. Then $q_{ik} \in Q_{rt}$. Then, by the definition of ordinal product of poset matrices, $a_{rs}, a_{st} \in M_m$; and $a_{rs} = a_{st} = 1$. Since M_m is transitive, $a_{rt} = 1$. Therefore, $Q_{rt} = O_n$ and clearly $q_{ik} = 1$.

Thus $Q_{m\times n}$ is transitive and hence is a poset matrix.
We now show that $Q_{m\times n}$ represents $\mathbf{A} \otimes \mathbf{B} = \langle A \times B; \leqslant_\otimes \rangle$, where $A \times B = \{(x_k, y_r) : 1 \le k \le m, 1 \le r \le n\}$. Since the mapping $(x_k, y_r) \mapsto z_i$, where $n(k-1) + r = i$, gives an one-to-one correspondence, $A \times B \cong \{z_i : 1 \le i \le m \times n\}$. Let $q_{ij} = 1$ in $Q_{m\times n}$ for some $1 \le i \le j \le m \times n$. Assign $r = i \mod n$, $s = j \mod n$, $k = \frac{i-r}{n} + 1$ and $l = \frac{j-s}{n} + 1$. Then $(x_k, y_r) \mapsto z_i$, $(x_l, y_s) \mapsto z_j$ and we have the following cases:

1. $k = l$. Then $Q_{kl} = N_n$ and $b_{rs} = q_{ij} \in Q_{kl} = N_n$. Then $x_k = x_l$ in A and, since N_n represents $\mathbf{B}$, $y_r \leqslant_B y_l$. Then, by the definition of ordinal product of posets, $(x_k, y_r) \leqslant_\otimes (x_l, y_s)$; i.e., $z_i \leqslant_\otimes z_j$.

2. $k < l$. Then $Q_{kl} = O_n$. By the definition of ordinal product of poset matrix, $a_{kl} \in M_m$ and $a_{kl} = 1$. Since M_m represents $\mathbf{A}$, $x_k \leqslant_A x_l$. Then, by the definition of ordinal product of posets, $(x_k, y_r) \leqslant_\otimes (x_l, y_s)$ i.e. $z_i \leqslant_\otimes z_j$.

For the converse, similarly, we can show that $z_i \leqslant_\otimes z_j$ for some $1 \le i, j \le m \times n$ implies $q_{ij} = 1$ in $Q_{m\times n}$. Hence $Q_{m\times n}$ represents $\mathbf{A} \otimes \mathbf{B}$. □

Proposition 4.4. *Let* $\mathbf{B}$ *be any poset. Then* $\mathbf{C}_m \otimes \mathbf{B} \cong \oplus^m \mathbf{B}$.

Proof. Let the poset matrix N_n represent the poset $\mathbf{B}$. We first show that $C_m \boxtimes N_n = \boxplus^m N_n$. By Theorem 4.3 and Theorem 4.1, both $C_m \boxtimes N_n$ and $\boxplus^m N_n$ are poset matrices. By the definition of ordinal product of poset matrices, the (i,j)-th block Q_{ij} of the matrix $C_m \boxtimes N_n = Q_{m\times n} = [Q_{ij}], 1 \le i, j \le m$ takes the following form:

$$Q_{ij} = \begin{cases} N_n \text{ if } i = j, \\ O_n \text{ if } i < j, \\ Z_n \text{ otherwise.} \end{cases}$$

By Theorem 4.1, the (i,j)-th block T_{ij} of the matrix $\boxplus_{k=1}^{m} N_{n_k} = T_t = [T_{ij}], 1 \le i, j \le m$, where $t = \sum_{i=1}^{m} n_i$, takes the following form:

$$T_{ij} = \begin{cases} N_{n_i} \text{ if } i = j, \\ O_{n_i,n_j} \text{ if } i < j, \\ Z_{n_j,n_i} \text{ otherwise.} \end{cases}$$

Then for $n_i = n, 1 \le i \le m$, we have $\boxplus^m N_n = T_{m\times n} = Q_{m\times n}$. This shows that $C_m \boxtimes N_n = \boxplus^m N_n$.
Now we show that $\mathbf{C}_m \otimes \mathbf{B} \cong \oplus^m \mathbf{B}$. Theorem 4.3 shows that $C_m \boxtimes N_n$ represents the poset $\mathbf{C}_m \otimes \mathbf{B}$ and Theorem 4.1 shows that $\boxplus^m N_n$ represents the poset $\oplus^m \mathbf{B}$. Then $C_m \boxtimes N_n = \boxplus^m N_n$ implies $\mathbf{C}_m \otimes \mathbf{B} \cong \oplus^m \mathbf{B}$. □

5 Recognition of factorable posets

Definition 5.1. *A poset* $\mathbf{F}$ *is said to be* factorable *if and only if their exist the nonsingleton posets* $\mathbf{A}$ *and* $\mathbf{B}$ *such that* $\mathbf{F} \cong \mathbf{A} \times \mathbf{B}$.

For example, the ladder poset $\mathbf{L}_3$, shown in the Figure 5, is factorable because $\mathbf{L}_3 \cong \mathbf{C}_2 \times \mathbf{C}_3 \cong \mathbf{C}_3 \times \mathbf{C}_2$. In general, the posets $\mathbf{L}_n$, for all $n \ge 2$, are factorable because $\mathbf{L}_n \cong \mathbf{C}_2 \times \mathbf{C}_n \cong \mathbf{C}_n \times \mathbf{C}_2$. We see that for any poset $\mathbf{A}$, the poset $n\mathbf{A}$ is factorable because $n\mathbf{A} \cong \mathbf{I}_n \times \mathbf{A}$.

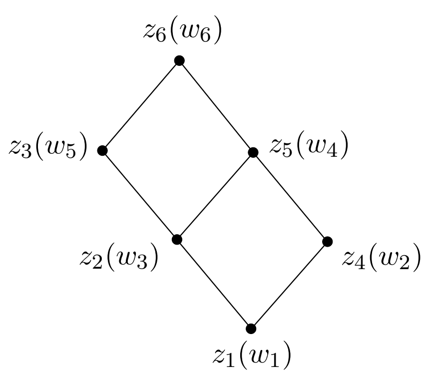

Figure 5: Hasse diagram of $\mathbf{L}_3$ with labeling.

We now define the property of transitive blocks of poset matrices in a poset matrix.

Definition 5.2. *Let M be an $(m \times n)$-by-$(m \times n)$ poset matrix consisting of the n-by-n blocks M_{ij}, $1 \le i, j \le m$ for some $m > 1$ and $n > 1$. Then M has the property of* transitive blocks of poset matrices *of length $\{m, n\}$ if and only if for all $1 \le i, j, k \le m$ the following conditions hold:*

1. *$M_{ii} = N_n$, a poset matrix,*
2. *$M_{ij} = Z_n$ for $i > j$; and $M_{ij} = N_n$ or $M_{ij} = Z_n$ for $i < j$,*
3. *$M_{ij} = M_{jk} = N_n$ implies $M_{ik} = N_n$.*

Example 5.1.

$$M = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ \hline 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\textit{(3,4)-relabeling}} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] = M^{'}$$

Here, although the poset matrix M seems not to satisfy the property of the transitive blocks of poset matrices, the poset matrix $M^{'}$ obtained by $(3, 4)$-relabeling of M satisfies the property of transitive blocks of poset matrices of length $\{2, 3\}$. Also, the poset matrix $M^{''}$, as in the following example, obtained by $(2, 3)$-relabeling and $(4, 5)$-relabeling of M satisfies the property of the transitive blocks of poset matrices of length $\{3, 2\}$.

We see that $M^{'} = C_2 \boxtimes C_3$ and $M^{''} = C_3 \boxtimes C_2$. We prove this result in general in the following example.

Example 5.2.

$$M = \left[\begin{array}{cc|cc|cc} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ \hline 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\textit{(2,3)-relabeling}} \left[\begin{array}{cc|cc|cc} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ \hline 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{\textit{(4,5)-relabeling}} \left[\begin{array}{cc|cc|cc} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ \hline 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] = M^{''}$$

Theorem 5.3. *A matrix satisfies the property of transitive blocks of poset matrices of length $\{m, n\}$ for some positive integers m and n if and only if it can be obtained as the Kronecker product of some poset matrices M_m and N_n.*

Proof. Let the matrix P be obtained as the Kronecker product of the poset matrices M_m and N_n. Then by the definition of Kronecker product, $P = M_m \otimes N_n$, and by Theorem 3.1, P is a block poset matrix. This shows that P is upper triangular having the poset matrix N_n as the diagonal blocks satisfying the first two conditions in Definition 5.2. Let $M_m = [a_{ij}]$, $1 \leq i, j \leq m$ and $P = [P_{ij}]$, $1 \leq i, j \leq m$ such that $P_{ij} = P_{jk} = N_n$ for some $1 \leq i < j \leq m$. Then by the definition of Kronecker product of poset matrices, we have $a_{ij} = a_{jk} = 1$ (Equation 3.1). Since M_m is transitive, $a_{ik} = 1$. Therefore, $P_{ik} = N_n$ which satisfies the last condition in Definition 5.2. This shows that P satisfies the property of transitive blocks of poset matrices of length $\{m, n\}$.
Conversely, we suppose that the matrix P satisfies the property of transitive blocks of poset matrices of length $\{m, n\}$ for some positive integers m and n and show, similarly, that P can be obtained as the Kronecker product of the poset matrices M_m and N_n. □

We observe that $M^{'}$ represents the factorable poset $\mathbf{L}_3 \cong \mathbf{C}_2 \times \mathbf{C}_3$ with labeling z_i as the i-th element and $M^{''}$ represents the factorable poset $\mathbf{L}_3 \cong \mathbf{C}_3 \times \mathbf{C}_2$ with labeling w_i as the i-th element (Figure 5). We establish this result in the following theorem, where we give a matrix recognition of factorable posets.

Theorem 5.4. *Let the poset matrix P represent the poset $\mathbf{F}$. Then $\mathbf{F}$ is a factorable poset if and only if P can be relabeled in such a form that it satisfies the property of transitive blocks of poset matrices.*

Proof. Let the poset $\mathbf{F}$ be factorable. Then there exist the nonsingleton posets $\mathbf{A}$ and $\mathbf{B}$ such that $\mathbf{F} \cong \mathbf{A} \times \mathbf{B}$. Let the poset matrix M_m represents $\mathbf{A}$ and the poset matrix N_n represents $\mathbf{B}$. Then, by Theorem 3.1, the poset matrix $M_m \otimes N_n$ represents the poset $\mathbf{A} \times \mathbf{B} \cong \mathbf{F}$. This shows that the poset matrix P can be relabeled such that $P = M_m \otimes N_n$. Then by Theorem 5.3, P satisfies the property of transitive blocks of poset matrices of length $\{m, n\}$.
Conversely, we suppose that the poset matrix P can be relabeled in such a form that it satisfies the property of transitive blocks of poset matrices and show, similarly, that the poset $\mathbf{F}$ is factorable. □

6 Recognition of composite posets

Definition 6.1. *A poset* $\mathbf{C}$ *is said to be* composite *if and only if their exist the nonsingleton posets* $\mathbf{A}$ *and* $\mathbf{B}$ *such that* $\mathbf{C} \cong \mathbf{A} \otimes \mathbf{B}$.

For example, the poset $2\mathbf{C}_2 \oplus \mathbf{1}$, shown in Figure 6, is composite because $2\mathbf{C}_2 \oplus \mathbf{1} \cong \mathbf{B}_{2,1} \otimes \mathbf{C}_2$. We see that for any poset $\mathbf{A}$, the poset $n\mathbf{A}$ is composite because $n\mathbf{A} \cong \mathbf{I}_n \otimes \mathbf{A}$.

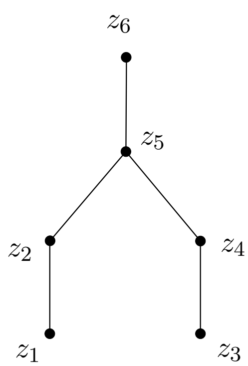

Figure 6: Hasse diagram of $2\mathbf{C}_2 \oplus \mathbf{1}$ with labeling.

We now define the property of transitive blocks of 1s in a poset matrix.

Definition 6.2. *Let* M *be an* $(m \times n)$*-by-*$(m \times n)$ *poset matrix consisting of the* n*-by-*n *blocks* M_{ij}, $1 \le i, j \le m$ *for some* $m > 1$ *and* $n > 1$. *Then* M *has the property of* transitive blocks of 1s *of length* $\{m, n\}$ *if and only if for all* $1 \le i, j, k \le m$ *the following conditions hold:*

1. $M_{ii} = N_n$, *a poset matrix,*
2. $M_{ij} = Z_n$ *for* $i > j$*; and* $M_{ij} = O_n$ *or* $M_{ij} = Z_n$ *for* $i < j$,
3. $M_{ij} = M_{jk} = O_n$ *implies* $M_{ik} = O_n$.

Example 6.1.

$$N = \left[\begin{array}{cc|cc|cc} 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ \hdashline 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ \hdashline 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\textit{(2,3)-relabeling}} \left[\begin{array}{cc|cc|cc} 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ \hdashline 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ \hdashline 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] = N'$$

Here, although the poset matrix N in the above example seems not to satisfy the property of the transitive blocks of 1s, the poset matrix $N^{'}$ obtained by $(2,3)$-relabeling of N satisfies the property of transitive blocks of 1s of length $\{3,2\}$. We see that $N^{'} = C_3 \boxtimes C_2$. We prove this result in general in the following theorem.

Theorem 6.3. *A matrix satisfies the property of transitive blocks of 1s of length $\{m,n\}$ for some positive integers m and n if and only if it can be obtained as the ordinal product of some poset matrices M_m and N_n.*

Proof. Let the matrix Q be obtained as the ordinal product of the poset matrices M_m and N_n. Then by the definition of the ordinal product of poset matrices, $Q = M_m \boxtimes N_n$, and by Theorem 4.3, Q is a block poset matrix. This shows that Q is upper triangular having the poset matrix N_n as the diagonal blocks satisfying the first two conditions in Definition 6.2. Let $M_m = [a_{ij}]$, $1 \leq i,j \leq m$ and $Q = [Q_{ij}]$, $1 \leq i,j \leq m$ such that $Q_{ij} = Q_{jk} = O_n$ for some $1 \leq i < j \leq m$. Then by the definition of ordinal product of poset matrices, we have $a_{ij} = a_{jk} = 1$ (Equation 4.2). Since M_m is transitive, $a_{ik} = 1$. Therefore, $Q_{ik} = O_n$ which satisfies the last condition in Definition 6.2. This shows that Q satisfies the property of transitive blocks of 1s of length $\{m,n\}$. Conversely, we suppose that the matrix Q satisfies the property of transitive blocks of 1s of length $\{m,n\}$ for some positive integers m and n and show, similarly, that the matrix Q can be obtained as the ordinal product of the poset matrices M_m and N_n. □

We observe that the poset matrix $N^{'}$, as in the previous example, represents the composite poset $2\mathbf{C}_2 \oplus \mathbf{1} \cong \mathbf{B}_{2,1} \otimes \mathbf{C}_2$. We establish this result in the following theorem, where we give a matrix recognition of composite posets.

Theorem 6.4. *Let the poset matrix Q represent the poset $\mathbf{C}$. Then $\mathbf{C}$ is a composite poset if and only if Q can be relabeled in such a form that it satisfies the property of transitive blocks of 1s.*

Proof. Let the poset $\mathbf{C}$ be a composite poset. Then there exist the nonsingleton posets $\mathbf{A}$ and $\mathbf{B}$ such that $\mathbf{C} \cong \mathbf{A} \otimes \mathbf{B}$. Let the poset matrix M_m represents $\mathbf{A}$ and the poset matrix N_n represents $\mathbf{B}$. Then, by Theorem 4.3, the poset matrix $M_m \boxtimes N_n$ represents the poset $\mathbf{A} \otimes \mathbf{B} \cong \mathbf{F}$. This shows that the poset matrix Q can be relabeled such that $Q = M_m \boxtimes N_n$. Then by Theorem 6.3, Q satisfies the property of transitive blocks of 1s of length $\{m,n\}$.

Conversely, we suppose that the poset matrix Q can be relabeled in such a form that it satisfies the property of transitive blocks of 1s and show, similarly, that the poset **C** is a composite poset. □

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