International Journal of Mathematics and
Computer Science, **16**(2021), no. 4, 1203–1215

Separability of Intuitionistic Fuzzy Relations

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(Received January 31, 2021, Accepted March 4, 2021)

Abstract

In this paper, we introduce the concept of separability of an intuitionistic fuzzy relation (IFR). First, we explain the concept of a fuzzy relation and IFR. Some operations and relations on IFRs are defined and examined. Then we introduc the definition of separability on IFR as a generalization of separability on a fuzzy relation. We present some basic properties and identify separability for some operations with examples. We characterize the separability of IFR. Finally, we construct an algorithm to check the separability of IFR.

1 Introduction and Motivation

Cantor introduced the theory of a set which we refer to as a “classic or crisp set” in which the notion of on element is bivalent, 0 or 1. In 1965, Zadeh introduced the concept of a fuzzy set and suggested the idea that the membership degree is between 0 and 1. The concept of a fuzzy set is of fundamental importance in Fuzzy Theory. Later, many researchers developed many concepts based on the fuzzy concept.

Key words and phrases: Intuitionistic Fuzzy Set, Intuitionistic Fuzzy Relation, Separability.
AMS (MOS) Subject Classifications: 03E72, 08A72, 94D05.
ISSN 1814-0432, 2021, http://ijmcs.future-in-tech.net

The classical relation was generalized using the fuzzy relation by admitting a partial association among elements of a universe or discourse [1]. Moreover, the fuzzy relation was generalized to IFR which seems to be natural and useful in various applications [2-6]. For many cases, the IFR states the actual situation more adequately than the classical ones.

Some types of relations, such as fuzzy equivalent relation, similarity relation, fuzzy ordering relation, and separable fuzzy relation are of special importance. In 2017, Grzegorzewski [1] introduced the separable fuzzy relation. Since the concept of fuzzy relation has been generalized to IFR, it is therefore interesting to develop the new concept of separability on IFR. In this article, we construct the definition of a separable IFR as a generalization of a separable fuzzy relation and develop a new concept related to the concept of separability of IFR. We look at some examples, properties and some characterizations.

2 Preliminaries: Relations and Operations on Intuitionistic Fuzzy Sets

In 1983, Atanassov generalized the concept of a fuzzy set to an intuitionistic fuzzy set (IFS) [8-10]. Many researchers [11-13] discussed some applications of IFS. Following are the definitions of a fuzzy set, IFS, and the relations and operations on IFS with some examples.

Definition 2.1. *Let X be a crisp set. A Fuzzy Set (FS) of X is defined as a set of ordered pairs $A = \{(x, \mu_A(x)) : x \in X\}$, where $0 \leq \mu_A(x) \leq 1$, for all $x \in X$. $\mu_A(x)$ is called the membership function or grade of membership of x in A.*

Example 2.2. *Let $X = \{a, b, c, d, e\}$. Clearly, $A = \{(a, 0.8), (b, 0.3), (c, 0.9), (d, 0.6), (e, 0.1)\}$ is an example of a fuzzy set of X.*

Every crisp set X can be represented as a FS $A = \{(x, 1), x \in \text{X}\}$.

Definition 2.3. *Let X be a crisp set. An Intuitionistic Fuzzy Set (IFS) $\mathcal{A}$ of X is defined as $\mathcal{A} = \{(x, \mu_A(x), v_A(x)) : x \in X\}$, where $0 \leq \mu_A(x) \leq 1, 0 \leq v_A(x) \leq 1$, and $0 \leq \mu_A(x) + v_A(x) \leq 1$, for all $x \in X$.*

The functions μ_A and v_A are called the membership and the non-membership functions respectively. The $\mu_A(x)$ is called the degree of membership of x in

$\mathcal{A}$, while the $v_A(x)$ is called the degree of non-membership of x in $\mathcal{A}$. The amount $\pi_A(x) = 1 - \mu_A(x) - v_A(x)$ is called the degree of indeterminacy or hesitation part.

Example 2.4. *Let* $X = \{x_1, x_2, x_3\}$ *and* $Y = \{y_1, y_2, y_3, y_4\}$*. The following are two examples of IFS of* X *and* Y *respectively.*
(1) $\mathcal{A} = \{(x_1, 0.6, 0.3), (x_2, 0.5, 0.4), (x_3, 0.9, 0.1)\}$,
(2) $\mathcal{B} = \{(y_1, 0.3, 0.7), (y_2, 0.8, 0.1), (y_3, 0.1, 0.7), (y_4, 0.6, 0.3)\}$.
We have $\pi_A(x_1) = 0.1, \pi_A(x_3) = 0, \pi_B(y_1) = 0, \pi_B(y_3) = 0.2$.

Every fuzzy set $A = \{(x, \mu_A(x)) : x \in X\}$ can be viewed as an IFS $\mathcal{A} = \{(x, \mu_A(x), 1 - \mu_A(x)) : x \in X\}$.

Definition 2.5. *Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be IFSs of* X*. We define the following relations and operations*
(1) $\mathcal{A} \subset \mathcal{B}$*(or* $\mathcal{B} \supset \mathcal{A}$*) iff* $(\forall x \in X)(\mu_A(x) \leq \mu_B(x))$ *and* $(v_A(x) \geq v_B(x))$.
(2) $\mathcal{A} = \mathcal{B}$ *iff* $(\forall x \in X)(\mu_A(x) = \mu_B(x))$ *and* $(v_A(x) = v_B(x))$.
(3) $\mathcal{A}^{\mathcal{C}} = \{(x, v_A(x), \mu_A(x)) : x \in X\}$.
(4) $\mathcal{A} \cap \mathcal{B} = \{< x, min(\mu_A(x), \mu_B(x)), max(v_A(x), v_B(x)) >: x \in X\}$.
(5) $\mathcal{A} \cup \mathcal{B} = \{< x, max(\mu_A(x), \mu_B(x)), min(v_A(x), v_B(x)) >: x \in X\}$.

Example 2.6. *Considering Example 2.4,*
if $\mathcal{P} = \{(x_1, 0.4, 0.5), (x_2, 0.5, 0.5), (x_3, 0.7, 0.2)\}$*, then* $\mathcal{A} \subset \mathcal{P}$.
$\mathcal{P}^{\mathcal{C}} = \{(x_1, 0.5, 0.4), (x_2, 0.5, 0.5), (x_3, 0.2, 0.7)\}$. $\mathcal{A} \cap \mathcal{P} = \{(x_1, 0.4, 0.5), (x_2, 0.5, 0.5), (x_3, 0.7, 0.2)\}$. $\mathcal{A} \cup \mathcal{P} = \{(x_1, 0.6, 0.3), (x_2, 0.5, 0.4), (x_3, 0.9, 0.1)\}$.

3 Domain and Codomain of IFR

A Fuzzy relation from X to Y is a fuzzy set from the cartesian product $X \times Y$. In this section, we first describe the concept and then we present some examples for illustration. Next, we present the definition of domain and codomain which are basic in order to construct the definition of separability of IFR. We conclude this section by giving some examples.

Definition 3.1. *Let* X *and* Y *be crisp sets. A fuzzy relation* R *from* X *to* Y *is a fuzzy set from* $X \times Y$. $R = \{< (x, y), \mu_R(x, y) >: (x, y) \in X \times Y\}$, *where* $0 \leq \mu_R(x, y) \leq 1$, *for all* $(x, y) \in X \times Y$.

Example 3.2. *Considering* X *and* Y *in Example 2.4, the following are two examples of fuzzy relations from* X *to* Y
$(1) R_1 = \{< (x_1, y_1), 0.6 >, < (x_1, y_2), 0.7 >, < (x_1, y_3), 0.2 >, < (x_1, y_4), 0.7 >$

$, < (x_2, y_1), 0.4 >, < (x_2, y_2), 0.5 >, < (x_2, y_3), 0.9 >, < (x_2, y_4), 0.5 >, < (x_3, y_1), 0.1 >, < (x_3, y_2), 0.8 >, < (x_3, y_3), 0.6 >, < (x_3, y_4), 0.6 >\}$.

(2)$R_2 = \{< (x_1, y_1), 1 >, < (x_1, y_2), 0.5 >, < (x_1, y_3), 0.7 >, < (x_1, y_4), 0 >, < (x_2, y_1), 0.7 >, < (x_2, y_2), 0.3 >, < (x_2, y_3), 0.7 >, < (x_2, y_4), 0.5 >, < (x_3, y_1), 0.1 >, < (x_3, y_2), 0.8 >, < (x_3, y_3), 0.4 >, < (x_3, y_4), 0.3 >\}$.
We can express these fuzzy relations as matrices:

$$[R_1] = \begin{bmatrix} 0.6 & 0.7 & 0.2 & 0.7 \\ 0.4 & 0.5 & 0.9 & 0.5 \\ 0.1 & 0.8 & 0.6 & 0.6 \end{bmatrix}, [R_2] = \begin{bmatrix} 1 & 0.5 & 0.7 & 0 \\ 0.7 & 0.3 & 0.7 & 0.5 \\ 0.1 & 0.8 & 0.4 & 0.3 \end{bmatrix}.$$

Definition 3.3. *Let X and Y be crisp sets. An IFR R from X to Y is an IFS of $X \times Y$. The membership function of R and the non-membership function of R are μ_R and v_R respectively. We write*
$R(X- > Y) = \{< (x, y), \mu_R(x, y), v_R(x, y) >: (x, y) \in X \times Y\}$.

From now on, we assume that the sets X and Y are finite; say, $X = \{x_1, x_2, x_3, ..., x_n\}$ and $Y = \{y_1, y_2, y_3, ..., y_m\}$. The IFR R from X to Y, $R(X- > Y)$ can be represented as a matrix $[R] = [(u_{ij}, v_{ij})]$, where $u_{ij} = \mu_R(x_i, y_j)$ and $v_{ij} = v_R(x_i, y_j)$, $i = 1, 2, 3, ..., n$ and $j = 1, 2, 3, ..., m$. We write $[R_u] = [u_{ij}]$, and $[R_v] = [v_{ij}]$.

Example 3.4. *Let X and Y be as in Example 2.4. Thus, for example,*
$R(X- > Y) = \{< (x_1, y_1), 0.4, 0.5 >, < (x_1, y_2), 0.8, 0.1 >, < (x_1, y_3), 0.8, 0.2 >, < (x_1, y_4), 0.5, 0.4 >, < (x_2, y_1), 0.7, 0.2 >, < (x_2, y_2), 0.4, 0.6 >, < (x_2, y_3), 0.9, 0.1 >, < (x_2, y_4), 0.6, 0.3 >, < (x_3, y_1), 0.2, 0.6 >, < (x_3, y_2), 0.8, 0.1 >, < (x_3, y_3), 0.6, 0.4 >, < (x_3, y_4), 0.3, 0.6 >\}$.
with

$$[R] = \begin{bmatrix} (0.4, 0.5) & (0.8, 0.1) & (0.8, 0.2) & (0.5, 0.4) \\ (0.7, 0.2) & (0.4, 0.6) & (0.9, 0.1) & (0.6, 0.3) \\ (0.2, 0.6) & (0.8, 0.1) & (0.6, 0.4) & (0.3, 0.6) \end{bmatrix}, while$$

$$[R_u] = \begin{bmatrix} 0.4 & 0.8 & 0.8 & 0.5 \\ 0.7 & 0.4 & 0.9 & 0.6 \\ 0.2 & 0.8 & 0.6 & 0.3 \end{bmatrix}, [R_v] = \begin{bmatrix} 0.5 & 0.1 & 0.2 & 0.4 \\ 0.2 & 0.6 & 0.1 & 0.3 \\ 0.6 & 0.1 & 0.4 & 0.6 \end{bmatrix}.$$

Definition 3.5. *Let R be an IFR from X to Y. The complement of the relation R is an IFR from X to Y, R^T, where $\mu_{R^T} = v_R$ and $v_{R^T} = \mu_R$. We can write, $R^T(X- > Y) = \{< (x, y), v_R(x, y), \mu_R(x, y); (x, y) \in X \times Y\}$.*

Example 3.6. *If R is as in Example 3.4, then*

$$[R^T] = \begin{bmatrix} (0.5, 0.4) & (0.1, 0.8) & (0.2, 0.8) & (0.4, 0.5) \\ (0.2, 0.7) & (0.6, 0.4) & (0.1, 0.9) & (0.3, 0.6) \\ (0.6, 0.2) & (0.1, 0.8) & (0.4, 0.6) & (0.6, 0.3) \end{bmatrix}.$$

Definition 3.7. *Let R and S be IFRs from X to Y. The union of R and S, $R \cup S$, is an IFR from X to Y whose membership and non-membership functions are $\mu_{R\cup S}(x, y) = max\{\mu_R(x, y), \mu_S(x, y)\}$ and $v_{R\cup S}(x, y) = min\{v_R(x, y), v_S(x, y)\}$, for all $(x, y) \in X \times Y$.*

Definition 3.8. *Let R and S be IFRs from X to Y. The intersection of R and S, $R\cap S$, is IFR from X to Y whose the membership and non-membership functions are $\mu_{R\cap S}(x, y) = min\{\mu_R(x, y), \mu_S(x, y)\}$ and $v_{R\cap S}(x, y) = max\{v_R(x, y), v_S(x, y)\}$, for all $(x, y) \in X \times Y$.*

It is easy to see that the " $\cup$ " and " $\cap$ " are well defined.

Example 3.9. *Consider X, Y and R as in Example 3.4.*
Let S be an IFR from X to Y whose matrix S is

$$[S] = \begin{bmatrix} (0.8, 0.2) & (0.3, 0.5) & (0.7, 0.2) & (0.6, 0.4) \\ (0.5, 0.4) & (0.4, 0.6) & (0.5, 0.4) & (0.2, 0.7) \\ (0.4, 0.6) & (0.8, 0.2) & (0.3, 0.6) & (0.3, 0.7) \end{bmatrix}.$$

Then the matrices of $R \cup S$ and $R \cap S$ are as follows:

$$[R \cup S] = \begin{bmatrix} (0.8, 0.2) & (0.8, 0.1) & (0.8, 0.2) & (0.6, 0.4) \\ (0.7, 0.2) & (0.4, 0.6) & (0.9, 0.1) & (0.6, 0.3) \\ (0.4, 0.6) & (0.8, 0.1) & (0.6, 0.4) & (0.3, 0.6) \end{bmatrix},$$

$$[R \cap S] = \begin{bmatrix} (0.4, 0.5) & (0.3, 0.5) & (0.7, 0.2) & (0.5, 0.4) \\ (0.5, 0.4) & (0.4, 0.6) & (0.5, 0.4) & (0.2, 0.7) \\ (0.2, 0.6) & (0.8, 0.1) & (0.3, 0.6) & (0.3, 0.7) \end{bmatrix}.$$

Definition 3.10. *Given an IFR from X to Y, $R(X- > Y)$, the Domain of R is defined as an IFS of X whose membership and non-membership functions of domain R are: $\mu_{dom(R)}(x) = max_{y\in Y}(\mu_R(x, y))$ and $v_{dom(R)}(x) = min_{y\in Y}(v_R(x, y))$, for all $x \in X$.*

Definition 3.11. *Given an IFR from X to Y, $R(X- > Y)$, the Codomain of R is defined as an IFS of Y whose the membership and non-membership functions of codomain R are: $\mu_{cod(R)}(y) = max_{x\in X}(\mu_R(x, y))$ and $v_{cod(R)}(y) = min_{x\in X}(v_R(x, y))$, for all $y \in Y$.*

Example 3.12. *In Example 3.4,*
$\mu_{dom(R)}(x_1) = max\{\mu_R(x_1, y_1), \mu_R(x_1, y_2), \mu_R(x_1, y_3), \mu_R(x_1, y_4)\}$.
$\mu_{dom(R)}(x_1) = max\{0.4, 0.8, 0.8, 0.5\} = 0.8, \mu_{dom(R)}(x_2) = 0.9$,
and $\mu_{dom(R)}(x_3) = 0.8$.
$v_{dom(R)}(x_1) = min\{v_R(x_1, y_1), v_R(x_1, y_2), v_R(x_1, y_3), v_R(x_1, y_4)\} = 0.1$.
$v_{dom(R)}(x_2) = 0.1$, *and* $v_{dom(R)}(x_3) = 0.1$.
$\mu_{cod(R)}(y_1) = max\{\mu_R(x_1, y_1), \mu_R(x_2, y_1), \mu_R(x_3, y_1)\} = 0.7$.
$\mu_{cod(R)}(y_2) = 0.8, \mu_{cod(R)}(y_3) = 0.9$, *and* , $\mu_{cod(R)}(y_4) = 0.6$.
$v_{cod(R)}(y_1) = min\{v_R(x_1, y_1), v_R(x_2, y_1), v_R(x_3, y_1)\} = 0.2$.
$v_{cod(R)}(y_2) = 0.1$, $v_{cod(R)}(y_3) = 0.1$, *and* $v_{cod(R)}(y_4) = 0.3$.

Thus, domain R=$\{(x_1, 0.8, 0.1), (x_2, 0.9, 0.1), (x_3, 0.8, 0.1)\}$ *and codomain R=*$\{(y_1, 0.7, 0.2), (y_2, 0.8, 0.1), (y_3, 0.9, 0.1), (y_4, 0.6, 0.3)\}$.

The following corollary is easy to prove:

Corollary 3.13. *(1)* $\mu_{dom(R)}(x)$ *is equal to maximum of the corresponding row of* $[R_u]$
(2) $v_{dom(R)}(x)$ *is equal to the minimum of the corresponding row of* $[R_v]$
(3) $\mu_{cod(R)}(y)$ *is equal to the maximum of the corresponding column of* $[R_u]$*, and*
(4) $v_{cod(R)}(y)$ *is equal to the minimum of the corresponding column of* $[R_v]$.

4 Separability of IFR

In this section, we introduce the definition of a separable IFR which is a generalization of Grzegorzewski's separable fuzzy relation [1]. We use an additional condition dealing with membership and non-membership functions of domain and codomain. We then identify and prove some properties and present some examples.

Definition 4.1. *An IFR* $R(X- > Y)$ *is separable if and only if*
$\mu_R(x, y) = min\{\mu_{dom(R)}(x), \mu_{cod(R)}(y)\}$ *and*
$v_R(x, y) = max\{v_{dom(R)}(x), v_{cod(R)}(y)\}$, *for all* $(x, y) \in X \times Y$.

Example 4.2. *Consider R as in Example 3.4. The IFR R is not separable since* $\mu_R(x_1, y_1) = 0.4$, *but* $min\{\mu_{dom(R)}(x_1), \mu_{cod(R)}(y_1)\} = 0.7$.

Example 4.3. *Let* $X = \{x_1, x_2, x_3, x_4\}$ *and let* $Y = \{y_1, y_2, y_3, y_4, y_5\}$. *The*

IFR from X to Y, $R(X->Y)$ is given as

$$[R] = \begin{bmatrix} (0.3,0.7) & (0.4,0.6) & (0.4,0.6) & (0,1) & (0.4,0.6) \\ (0.3,0.7) & (0.7,0.3) & (0.5,0.5) & (0,1) & (0.7,0.3) \\ (0.3,0.7) & (1,0) & (0.5,0.5) & (0,1) & (0.9,0.1) \\ (0.1,0.9) & (0.1,0.9) & (0.1,0.9) & (0,1) & (0.1,0.9) \end{bmatrix}.$$

Indeed, the domain R=$\{(x_1,0.4,0.6),(x_2,0.7,0.3),(x_3,1,0),(x_4,0.1,0.9)\}$ and the codomain R=$\{(y_1,0.3,0.7),(y_2,1,0),(y_3,0.5,0.5),(y_4,0,1),(y_5,0.9,0.1)\}$. We can check that R is separable.

Theorem 4.4. *If $R(X->Y))$ is IFR with $R=\emptyset$, then R is separable.*

Proof. Since $R=\emptyset$, $\mu_R(x,y)=0$ and $v_R(x,y)=0$, for all $(x,y)\in X\times Y$. Therefore, $\mu_R(x,y)=min\{\mu_{dom(R)}(x),\mu_{cod(R)}(y)\}$ and $v_R(x,y)=max\{v_{dom(R)}(x),v_{cod(R)}(y)\}$, for all $(x,y)\in X\times Y$.

Corollary 4.5. *Every classic relation R from X to Y with domain $(R)=X\times Y$ is separable.*

Proof. We can write $R=\{<(x,y),1,0>: x\in X, y\in Y\}$. We conclude that R is separable.

Theorem 4.6. *If $R(X->Y))$ is IFR with $v_R(x,y)=1$, for all $(x,y)\in X\times Y$, then R is separable.*

Proof. Since $v_R(x,y)=1$ for all $(x,y)\in X\times Y$, $\mu_R(x,y)=0$ for all $(x,y)\in X\times Y$. Therefore, $min\{\mu_{dom(R)}(x),\mu_{cod(R)}(y)\}$=min $\{0,0\}=0=\mu_R(x,y)$ and $max\{v_{dom(R)}(x),v_{cod(R)}(y)\}=max\{1,1\}=0=v_R(x,y)$, for all $(x,y)\in X\times Y$.

The separability of $R(X->Y)$ does not necessarily imply the separability of R^T as the following example shows.

Example 4.7. *Consider R as Example 4.3.*

$$[R^T] = \begin{bmatrix} (0.7,0.3) & (0.6,0.4) & (0.6,0.4) & (1,0) & (0.6,0.4) \\ (0.7,0.3) & (0.3,0.7) & (0.5,0.5) & (1,0) & (0.3,0.7) \\ (0.7,0.3) & (0,1) & (0.5,0.5) & (1,0) & (0.1,0.9) \\ (0.9,0.1) & (0.9,0.1) & (0.9,0.1) & (1,0) & (0.9,0.1) \end{bmatrix}.$$

Domain R^T=$\{(x_1,1,0),(x_2,1,0),(x_3,1,0),(x_4,1,0)\}$ and codomain R^T=$\{(y_1,0.9,0.1),(y_2,0.9,0.1),(y_3,0.9,0.1),(y_4,0.9,0.1),(y_5,0.9,0.1)\}$. Obviously, R^T is not separable.

Example 4.8. *Let $X = \{x_1, x_2, x_3\}$ and let $Y = \{y_1, y_2, y_3\}$. The matrices of IFRs P and Q from X to Y are given below.*

$$[P] = \begin{bmatrix} (0.6, 0.4) & (0.4, 0.6) & (0.6, 0.4) \\ (0.8, 0.2) & (0.4, 0.6) & (0.7, 0.3) \\ (0.9, 0.1) & (0.4, 0.6) & (0.7, 0.3) \end{bmatrix}, [Q] = \begin{bmatrix} (0.7, 0.3) & (0.6, 0.4) & (0.5, 0.5) \\ (0.5, 0.5) & (0.5, 0.5) & (0.5, 0.5) \\ (0.3, 0.7) & (0.7, 0.3) & (0.3, 0.7) \end{bmatrix}.$$

We have

$$[P \cup Q] = \begin{bmatrix} (0.7, 0.3) & (0.6, 0.4) & (0.5, 0.5) \\ (0.5, 0.5) & (0.5, 0.5) & (0.5, 0.5) \\ (0.3, 0.7) & (0.7, 0.3) & (0.3, 0.7) \end{bmatrix}.$$

$$[P \cap Q] = \begin{bmatrix} (0.6, 0.4) & (0.4, 0.6) & (0.6, 0.4) \\ (0.8, 0.2) & (0.4, 0.6) & (0.7, 0.3) \\ (0.9, 0.1) & (0.4, 0.6) & (0.7, 0.3) \end{bmatrix}.$$

Domain of $P \cup Q$=$\{(x_1, 0.7, 0.3), (x_2, 0.8, 0.2), (x_3, 0.9, 0.1)\}$
and codomain of $P \cap Q$=$\{(y_1, 0.9, 0.1), (y_2, 0.6, 0.4), (y_3, 0.7, 0.3)\}$.
Domain of $P \cap Q$=$\{(x_1, 0.6, 0.4), (x_2, 0.8, 0.2), (x_3, 0.9, 0.1)\}$
and codomain of $P \cap Q$=$\{(y_1, 0.9, 0.1), (y_2, 0.4, 0.6), (y_3, 0.7, 0.3)\}$.
(1) $P \cup Q$ is not separable because $\mu_{P \cup Q}(x_1, y_1) = 0.6$, but $min\{\mu_{dom(P \cup Q)}(x_1), \mu_{cod(P \cup Q)}(y_2)\} = min\{0.7, 0.9\} = 0.7$.
(2) One can check that $P \cap Q$ is separable.

Note that the first result in Example 4.8 shows that the operation "$\cup$" on IFR do not preserve separability.

Lemma 4.9. *Let $R(X- > Y)$ be an IFR, where $[R_u]$ consists only of the element* 0. *Then R is not separable.*

Proof. $[R_u] = [u_{ij}]$. Assume that $u_{ij} = 0$ and $u_{kl} \neq 0$, for $k \neq i$ or $l \neq j$. This means that $\mu_R(x_i, y_j) = 0$. Obviously, $\mu_{dom(R)}(x_i) > 0$ and $\mu_{cod(R)}(y_j) > 0$.
Therefore, $min\{\mu_{dom(R)}(x_i), \mu_{cod(R)}(y_j)\} > 0$.
Thus, $min\{\mu_{dom(R)}(x_i), \mu_{cod(R)}(y_j)\} \neq \mu_R(x_i, y_j)$. Consequently, R is not separable.

Lemma 4.10. *Let $R(X- > Y)$ be an IFR, where $[R_v]$ consists only of the element* 1. *Then R is not separable.*

Proof. $[R_v] = [v_{ij}]$. Assume that $v_{ij} = 1$ and $v_{kl} \neq 1$, for $k \neq i$ or $l \neq j$. It means $v_R(x_i, y_j) = 1$. Obviously, $v_{dom(R)}(x_i) < 1$ and $v_{cod(R)}(y_j) < 1$. Therefore, $max\{\mu_{dom(R)}(x_i), \mu_{cod(R)}(y_j)\} < 1$.

Thus, $max\{v_{dom(R)}(x_i), v_{cod(R)}(y_j)\} \neq v_R(x_i, y_j)$. So R is not separable.

The next two examples illustrate the previous lemmas.

Example 4.11. *Let $X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2, y_3, y_4\}$. The IFR from X to Y, $R(X- > Y)$ is given as*

$$[R] = \begin{bmatrix} (0.2, 0.7) & (0.5, 0.4) & (0.4, 0.6) & (0.2, 0.7) \\ (0.3, 0.7) & (0.6, 0.3) & (0, 0.7) & (0.4, 0.6) \\ (0.4, 0.6) & (0.3, 0.6) & (0.5, 0.5) & (0.8, 0.1) \end{bmatrix}.$$

We have

$$[R_u] = \begin{bmatrix} 0.2 & 0.5 & 0.4 & 0.2 \\ 0.3 & 0.6 & 0 & 0.4 \\ 0.4 & 0.3 & 0.5 & 0.8 \end{bmatrix}.$$

There is only an element 0; that is, u_{23}. So $\mu_R(x_2, y_3) = 0$.
$Min\{\mu_{dom(R)}(x_2), \mu_{cod(R)}(y_3)\} = min\{0.6, 0.5\} = 0.5 \neq \mu_R(x_2, y_3) = 0$.
As a result, R is not separable.

Example 4.12. *Let $X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2, y_3, y_4\}$. The IFR from X to Y, $R(X- > Y)$ is given as*

$$[R] = \begin{bmatrix} (0.4, 0.5) & (0, 0.8) & (0.4, 0.6) & (0.3, 0.7) \\ (0.6, 0.3) & (0.6, 0.3) & (0.2, 0.7) & (0.4, 0.6) \\ (0.4, 0.6) & (0, 1) & (0.4, 0.5) & (0.9, 0.1) \end{bmatrix}.$$

We have

$$[R_v] = \begin{bmatrix} 0.5 & 0.8 & 0.6 & 0.7 \\ 0.3 & 0.3 & 0.7 & 0.6 \\ 0.6 & 1 & 0.5 & 0.1 \end{bmatrix}.$$

There is only an element 1; that is, v_{32}. So $v_R(x_3, y_2) = 1$.
$Max\{v_{dom(R)}(x_3), v_{cod(R)}(y_2)\} = max\{0.1, 0.3\} = 0.3 \neq \mu_R(x_3, y_2) = 1$.
As a result, R is not separable.

The following corollary follows easily by using Definition 3.1 and Corollary 3.13.

Corollary 4.13. *An IFR R is separable if and only if for all $(x, y) \in X \times Y$:*
(1) $\mu_R(x, y)$ is equal to the maximum of the corresponding row of matrix $[R_u]$ or equal to the maximum of the corresponding column of matrix $[R_u]$;
and
(2) $v_R(x, y)$ is equal to the minimum of the corresponding row of matrix $[R_v]$ or equal to minimum of the corresponding column of matrix $[R_v]$.

5 Separability Checking Algorithm

Corollary 4.13 motivates developing an algorithm to check separability of an IFR R. We have to check the matrix of R, by checking the matrices $[R_u]$ and $[R_v]$. We can either check by row or column expansion. Below, we use row expansion.

Checking of $[R_u]$:
The i^{th} row:
Step 1: Start $j = 1$.
Step 2: Write element u_{ij} and find $max_{l\in\{1,2,3,\ldots,m\}}(u_{il})$.
Step 3 (1) : If $u_{ij} = max_{l\in\{1,2,3,\ldots,m\}}(u_{il})$, then write $j_b = j + 1$.
If $j_b \le m$, then write $j = j_b$, then go to Step 2.
If $j_b > m$, then All of the elements of the i^{th} row satisfy the requirement, then STOP.
Step 3 (2) : If $u_{ij} \ne max_{l\in\{1,2,3,\ldots,m\}}(u_{il})$, then find $max_{k\in\{1,2,3,\ldots,n\}}(u_{kj})$.
Step 4 (1): If $u_{ij} = max_{k\in\{1,2,3,\ldots,n\}}(u_{kj})$, then write $j_b = j + 1$.
If $j_b \le m$, then write $j = j_b$, then go to Step 2.
If $j_b > m$, then All of the elements of the i^{th} row satisfy the requirement, then STOP.
Step 4 (2) : If $u_{ij} \ne max_{k\in\{1,2,3,\ldots,n\}}(u_{kj})$, then R is not separable. STOP.
Checking of $[R_v]$:
The i^{th} row:
Step 1: Start $j = 1$.
Step 2: Write element v_{ij} and find $min_{l\in\{1,2,3,\ldots,m\}}(v_{il})$.
Step 3 (1) : If $v_{ij} = min_{l\in\{1,2,3,\ldots,m\}}(v_{il})$, then find $j_b = j + 1$.
If $j_b \le m$, write $j = j_b$, then go to Step 2.
If $j_b > m$, then All of the elements of the i^{th} row satisfy the requirement, then STOP.
Step 3 (2) : If $v_{1j} \ne min_{l\in\{1,2,3,\ldots,m\}}(v_{il})$, then write $min_{k\in\{1,2,3,\ldots,n\}}(v_{kj})$.
Step 4 (1): If $v_{1j} = min_{k\in\{1,2,3,\ldots,n\}}(v_{kj})$, then write $j_b = j + 1$.
If $j_b \le m$, then write $j = j_b$, then go to Step 2.
If $j_b > m$, then All of the elements of the i^{th} row satisfy the requirement, then STOP.
Step 4 (2) : If $v_{1j} \ne min_{k\in\{1,2,3,\ldots,n\}}(v_{kj})$, then R is not separable, then STOP.
The relation R is separable if all rows of $[R_u]$ and $[R_v]$ satisfy the requirement.

Example 5.1. *Let $X = \{x_1, x_2\}$ and $Y = \{y_1, y_2, y_3\}$. The IFR from X to Y, $R(X- > Y)$ is given as*

$$[R] = \begin{bmatrix} (0.6, 0.4) & (0.5, 0.5) & (0.4, 0.5) \\ (0.7, 0.2) & (0.5, 0.5) & (0.6, 0.3) \end{bmatrix}.$$

We have,

$$[R_u] = \begin{bmatrix} 0.6 & 0.5 & 0.4 \\ 0.7 & 0.5 & 0.5 \end{bmatrix}, [R_v] = \begin{bmatrix} 0.4 & 0.5 & 0.5 \\ 0.2 & 0.1 & 0.3 \end{bmatrix}.$$

Checking of $[R_u]$*:*
For $i = 1$: *Start* $j = 1$.
$u_{11} = 0.6 = max_{k\in\{1,2,3\}}(u_{1k}); j = 2 \leq 3.$
$u_{12} = 0.5 \neq max_{k\in\{1,2,3\}}(u_{1k}).$
$max_{k\in\{1,2\}}(u_{k2}) = 0.5 = u_{12}; j = 3 \leq 3..$
$u_{13} = 0.4 \neq max_{k\in\{1,2,3\}}(u_{1k}).$
$max_{k\in\{1,2\}}(u_{k3}) = 0.5.$
$u_{13} \neq max_{k\in\{1,2\}}(u_{k3})$
The algorithm shows that R is not separable.

Example 5.2. *Let* $X = \{x_1, x_2\}$ *and* $Y = \{y_1, y_2, y_3\}$*. The IFR from* X *to* Y*,* $R(X- > Y)$ *is given as*

$$[Q] = \begin{bmatrix} (0.5, 0.4) & (0.4, 0.4) & (0.5, 0.5) \\ (0.8, 0.1) & (0.4, 0.4) & (0.5, 0.5) \end{bmatrix}$$

We have

$$[Q_u] = \begin{bmatrix} 0.5 & 0.4 & 0.5 \\ 0.8 & 0.4 & 0.5 \end{bmatrix}, [Q_v] = \begin{bmatrix} 0.4 & 0.4 & 0.5 \\ 0.1 & 0.4 & 0.5 \end{bmatrix}.$$

Checking of $[Q_u]$*:*
For $i = 1$: *Start* $j = 1$.
$u_{11} = 0.5 = max_{k\in\{1,2,3\}}(u_{1k}); j = 2 \leq 3.$
$u_{12} = 0.4 \neq max_{k\in\{1,2,3\}}(u_{1k}).$
$max_{k\in\{1,2,3\}}(u_{k2}) = 0.4 = u_{12}; j = 3 \leq 3.$ $u_{13} = 0.5 = max_{k\in\{1,2,3\}}(u_{1k}).$
Conclusion: the first row satisfies the requirement.
For $i = 2$: *Start* $j = 1$.
$u_{21} = 0.8 = max_{k\in\{1,2,3\}}(u_{1k}); j = 2 \leq 3.$
$u_{22} = 0.4 \neq max_{k\in\{1,2,3\}}(u_{1k}).$
$max_{k\in\{1,2\}}(u_{k2}) = 0.4 = u_{22}; j = 3 \leq 3.$
$u_{23} = 0.5 = max_{k\in\{1,2,3\}}(u_{2k}); j = 4 > 3.$
Conclusion: the second row satisfies the requirement.
checking of $[Q_v]$*:*
For $i = 1$: *Start* $j = 1$.
$v_{11} = 0.4 \neq max_{k\in\{1,2,3\}}(v_{1k}).$
$max_{k\in\{1,2\}}(v_{k1}) = 0.4 = v_{11}; j = 2 \leq 3.$
$v_{12} = 0.4 \neq max_{k\in\{1,2,3\}}(v_{1k}).$

$max_{k\in\{1,2\}}(v_{k2}) = 0.4 = v_{12}; j = 3 \leq 3.$
$v_{13} = 0.5 = max_{k\in\{1,2,3\}}(v_{1k}).$
Conclusion: the first row satisfies the requirement.
For $i = 2$: *Start* $j = 1$.
$v_{21} = 0.1 = max_{k\in\{1,2,3\}}(v_{2k}); j = 2 \leq 3.$
$v_{22} = 0.4 = max_{k\in\{1,2,3\}}(v_{2k}); j = 3 \leq 3.$
$v_{23} = 0.4 = max_{k\in\{1,2,3\}}(v_{2k}).$
Conclusion: the second row satisfies the requirement.
The algorithm shows that that R is separable.

6 Conclusion

For many cases in approximate reasoning and application of decision making, knowing whether or not an IFR relating some variables (attributes) is separable was quite important. Therefore, a necessary and sufficient condition for separability of IFR was desirable. The algorithm for checking separability of IFR that we have suggested was a contribution to that end. The algorithm simplifies checking the separability and can be formulated as program by using some computer software to operate on input of elements of the matrix [R] only. For future work, we suggest using the concept of distance of IFR to generalize concept of separability.

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