International Journal of Mathematics and
Computer Science, **16**(2021), no. 4, 1295–1302

Certain results on a class of functions with negative coefficients

Matthew O. Oluwayemi[1,3], **Joshua O. Okoro**[2,3]

[1]Landmark University SDG 4 (Quality Education Research Group)
Landmark University
Omu-Aran, Nigeria

[2]Landmark University SDG 13 (Climate Action Research Group)
Landmark University
Omu-Aran, Nigeria

[3]Department of Mathematics
Landmark University
P.M.B. 1001
Omu-Aran, Nigeria

email: oluwayemimatthew@gmail.com, okoro.joshua@lmu.edu.ng

(Received December 21, 2020, Accepted March 10, 2021)

Abstract

We study a family of univalent functions defined as $TK^{m}_{\alpha,\lambda,\gamma,\beta}(n,\delta,e_m)$ with fixed finitely many negative coefficients of the form:

$$f(z) = z - \sum_{m=2}^{j} \frac{(1-\delta)e_m}{(m-\delta)\left[\left(\frac{\alpha+\lambda}{\gamma+\beta}\right)(m-1)+1\right]^n} z^m - \sum_{k=j+1}^{\infty} a_k z^k.$$

Moreover, we investigate some properties of this family.

Key words and phrases: Fractional calculus, weighted average, radii of starlikeness and convexity.
AMS (MOS) Subject Classifications: 30C45, 30C50.
ISSN 1814-0432, 2021, http://ijmcs.future-in-tech.net

1 Introduction

Let A denote the class of functions of the form

$$f(z) = z + \sum_{k=t+1}^{\infty} a_k z^k, t \in \mathbb{N} \tag{1.1}$$

which are analytic and univalent in the unit disk $U = \{z : |z| \leq 1\}$.
We denote by T the subclass of A consisting of functions $f(z)$ which are analytic and univalent in the unit disk U and which are of the form

$$f(z) = z - \sum_{k=t+1}^{\infty} a_k z^k; \quad a_k \geq 0. \tag{1.2}$$

Using a new differential operator $D^m_{\alpha,\lambda,\gamma,\beta}$, Alamri and Darus [1] introduced a certain class of univalent functions with negative coefficients and geometric properties of the class of function were established. Motivated by [1] and [8], we study a certain univalent class with fixed finitely many negative coefficients associated with the class $TK^m_{\alpha,\lambda,\gamma,\beta}$. Moreover, we investigate some additional properties of $f \in TK^m_{\alpha,\lambda,\gamma,\beta}$. Furthermore, we investigate some geometric properties of some univalent functions in the unit disk.
The following Lemma is important to establish our results.

Lemma 1.1 [1]
Let $f(z) = z - \sum_{k=2}^{\infty} a_k z^k$. Then $f \in TK^m_{\alpha,\lambda,\gamma,\beta}$ if and only if

$$\sum_{k=n+1}^{\infty} (k-\delta) \left[\left(\frac{\alpha+\lambda}{\gamma+\beta} \right) (k-1) + 1 \right]^m |a_k| \leq 1 - \delta,$$

where $\alpha, \lambda, \gamma, \beta \geq 0$, $0 \leq \delta \leq 1$, $n \in N$, $m \in N_0$ and $\gamma + \beta \neq 0$. The function is sharp for

$$f(z) = z - \frac{1-\delta}{(k-\delta)\left[\left(\frac{\alpha+\lambda}{\gamma+\beta} \right) (k-1) + 1 \right]^m} z^n.$$

Now we introduce the class $TK^m_{\alpha,\lambda,\gamma,\beta}(n, \delta, e_m)$ as a subclass of $TK^m_{\alpha,\lambda,\gamma,\beta}(n, \delta)$ with fixed finitely many negative coefficients of the form:

$$f(z) = z - \sum_{m=2}^{j} \frac{(1-\delta)e_m}{(m-\delta)\left[\left(\frac{\alpha+\lambda}{\gamma+\beta} \right) (m-1) + 1 \right]^n} z^m - \sum_{k=j+1}^{\infty} a_k z^k, \tag{1.3}$$

where $e_m = \frac{(m-\delta)\left[\left(\frac{\alpha+\lambda}{\gamma+\beta}\right)(m-1)+1\right]^n}{(1-\delta)} a_m$.
Many authors such as [6], [2], [5], [8] and [9] studied various cases of functions with finitely many fixed negative coefficients. Other authors such as [4], [3], [7], [10] and [11] also investigated geometric properties of certain subclasses univalent functions. This study however considers two classes of univalent functions with negative coefficients as $TK^m_{\alpha,\lambda,\gamma,\beta}(n,\delta,e_m)$ and $TK^m_{\alpha,\lambda,\gamma,\beta}(n,\delta)$ earlier defined in [1]. Furthermore, we pointed out that
$TK^m_{\alpha,\lambda,\gamma,\beta}(n,\delta,e_m) \subset TK^m_{\alpha,\lambda,\gamma,\beta}(n,\delta)$.

2 Main results

Theorem 2.1. *Let $f(z)$ be defined by* (1.2). *Then,* $f(z) \in TK^m_{\alpha,\lambda,\gamma,\beta}(n,\delta,e_m)$ *if and only if* $\sum_{k=j+1}^{\infty} \frac{(k-\delta)\left[\left(\frac{\alpha+\lambda}{\gamma+\beta}\right)(k-1)+1\right]^n}{(1-\delta)} a_k < 1 - \sum_{m=2}^{j+1} e_m$.

Proof. Suppose

$$a_m = \frac{(1-\delta)e_m}{(m-\delta)\left[\left(\frac{\alpha+\lambda}{\gamma+\beta}\right)(m-1)+1\right]^n}. \tag{2.4}$$

Then, $TK^m_{\alpha,\lambda,\gamma,\beta}(n,\delta,e_m) \subset TK^m_{\alpha,\lambda,\gamma,\beta}(n,\delta)$. if and only if

$$\sum_{m=2}^{j+1} \frac{(m-\delta)\left[\left(\frac{\alpha+\lambda}{\gamma+\beta}\right)(m-1)+1\right]^n}{(1-\delta)} a_m + \sum_{k=j+1}^{\infty} \frac{(k-\delta)\left[\left(\frac{\alpha+\lambda}{\gamma+\beta}\right)(k-1)+1\right]^n}{(1-\delta)} a_k < 1$$

which implies $\sum_{m=2}^{j+1} e_m + \sum_{k=j+1}^{\infty} \frac{(k-\delta)\left[\left(\frac{\alpha+\lambda}{\gamma+\beta}\right)(k-1)+1\right]^n}{(1-\delta)} a_k < 1$.
Thus, $\sum_{k=j+1}^{\infty} \frac{(k-\delta)\left[\left(\frac{\alpha+\lambda}{\gamma+\beta}\right)(k-1)+1\right]^n}{(1-\delta)} a_k < 1 - \sum_{m=2}^{j+1} e_m$.

Corollary 2.2. *Consider the function $f(z)$ defined by* (1.3) *and belonging to the class* $TK^m_{\alpha,\lambda,\gamma,\beta}(n,\delta,e_m)$. *Then,*

$$\sum_{k=j+1}^{\infty} \frac{(k-\delta)\left[\left(\frac{\alpha+\lambda}{\gamma+\beta}\right)(k-1)+1\right]^n}{(1-\delta)} a_k < 1 - \sum_{m=2}^{j+1} e_m$$

$$a_k \le \frac{[1-\delta]\left(1-\sum_{m=2}^{j+1} e_m\right)}{(k-\delta)\left[\left(\frac{\alpha+\lambda}{\gamma+\beta}\right)(k-1)+1\right]^n}. \tag{2.5}$$

Equality holds and is sharp for the function

$$f(z) = z - \sum_{m=2}^{j+1} \frac{(1-\delta)e_m}{(m-\delta)\left[\left(\frac{\alpha+\lambda}{\gamma+\beta}\right)(m-1)+1\right]^n} z^m + \frac{[1-\delta]\left(1-\sum_{m=2}^{j+1} e_m\right)}{(k-\delta)\left[\left(\frac{\alpha+\lambda}{\gamma+\beta}\right)(k-1)+1\right]^n} z^k. \tag{2.6}$$

Corollary 2.3. *Consider the function $f(z)$ defined by (1.3) and in the class $TK^m_{\alpha,\lambda,\gamma,\beta}(n, 0, e_m)$ Then,*

$$\sum_{k=j+1}^{\infty} k\left[\left(\frac{\alpha+\lambda}{\gamma+\beta}\right)(k-1)+1\right]^n a_k < 1 - \sum_{m=2}^{j+1} e_m$$

$$a_k \leq \frac{1-\sum_{m=2}^{j+1} e_m}{k\left[\left(\frac{\alpha+\lambda}{\gamma+\beta}\right)(k-1)+1\right]^n}.$$

The result is sharp with the extremal functions as

$$f(z) = z - \sum_{m=2}^{j+1} \frac{e_m}{m\left[\left(\frac{\alpha+\lambda}{\gamma+\beta}\right)(m-1)+1\right]^n} z^m + \frac{1-\sum_{m=2}^{j+1} e_m}{k\left[\left(\frac{\alpha+\lambda}{\gamma+\beta}\right)(k-1)+1\right]^n} z^k.$$

Corollary 2.4. *Consider the function $f(z)$ defined by (1.3) and in the class $TK^m_{0,\lambda,0,1}(n, \delta, e_m)$. Then,*

$$\sum_{k=j+1}^{\infty} \frac{(k-\delta)\left[\left(\frac{\alpha+\lambda}{\gamma+\beta}\right)(k-1)+1\right]^n}{(1-\delta)} a_k < 1 - \sum_{m=2}^{j+1} e_m$$

$$a_k \leq \frac{[1-\delta]\left(1-\sum_{m=2}^{j+1} e_m\right)}{(k-\delta)\left[\lambda(k-1)+1\right]^n}.$$

The result is sharp with the extremal functions as

$$f(z) = z - \sum_{m=2}^{j+1} \frac{(1-\delta)e_m}{(m-\delta)\left[\lambda(m-1)+1\right]^n} z^m + \frac{[1-\delta]\left(1-\sum_{m=2}^{j+1} e_m\right)}{(k-\delta)\left[\lambda(k-1)+1\right]^n} z^k.$$

Corollary 2.5. *Consider the function $f(z)$ defined by (1.3) and in the class $TK^m_{1,1,\gamma,\beta}(n, \delta, e_m)$. Then,*

$$\sum_{k=j+1}^{\infty} \frac{(k-\delta)\left[\left(\frac{2}{\gamma+\beta}\right)(k-1)+1\right]^n}{(1-\delta)} a_k < 1 - \sum_{m=2}^{j+1} e_m.$$

$$a_k \leq \frac{[1-\delta]\left(1-\sum_{m=2}^{j+1} e_m\right)}{(k-\delta)\left[\left(\frac{2}{\gamma+\beta}\right)(k-1)+1\right]^n}.$$

The result is sharp with the extremal functions as

$$f(z) = z - \sum_{m=2}^{j+1} \frac{(1-\delta)e_m}{(m-\delta)\left[\left(\frac{2}{\gamma+\beta}\right)(m-1)+1\right]^n} z^m + \frac{[1-\delta]\left(1-\sum_{m=2}^{j+1} e_m\right)}{(k-\delta)\left[\left(\frac{2}{\gamma+\beta}\right)(k-1)+1\right]^n} z^k.$$

Corollary 2.6. *Consider the function* $f(z)$ *defined by* (1.3) *and in the class* $TK^m_{\alpha,\lambda,1,1}(n,\delta,e_m)$. *Then,*

$$\sum_{k=j+1}^{\infty} \frac{(k-\delta)\left[\left(\frac{\alpha+\lambda}{2}\right)(k-1)+1\right]^n}{(1-\delta)} a_k < 1 - \sum_{m=2}^{j+1} e_m$$

which implies

$$a_k \leq \frac{[1-\delta]\left(1-\sum_{m=2}^{j+1} e_m\right)}{(k-\delta)\left[\left(\frac{\alpha+\lambda}{2}\right)(k-1)+1\right]^n}.$$

The result is sharp with the extremal functions as

$$f(z) = z - \sum_{m=2}^{j+1} \frac{(1-\delta)e_m}{(m-\delta)\left[\left(\frac{\alpha+\lambda}{2}\right)(m-1)+1\right]^n} z^m + \frac{[1-\delta]\left(1-\sum_{m=2}^{j+1} e_m\right)}{(k-\delta)\left[\left(\frac{\alpha+\lambda}{2}\right)(k-1)+1\right]^n} z^k. \quad (2.7)$$

Corollary 2.7. *Consider the function* $f(z)$ *defined by* (1.3) *and in the class* $TK^m_{1,1,1,1}(n,\delta,e_m)$. *Then,*

$$\sum_{k=j+1}^{\infty} \frac{(k-\delta)\left[(k-1)+1\right]^n}{(1-\delta)} a_k < 1 - \sum_{m=2}^{j+1} e_m$$

so that

$$a_k \leq \frac{[1-\delta]\left(1-\sum_{m=2}^{j+1} e_m\right)}{(k-\delta)\left[(k-1)+1\right]^n}.$$

The result is sharp for the function

$$f(z) = z - \sum_{m=2}^{j+1} \frac{(1-\delta)e_m}{(m-\delta)\left[(m-1)+1\right]^n} z^m + \frac{[1-\delta]\left(1-\sum_{m=2}^{j+1} e_m\right)}{(k-\delta)\left[(k-1)+1\right]^n} z^k.$$

Corollary 2.8. *Consider the function $f(z)$ defined by* (1.3) *and is in the class $TK^m_{1,1,1,1}(1, 0, e_m)$. Then, $\sum_{k=j+1}^{\infty} k^2 a_k < 1 - \sum_{m=2}^{j+1} e_m$ which implies*

$$a_k \leq \frac{1}{k^2}\left(1 - \sum_{m=2}^{j+1} e_m\right).$$

The result is sharp for the function

$$f(z) = z - \sum_{m=2}^{j+1} \frac{e_m}{m^2} z^m + \frac{1}{k^2}\left(1 - \sum_{m=2}^{j+1} e_m\right) z^k.$$

Theorem 2.9. *Let $f_i(z)$ $(i = 1, 2, 3, \dots q)$ defined by*

$$f_i(z) = z - \sum_{m=2}^{j} \frac{(1-\delta)e_m}{(m-\delta)\left[\left(\frac{\alpha+\lambda}{\gamma+\beta}\right)(m-1)+1\right]^n} z^m - \sum_{k=j+1}^{\infty} a_{k,i} z^k$$

be in class $TK^m_{\alpha,\lambda,\gamma,\beta}(n, \delta, e_m.)$ Then

$$F(z) = z - \sum_{m=2}^{j} \frac{(1-\delta)e_m}{(m-\delta)\left[\left(\frac{\alpha+\lambda}{\gamma+\beta}\right)(m-1)+1\right]^n} z^m - \sum_{k=j+1}^{\infty} d_k z^k$$

is also in class $TK^m_{\alpha,\lambda,\gamma,\beta}(n, \delta, e_m)$, where $d_k = \frac{1}{i}\sum_{i=1}^{q}$.

Proof. $\sum_{k=t+1}^{\infty} \frac{(k-\delta)\left[\left(\frac{\alpha+\lambda}{\gamma+\beta}\right)(k-1)+1\right]^n}{(1-\delta)} d_k = \sum_{k=t+1}^{\infty} \frac{(k-\delta)\left[\left(\frac{\alpha+\lambda}{\gamma+\beta}\right)(k-1)+1\right]^n}{i(1-\delta)} \left(\sum_{i=1}^{q} a_{i,k}\right)$

$$\frac{1}{l}\sum_{i=1}^{q}\left[\sum_{k=j+1}^{\infty} \frac{(k-\delta)\left[\left(\frac{\alpha+\lambda}{\gamma+\beta}\right)(k-1)+1\right]^n}{(1-\delta)} a_{k,i}\right] = \frac{1}{l}\sum_{i=1}^{q}\left(1 - \sum_{m=2}^{t} e_m\right) = 1 - \sum_{m=2}^{t} e_m$$

which completes the proof. Hence, the class $TK^m_{\alpha,\lambda,\gamma,\beta}(n, \delta, e_m)$ is closed under arithmetic mean. We also consider the radii properties for the class $TK^m_{\alpha,\lambda,\gamma,\beta}(n, \delta, e_m)$.

Theorem 2.10. *Let the function $f(z)$ be defined by* (1.2) *and is in the class $TK^m_{\alpha,\lambda,\gamma,\beta}(n, \delta)$.*
Then $f(z)$ is starlike of order σ $(0 \leq \sigma < 1)$ in $|z| < r_1$, where r_1 is $\inf_{k\geq 2}\left\{\frac{(1-\sigma)k\gamma(s)\{(3\pi\mu-\beta\pi)[1+(k-1)(\omega-\lambda)\alpha]-\beta\xi\}[1+(k-1)(\omega-\lambda)\alpha]^n}{(k-\sigma)[3\gamma\pi\mu-\gamma(1-\xi)(\pi\beta-1)]}\right\}^{\frac{1}{k-1}}$.

Proof. We need to show that $\left|\frac{zf'(z)}{f(z)} - 1\right| < 1-\sigma,\ |z| < r_1,$
so that

$$\left|\frac{zf'(z)}{f(z)} - 1\right| = \left|\frac{-\sum_{k=2}^{\infty}(k-1)a_k z^{k-1}}{1-\sum_{k=2}^{\infty} a_k z^{k-1}}\right| \leq \frac{\sum_{k=2}^{\infty}(k-1)a_k |z|^{k-1}}{(1-\sum_{k=2}^{\infty} a_k |z|^{k-1})} < 1-\sigma;$$

that is, $\left|\frac{zf'(z)}{f(z)} - 1\right| \leq 1-\sigma$ if

$$\left(\frac{k-\sigma}{1-\sigma}\right) a_k |z|^{k-1} \leq 1. \tag{2.8}$$

From Theorem 2.1, it follows that

$$|z| \leq \left\{\frac{(1-\sigma)(k-\delta)\left[\left(\frac{\alpha+\lambda}{\gamma+\beta}\right)(k-1)+1\right]^n}{(k-\sigma)(1-\delta)\left(1-\sum_{m=2}^{j+1} e_m\right)}\right\}^{\frac{1}{k-1}} ;\ |z| < r_1.$$

Hence,

$$r_1 = inf_k \left\{\frac{(1-\sigma)(k-\delta)\left[\left(\frac{\alpha+\lambda}{\gamma+\beta}\right)(k-1)+1\right]^n}{(k-\sigma)(1-\delta)\left(1-\sum_{m=2}^{j+1} e_m\right)}\right\}^{\frac{1}{k-1}} ,k \geq 2.$$

The equality holds for

$$f(z) = z - \frac{(k-\delta)(1-\sigma)\left[\left(\frac{\alpha+\lambda}{\gamma+\beta}\right)(k-1)+1\right]^n}{(1-\delta)(k-\sigma)\left(1-\sum_{m=2}^{j+1} e_m\right)} z^k, \quad k \geq 2.$$

Acknowledgment. The authors would like to acknowledge the support of Landmark University, Nigeria and the authors whose work motivated and contributed to the emergence of this research.

References

[1] M. Alamri, M. Darus, On univalent function with negative coefficients by using differential operator, *American Institute of Physics Conference Proceedings*, (2014), 1614, 871–883, Doi: 10.1063/1.4895316.

[2] A. R. S. Juma, S. R. Kulkarni, Applications of Generalized Ruscheweyh derivatives to univalent functions with finitely many coefficients, *Surveys in Mathematics and its Applications*, **4**, (2009), 77–88.

[3] O. A. Fadipe-Joseph, M. O. Oluwayemi, Sufficient conditions for class C_α, *Int. J. Math. Comput. Sci.*, **16,** no. 2, (2021), 515–524.

[4] O. A. Fadipe-Joseph, B. O. Moses, M. O. Oluwayemi, Certain New Classes of analytic functions defined by using sigmoid function, *Advances in Mathematics: Scientific Journal*, **5**, (2016), 83—89.

[5] Sh. Najafzadeh, Application of Sălăgean and Ruscheweyh operators on univalent functions with finitely many coefficients, *Fractional Calculus and Applied Analysis*, **13**, no. 5, (2010), 1–5.

[6] Sh. Najafzadeh, Univalent functions with fixed finitely many coefficients involving Sălăgean operator, Int. J. Nonlinear Anal. Appl., **1,** no. 1, (2010), 1–5.

[7] M. O. Oluwayemi, O. A. Fadipe-Joseph, New subclasses of univalent functions defined by using a linear combination of a generalized Sălăgean and Ruscheweyh operators, *Int. J. Math. Anal. Optim.*, (2017), 187–200.

[8] M. O. Oluwayemi, O. A. Fadipe-Joseph, Sh. Najafzadeh, On a Subclass of Analytic functions with fixed finitely many coefficients based on Sălăgean operator and modified sigmoid function, *Advances in Mathematics*, **10,** no. 6, (2021), 2807–2820, DOI: https://doi.org/10.37418/amsj.10.6

[9] M. O. Oluwayemi, O. A. Fadipe-Joseph, A New Class of Functions with Finitely many Fixed Points, *Abstract and Applied Analysis,* (2021), to appear.

[10] M. O. Oluwayemi, Imran Faisal, A new family of analytic functions associated with multiplier transformation, *Scientific African*, **12,** (2021), 1–9, e00754.

[11] M. O. Oluwayemi, A. Alb Lupas, K. O. Dopamu, On a family of univalent functions associated with multiplier transformations, *International Journal of Mechanical Engineering and Technology*, **9,** no. 12, (2018), 425–438.