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A note on almost subsemigroups of semigroups

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Abstract

In this paper, we introduce the concept of almost subsemigroups of semigroups and their fuzzifications. The basic properties of almost subsemigroups of semigroups and the relationship between almost subsemigroups and their fuzzifications are examined.

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1 Introduction and Preliminaries

Almost ideals of semigroups were first introduced and explored by Grosek and Satko [2]. After that, Bogdanovic [1] investigated almost bi-ideals of semigroups. At present, almost ideals of semigroups are extensively studied [3, 12]. Moreover, almost ideals were extended to examine in other algebraic structures; for instance, Γ-semigroups [8], ternary semigroups [9], semihypergroups [6, 10, 11] and LA-semihypergroups [7]. In 1965, Zadeh [14] introduced the concept of fuzzy sets. Later, fuzzy sets were applied to various fields of science and mathematics. One of those fields which we are interested in is the study of fuzzy almost ideals of semigroups [4, 12, 13]. A fuzzy subset of a set S is a membership function from S into the closed interval $[0, 1]$. Many interesting results in fuzzy semigroups can be seen in [5]. For any two fuzzy subsets f and g of S,

1. $f \cap g$ is a fuzzy subset of S defined by
$$(f \cap g)(x) = \min\{f(x), g(x)\} = f(x) \wedge g(x)$$
for all $x \in S$,

2. $f \cup g$ is a fuzzy subset of S defined by
$$(f \cup g)(x) = \max\{f(x), g(x)\} = f(x) \vee g(x)$$
for all $x \in S$ and

3. $f \subseteq g$ if $f(x) \leq g(x)$ for all $x \in S$.

For a fuzzy subset f of S, the support of f is defined by
$$supp(f) = \{x \in S \mid f(x) \neq 0\}.$$

The *characteristic mapping* of a subset A of S is a fuzzy subset of S defined by
$$C_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{otherwise.} \end{cases}$$

Let $F(S)$ be the set of all fuzzy subsets of a semigroup S. For each $f, g \in F(S)$, a fuzzy subset $f \circ g$ is called a *product* of f and g such that for any $x \in S$,
$$(f \circ g)(x) = \begin{cases} \sup\limits_{x=ab}\{\min\{f(a), g(b)\}\} & \text{if } x = ab \text{ for some } a, b \in S, \\ 0 & \text{otherwise.} \end{cases}$$

Then $F(S)$ is a semigroup with the product $\circ$.

There are many objectives of this research paper. We define the notion of almost subsemigroups of semigroups by using the concepts of almost ideals of semigroups. We also define the notion of fuzzy almost subsemigroups of semigroups. Moreover, we investigate the basic properties of almost subsemigroups. Furthermore, we explore the relationships between almost subsemigroups and fuzzy almost subsemigroups of semigroups.

2 Main results

2.1 Almost subsemigroups

In this section, we introduce the new definition namely an almost subsemigroup, and provide its basic properties.

Definition 2.1. A nonempty set A of a semigroup S is called an *almost subsemigroup* of S if $A^2 \cap A \neq \emptyset$.

Let A be any subsemigroup of a semigroup S. Then $A^2 \subseteq A$. This implies that $A^2 \cap A \neq \emptyset$ and hence A is an almost subsemigroup of S. Therefore, we conclude that every subsemigroup of a semigroup S is an almost subsemigroup of S.

Example 2.1. Consider a semigroup $(\mathbb{N}, \cdot)$. Let $A = \{2, 4\}$ and $B = \{4, 16\}$. It is clear that A and B are almost subsemigroups but are not subsemigroups of $\mathbb{N}$. Nevertheless, $A \cap B = \{4\}$ is not an almost subsemigroup of $\mathbb{N}$.

From Example 2.1, we obtain the following conclusions:

(1) Any almost subsemigroup of a semigroup S need not be a subsemigroup of S.

(2) The intersection of almost subsemigroups of a semigroup S need not be an almost subsemigroup of S.

Theorem 2.2. *Let A and B be nonempty subsets of a semigroup S such that $A \subseteq B$. If A is an almost subsemigroup of S, then B is also an almost subsemigroup of S.*

Proof. Assume that A is an almost subsemigroup of S. Hence $A^2 \cap A \neq \emptyset$ and so $A^2 \cap A \subseteq B^2 \cap B$ because $A \subseteq B$. This implies that $B^2 \cap B \neq \emptyset$. This completes the proof. $\square$

Corollary 2.3. *The union of almost subsemigroups of a semigroup S is also an almost subsemigroup of S.*

Proposition 2.4. *Let a be any element of a semigroup S.*

(1) If a is an idempotent, then $\{a\}$ is an almost subsemigroup of S.

(2) $\{a, a^2\}$ is an almost subsemigroup of S.

Theorem 2.5. *Every semigroup S such that $|S| > 1$ has a proper almost subsemigroup.*

Proof. Suppose that S has no a proper almost subsemigroup. Thus S has no an idempotent by Proposition 2.4(1). By Proposition 2.4(2), we obtain that S contains only two elements, say $S = \{a, b\}$. Since S has no an idempotent, $a^2 = b$ and $b^2 = a$. Consider $ab = aa^2 = a^2a = ba$. If $ab = a$, then $a = b^2 = a^2b = a^2$. Similarly, if $ab = b$, then $b = a^2 = ab^2 = b^2$. These are the contradictions. Therefore, S has a proper almost subsemigroup. $\square$

Example 2.2. Consider the semigroup $(\mathbb{N}, +)$. Let $a \in \mathbb{N}$. Define the sequence a_k by $a_1 = a$ and $a_k = 2a_{k-1} + 1$ for all $k > 1$. For example, if $a = 1$, then $\{a_k \mid k \in \mathbb{N}\} = \{1, 3, 7, 15, \ldots\}$. It is easy to see that $\{a_k \mid k \in \mathbb{N}\}$ is not an almost subsemigroup of $\mathbb{N}$. This shows that the semigroup $(\mathbb{N}, +)$ has an infinite number of infinite almost subsemigroups.

2.2 Fuzzy almost subsemigroups

In this section, we introduce the notion of fuzzy almost subsemigroups and give their basic properties. In addition, the relationships between almost subsemigroups and their fuzzifications are provided.

Definition 2.6. A fuzzy subset f of a semigroup S is called an *fuzzy almost subsemigroup* of S if $(f \circ f) \cap f \neq 0$.

Let f be any nonzero fuzzy subsemigroup of a semigroup S. Then $f \circ f \subseteq f$ (see Lemma 2.3.2, [5]). Hence $(f \circ f) \cap f \neq 0$. This implies that f is a fuzzy almost subsemigroup of S. We conclude that every nonzero fuzzy subsemigroup of a semigroup S is a fuzzy almost subsemigroup of S.

Example 2.3. Consider the semigroup $(\mathbb{N}, \cdot)$. Let f and g be fuzzy subsets of S as follows:

$$f(x) = \begin{cases} 0.5 & \text{if } x = 1, 2, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$g(x) = \begin{cases} 0.4 & \text{if } x = 2, 4, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that f and g are fuzzy almost subsemigroups but are not fuzzy subsemigroups of $\mathbb{N}$. However, $f \cap g$ is not a fuzzy almost subsemigroup of $\mathbb{N}$.

From Example 2.3, we have the following conclusions:

(1) Any fuzzy almost subsemigroup of a semigroup S need not be a fuzzy subsemigroup of S.

(2) The intersection of fuzzy almost subsemigroups of a semigroup S need not be a fuzzy almost subsemigroup of S.

Theorem 2.7. *Let f and g be fuzzy subsets of a semigroup S such that $f \subseteq g$. If f is a fuzzy almost subsemigroup of S, then g is also a fuzzy almost subsemigroup of S.*

Proof. Assume that f is a fuzzy almost subsemigroup of S. So $(f \circ f) \cap f \neq 0$ and hence $(f \circ f) \cap f \subseteq (g \circ g) \cap g$ because $f \subseteq g$. Thus $(g \circ g) \cap g \neq 0$. □

Corollary 2.8. *The union of fuzzy almost subsemigroups of a semigroup S is also a fuzzy almost subsemigroup of S.*

At the end of this section, we deal with the relationship between almost subsemigroups and fuzzy almost subsemigroups of semigroups.

Theorem 2.9. *Let A be a nonempty subset of a semigroup S. Then A is an almost subsemigroup of S if and only if C_A is a fuzzy almost subsemigroup of S.*

Proof. Suppose that A is an almost subsemigroup of S. Then $A^2 \cap A \neq \emptyset$, hence there exists $x \in A^2 \cap A$. So $x = yz$ for some $y, z \in A$ and $x \in A$. We obtain $[C_A \circ C_A](x) = 1$ and $C_A(x) = 1$. Thus $[(C_A \circ C_A) \cap C_A](x) = 1 \neq 0$. This implies that $(C_A \circ C_A) \cap C_A \neq 0$. Therefore, C_A is a fuzzy almost subsemigroup of S.

Conversely, assume that C_A is a fuzzy almost subsemigroup of S. We have $(C_A \circ C_A) \cap C_A \neq 0$. Then there exists $x \in S$ such that $[(C_A \circ C_A) \cap C_A](x) \neq 0$. This implies that $[C_A \circ C_A](x) \neq 0$ and $C_A(x) \neq 0$. It follows that $[C_A \circ C_A](x) = 1$ and $C_A(x) = 1$. Hence $x \in A^2$ and $x \in A$. Eventually, $A^2 \cap A \neq \emptyset$. Consequently, A is an almost subsemigroup of S. □

Theorem 2.10. *Let f be a fuzzy subset of a semigroup S. Then f is a fuzzy almost subsemigroup of S if and only if $supp(f)$ is an almost subsemigroup of S.*

Proof. Assume that f is a fuzzy almost subsemigroup of S. So $(f \circ f) \cap f \neq 0$. Thus there exists $x \in S$ such that $[(f \circ f) \cap f](x) \neq 0$. It follows that $f(x) \neq 0$ and $[f \circ f](x) \neq 0$. We obtain $x = yz$ for some $y, z \in S$ such that $f(y) \neq 0$ and $f(z) \neq 0$. This implies that $y, z \in supp(f)$. Thus, $[C_{supp(f)} \circ C_{supp(f)}](x) \neq 0$ and $C_{supp(f)}(x) \neq 0$. Hence, $[(C_{supp(f)} \circ C_{supp(f)}) \cap C_{supp(f)}](x) \neq 0$. We have $C_{supp(f)}$ is a fuzzy almost subsemigroup of S, and then $supp(f)$ is an almost subsemigroup of S by Theorem 2.9.

Conversely, suppose that $supp(f)$ is an almost subsemigroup of S. We obtain $C_{supp(f)}$ is a fuzzy almost subsemigroup of S by Theorem 2.9. Hence $[(C_{supp(f)} \circ C_{supp(f)}) \cap C_{supp(f)}] \neq 0$. Then there exists $x \in S$ such that $[(C_{supp(f)} \circ C_{supp(f)}) \cap C_{supp(f)}](x) \neq 0$, and hence $[C_{supp(f)} \circ C_{supp(f)}](x) \neq 0$ and $C_{supp(f)}(x) \neq 0$. Then there exist $y, z \in supp(f)$ such that $x = yz$. It follows that $f(y) \neq 0$ and $f(z) \neq 0$. Therefore, $[f \circ f](x) \neq 0$, and so $[(f \circ f) \cap f](x) \neq 0$. Consequently, f is a fuzzy almost subsemigroup of S. □

3 Conclusions

In this paper, we have introduced the concepts of almost subsemigroups and fuzzy almost subsemigroups of semigroups. From our study, we have found that an almost subsemigroup and its characteristic mapping are equivalent. Moreover, a fuzzy almost subsemigroup and its support are also equivalent.

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