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Rough Set Theory Applied to JU-algebras

Moin Akhtar Ansari

Department of Mathematics
College of Science
Jazan University
Jazan, Saudi Arabia

email: maansari@jazanu.edu.sa, moinakhtar83@gmail.com

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Abstract

This paper is based on the concept of roughness in JU-algebras and related terminologies. In this paper, we study lower and upper approximations of JU-subalgebras and JU-ideals. Moreover, we study rough sets and JU-algebras with their weak and strong ideals and prove that the lower/upper approximation of JU-subalgebra (resp., JU-ideals) is a JU-subalgebra (resp., JU-ideals). Furthermore, we show some related results.

1 Introduction

Pawlak [16] introduced the concept of rough set considering it a tool for dealing with granularity in knowledge. Later on, this notion emerged as another major power full mathematical approach for managing and handling different types of uncertainty in information systems that arise from inexact, noisy, or incomplete information. It turning out to be methodologically significant to the domains of artificial intelligence and cognitive sciences, especially in the representation of and reasoning with vague and/or imprecise knowledge, data classification, data analysis, machine learning, pattern recognition and knowledge discovery in connection with algebraic structures.

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Ameri et al. [1] applied rough set theory to hyper BCK-algebra. Dudek et al. [6] and Ma [11] applied rough set theory to BCI-algebras. Jun et al. considered roughness in BCK-algebra [7], lattice implication algebras [8] and BCC-algebra [9]. Mao and Zhou [12] studied the rough set theory in Pseudo-BCK-algebra. Torkzadeh and Ghorbani [17] studied rough filters in B-Algebras. As a work in computer science, Chen et al. [2] worked on data mining framework based on rough set theory to improve location selection decisions as a case study of a restaurant chain whereas Karimi [10] studied rough sets and gray sets.

Rough set theory applied to UP-algebras has been studied by Moin et al. [14]. Recently, Ansari et al. [15] introduced JU-algebras with their p-closure ideals. Moreover, Usman et al. [18] introduced Pseudo-valuations and pseudo-metric on JU-algebras. Furthermore, Ansari [13] studied roughness applied to generalized Γ-ideals of ordered LA Γ-Semigroups. Daniel ([3], [5]) studied comparative and Implicative filters of JU-algebras. In addition, Daniel [4] discovered some new results on JU-algerbas. In this section, we shall define JU-algebras, JU-subalgebras, JU-ideals and give some examples based on them.

Definition 1.1. *[15] An algebra $(X, \diamond, 1)$ of type $(2, 0)$ with a single binary operation $\diamond$ is said to be JU-algebra satisfying the following identities, for any $x, y, z \in X$,*
(JU_1) $(y \diamond z) \diamond [(z \diamond x) \diamond (y \diamond x)] = 1$,
(JU_2) $1 \diamond x = x$,
(JU_3) $x \diamond y = y \diamond x = 1$ *implies* $x = y$.

We call the constant 1 of X the fixed element of X. For convenience, we write X instead of $(X, \diamond, 1)$ to represent a JU-algebra. We define a relation " $\leq$ " in X by $y \leq x$ if and only if $x \diamond y = 1$.

Lemma 1.2. *[15] If X is a JU-algebra, then $(X, \leq)$ is a partial ordered set; i.e.,*
(J_4) $x \leq x$,
(J_5) $x \leq y, y \leq x$, *imply* $x = y$,
(J_6) $x \leq z, z \leq y$, *imply* $x \leq y$.

Example 1.3. *Let $X = \{0, a, b, c, d\}$ in which $\diamond$ is defined by the following* *table*

$\diamond$	0	a	b	c	d
0	0	a	b	c	d
a	0	0	b	c	d
b	0	a	0	c	c
c	0	0	b	0	b
d	0	0	0	0	0

It is easy to see that X is JU-algebra.

Definition 1.4. *[15] A subset S of JU-algebra X is called a JU-subalgebra of X if $x \diamond y \in S$, whenever $x, y \in S$.*

Definition 1.5. *[15] A non-empty subset A of a JU-algebra X is called a JU-ideal of X if it satisfies the following conditions:*
(1) $0 \in A$,
(2) $x \diamond (y \diamond z) \in A$, $y \in A$ *implies* $x \diamond z \in A$, *for all* $x, y, z \in X$.

Example 1.6. *Let $X = \{0, a, b, c, d, e\}$ and $\diamond$ is defined by the following table:*

$\diamond$	0	a	b	c	d	e
0	0	a	b	c	d	e
a	0	0	b	b	d	e
b	0	0	0	a	d	e
c	0	0	0	0	d	e
d	0	0	0	a	0	e
e	0	0	0	0	0	0

Clearly, $(X, \diamond, 0)$ is a JU-algebra. It is easy to show that $A = \{0, a\}$ and $B = \{0, a, b, c, d\}$ are JU-ideals of X.

Definition 1.7. *Let A be a nonempty subset of a JU-algebra X and $0 \in A$. Then,*
(1) *A is called a weak JU-ideal of X if $y \diamond x \in A$ and $y \in A$ imply that $x \in A$, for all $x, y \in X$;*
(2) *A is called a strong JU-ideal of X if $(y \diamond x) \cap A \neq \emptyset$ and $y \in A$ imply that $x \in A$, for all $x, y \in X$.*

2 Roughness in JU-algebras

Let J be a set, E an equivalence relation on J and $P(J)$ the power set of J. For all $x \in J$, let $[x]_E$ denote the equivalence class of x with respect to E.

Define the functions $\underline{E}, \overline{E} : P(J) \to P(J)$ as follows, $\forall\ S \in P(J)$,

$$\underline{E}(S) = \{x \in J : [x]_E \subseteq S\}$$

and

$$\overline{E}(S) = \{x \in J : [x]_E \cap S \neq \emptyset\}.$$

The pair (J, E) is called an approximation space. Let S be a subset of J. Then S is said to be definable if $\underline{E}(S) = \overline{E}(S)$ and rough otherwise. $\underline{E}(S)$ is called the lower approximation of S while $\overline{E}(S)$ is called the upper approximation.

Throughout this section, X is a JU-algebra. Let A be a JU-ideal of X. Define a relation ρ on X by $(x, y) \in \rho$ if and only if $x \diamond y \in A$ and $y \diamond x \in A$. Then ρ is an equivalence relation on X related to a JU-ideal A of X. Moreover, $(x, y) \in \rho$ and $(u, v) \in \rho$ imply $(x \diamond u, y \diamond v) \in \rho$.

Hence ρ is a congruence relation on X. Let A_x denote the equivalence class of x with respect to the equivalence relation ρ related to a JU-ideal A of X, and let X/A denote the collection of all equivalence classes; that is, $X/A = \{A_x : x \in X\}$. Then $A_0 = A$. If $A_x \diamond A_y$ is defined as the class containing $x \diamond y$ (that is, $A_x \diamond A_y = A_{x \diamond y}$), then $(X/A, \diamond, A_0)$ is a JU-algebra. Let ρ be an equivalence relation on X related to a JU-ideal A of X. For any nonempty subset S of X, the lower and upper approximation of S are denoted by $\underline{\rho}(A, S)$ and $\overline{\rho}(A, S)$ respectively; that is,

$$\underline{\rho}(A, S) = \{x \in X : A_x \subseteq S\}$$

and

$$\overline{\rho}(A, S) = \{x \in x : A_x \cap S \neq \emptyset\}.$$

If $A = S$, then $\underline{\rho}(A, S)$ and $\overline{\rho}(A, S)$ are denoted by $\underline{\rho}(A)$ and $\overline{\rho}(A)$, respectively.

Definition 2.1. *[16] Given an approximation space (U, ρ), a pair $(A, B) \in P(U) \times P(U)$ is called a rough set in (U, ρ) if and only if $(A, B) = Apr(X)$ for some $X \in P(U)$.*

Definition 2.2. *[16] Let (U, ρ) be an approximation space and let X be a non-empty subset of U.*
(i) If $\underline{Apr}(X) = \overline{Apr}(X)$, then X is called definable.
(ii) If $\underline{Apr}(X) = \emptyset$, then X is called empty interior.
(iii) If $\overline{Apr}(X) = U$, then X is called empty exterior.

Example 2.3. *Let $X = \{0, a, b, c, d\}$ be a JU-algebra with the Cayley's table as follows*

$\diamond$	0	a	b	c	d
0	0	a	b	c	d
a	0	0	0	0	a
b	0	c	0	c	d
c	0	a	b	0	a
d	0	0	0	0	0

Let $A = \{0, a\}$ be a JU-ideal of X ($A \lhd X$) and let ρ be an equivalence relation on X related to A. Then $A_0 = A_a = A$, $A_b = \{b\}$, $A_c = \{c\}$, and $A_d = \{d\}$. Hence,

$$\begin{array}{rcl} \underline{\rho}(A, \{0, a\}) &=& \{0, a\} \lhd X \\ \underline{\rho}(A, \{0, b\}) &=& \{b\} \\ \underline{\rho}(A, \{0, c\}) &=& \{c\} \\ \underline{\rho}(A, \{0, a, b, c\}) &=& \{0, a, b, c\} \lhd X \end{array}$$

and

$$\begin{array}{rcl} \overline{\rho}(A, \{0, a\}) &=& \{0, a\} \lhd X \\ \overline{\rho}(A, \{0\}) &=& \{0, a\} \\ \overline{\rho}(A, \{b\}) &=& \{0, b\} \\ \overline{\rho}(A, \{a, b, c\}) &=& \{0, a, b, c\} \lhd X \\ \overline{\rho}(A, \{0, b, c\}) &=& \{0, a, b, c\} \lhd X \\ \overline{\rho}(A, \{a, b, c, d\}) &=& \{0, a, b, c, d\} \lhd X. \end{array}$$

In Example 2.3, we know that there exists a non-JU-ideal S of X such that their lower and upper approximation are JU-ideals of X. Also we choose some non-JU-ideals S of X such that their lower and upper approximation are JU-ideals of X.

Proposition 2.4. *Let ρ and φ be equivalence relations on X related to JU-ideals A and B of X, respectively. If S and T are nonempty subsets of X, then*
(1) $\underline{\rho}(A, S) \subseteq S \subseteq \overline{\rho}(A, S)$;
(2) $\underline{\rho}(A, \emptyset) = \emptyset = \overline{\rho}(A, \emptyset)$
(3) $\overline{\rho}(A, S \cup T) = \overline{\rho}(A, S) \cup \overline{\rho}(A, T)$;
(4) $\underline{\rho}(A, S \cap T) = \underline{\rho}(A, S) \cap \underline{\rho}(A, T)$;

(5) $S \subseteq T$ *implies* $\underline{\rho}(A,S) \subseteq \underline{\rho}(A,T)$ *and* $\overline{\rho}(A,S) \subseteq \overline{\rho}(A,T)$;
(6) $\underline{\rho}(A,S) \cup \underline{\rho}(A,T) \subseteq \underline{\rho}(A,S \cup T)$;
(7) $\overline{\rho}(A,S \cap T) \subseteq \overline{\rho}(A,S) \cap \overline{\rho}(A,T)$;
(8) $\rho \subseteq \varphi$ *and* $A \subseteq B$ *implies* $\underline{\varphi}(B,S) \subseteq \overline{\rho}(A,S)$ *and* $\overline{\rho}(A,S) \subseteq \overline{\varphi}(B,S)$.

Proof.
(1) If $x \in \underline{\rho}(A,S)$, then $x \in A_x \subseteq S$. Hence, $\underline{\rho}(A,S) \subseteq S$. Since $x \in A_x$, if $x \in S$, then $A_x \cap S \neq \phi$. So $x \in \overline{\rho}(A,S)$. Consequently, $S \subseteq \overline{\rho}(A,S)$.
(2) is straightforward.
(3) Note that,

$$\begin{aligned} x \in \overline{\rho}(A,S \cup T) &\iff A_x \cap (S \cup T) \neq \phi \\ &\iff (A_x \cap S) \cup (A_x \cap T) \neq \phi \\ &\iff A_x \cap S \neq \phi \;\; or \; A_x \cap T \neq \phi \\ &\iff x \in \overline{\rho}(A,S) \; or \; a \in \overline{\rho}(A,T) \\ &\iff x \in \overline{\rho}(A,S) \cup \overline{\rho}(A,T). \end{aligned}$$

Thus

$$\overline{\rho}(A,S \cup T) = \overline{\rho}(A,S) \cup \overline{\rho}(A,T).$$

(4) Note that,

$$\begin{aligned} x \in \underline{\rho}(A,S \cap T) &\iff A_x \subseteq S \cap T \\ &\iff A_x \subseteq S \;\; and \;\; A_x \subseteq T \\ &\iff x \in \underline{\rho}(A,S) \; and \; x \in \underline{\rho}(A,T) \\ &\iff x \in \underline{\rho}(A,S) \cap \underline{\rho}(A,T). \end{aligned}$$

Thus

$$\underline{\rho}(A,S \cap T) = \underline{\rho}(A,S) \cap \underline{\rho}(A,T).$$

(5) Since $S \subseteq T$ if and only if $S \cap T = S$, by (3) we have

$$\underline{\rho}(A,S) = \underline{\rho}(A,S \cap T) = \underline{\rho}(A,S) \cap \underline{\rho}(A,T).$$

This implies that $\underline{\rho}(A,S) \subseteq \underline{\rho}(A,T)$. Note also that $S \subseteq T$ if and only if $S \cup T = T$, by (2) we have

$$\overline{\rho}(A,T) = \overline{\rho}(A,S \cup T) = \overline{\rho}(A,S) \cup \overline{\rho}(A,T).$$

This implies that $\overline{\rho}(A,S) \subseteq \overline{\rho}(A,T)$.
(6) Since $S \subseteq S \cup T$ and $T \subseteq S \cup T$, by (4) we have

$$\underline{\rho}(A,S) \subseteq \underline{\rho}(A,S \cup T) \quad \text{and} \quad \underline{\rho}(A,T) \subseteq \underline{\rho}(A,S \cup T).$$

This implies $\underline{\rho}(A,S)\cup\underline{\rho}(A,T)\subseteq\underline{\rho}(A,S\cup T)$.
(7) Since $S\cap T\subseteq S$ and $S\cap T\subseteq T$, by (4) we have

$$\overline{\rho}(A,S\cap T)\subseteq\overline{\rho}(A,S)\quad\text{and}\quad\overline{\rho}(A,S\cap T)\subseteq\overline{\rho}(A,T).$$

This implies $\overline{\rho}(A,S\cap T)\subseteq\overline{\rho}(A,S)\cap\overline{\rho}(A,T)$.
(8) Since $\rho\subseteq\varphi$. If $x\in\underline{\varphi}(B,S)$, then $B_x\subseteq S$. But $\rho\subseteq\varphi$. Therefore, $A_x\subseteq B_x\subseteq S$; that is, $A_x\subseteq S$. Thus $x\in\underline{\rho}(A,S)$. Henc,e

$$\underline{\varphi}(B,S)\subseteq\underline{\rho}(A,S).$$

Now, let x be any element of $\overline{\rho}(S)$. So $A_x\cap S\neq\phi$. Then there exists $y\in A_y\cap S$ such that $y\in A_y$ and $y\in S$. Hence $(y,x)\in\rho$; that is, $y\diamond x\in A$. Since $A\subseteq B$, $y\diamond x\in B$ and $x\diamond y\in B$. So $(y,x)\in\varphi$; that is, $y\in B_x$. Therefore, $y\in B_x\cap S$, which means that $x\in\varphi(B,S)$. This completes the proof.

Proposition 2.5. *Let ρ be an equivalence relation on X related to a JU-ideal A of X. If S is a nonempty subset of X, then*
(1) $\underline{\rho}(A,\underline{\rho}(A,S))=\underline{\rho}(A,S)$;
(2) $\overline{\rho}(A,\overline{\rho}(A,S))=\overline{\rho}(A,S)$;
(3) $\overline{\rho}(A,\underline{\rho}(A,S))=\underline{\rho}(A,S)$;
(4) $\underline{\rho}(A,\overline{\rho}(A,S))=\overline{\rho}(A,S)$;
(5) $\underline{\rho}(A,S)=(\overline{\rho}(A,S^c))^c$;
(6) $\overline{\rho}(A,S)=(\underline{\rho}(A,S^c))^c$;
(7) $\underline{\rho}(A,A_x)=X=\overline{\rho}(A,A_x)$, *for all* $x\in X$.

Proof.
The proof is straightforward.

Proposition 2.6. *Let ρ be an equivalence relation on X related to a JU-ideal A of X. If S is a nonempty subset of X, then*
(1) $\overline{\rho}\,(A,S)\diamond\overline{\rho}\,(A,T)\subseteq\overline{\rho}\,(A,S\diamond T)$;
(2) *If ρ is congruence relation, then* $\underline{\rho}\,(A,S)\diamond\underline{\rho}\,(A,T)\subseteq\underline{\rho}\,(A,S\diamond T)$.

Proof.
(1) Let c be any element of $\overline{\rho}(A,S)\diamond\overline{\rho}(A,T)$. Then $c=p\diamond q$ with $p\in\overline{\rho}(A,S)$ and $q\in\overline{\rho}(A,T)$. Thus there exist elements $x,y\in S$ such that

$$x\in A_p\cap S\quad\text{and}\quad y\in A_q\cap T.$$

Thus $x\in A_p$, $y\in A_q$, $x\in S$, and $y\in T$. Since ρ is a congruence on S, it follows that

$$x\diamond y\in A_p\diamond A_q\in A_{p\diamond q}.$$

On the other hand, since $x \diamond y \in S \diamond T$, $x \diamond y \in A_{p\diamond q} \cap S \diamond T$, and so $c = p \diamond q \in \overline{\rho}(A, S \diamond T)$. Thus

$$\overline{\rho}(A, S) \diamond \overline{\rho}(A, T) \subseteq \overline{\rho}(A, S \diamond T).$$

(2) Assume that ρ is complete. Let c be any element of $\underline{\rho}(A, S) \diamond \underline{\rho}(A, T)$. Then $c = p \diamond q$ with $p \in \underline{\rho}(A, S)$ and $q \in \underline{\rho}(A, T)$. It follows that $A_p \subseteq S$ and $A_q \subseteq T$. Since ρ is a congruence relation on S,

$$A_{p\diamond q} = A_p \diamond A_q \subseteq S \diamond T.$$

So $c = p \diamond q \in \underline{\rho}(A, S \diamond T)$. Thus

$$\underline{\rho}(A, S) \diamond \underline{\rho}(A, T) \subseteq \underline{\rho}(A, S \diamond T).$$

This completes the proof.

Proposition 2.7. *Let ρ and φ be equivalence relations on X related to JU-ideals A and B of X, respectively. If S and T are nonempty subsets of X, then*

(1) $\overline{\rho \cap \varphi}(A \cap B, S) \subseteq \overline{\rho}(A, S) \cap \overline{\varphi}(B, S)$;
(2) $\underline{\rho \cap \varphi}(A \cap B, S) \supseteq \underline{\rho}(A, S) \cap \underline{\varphi}(B, S)$.

Proof.
(1) Note that, $\rho \cap \varphi$ is also a congruence relation on S. Let $c \in \overline{\rho \cap \varphi}(A \cap B, S)$. Then $[A \cap B]_c \cap S \neq \phi$. Thus there exists an element $x \in [A \cap B]_c \cap S$. Since $(x, c) \in \rho \cap \varphi$,

$$(x, c) \in \rho \text{ and } (x, c) \in \varphi.$$

Thus $x \in A_c$ and $x \in B_c$. Since $x \in S$, $x \in A_c$, $x \in S$ and $x \in B_c$, $x \in S$. This implies that

$$x \in A_c \cap S \quad \text{and} \quad x \in B_c \cap S$$

$$A_c \cap S \neq \phi \quad \text{and} \quad B_c \cap S \neq \phi.$$

So $c \in \overline{\rho}(A, S)$ and $c \in \overline{\varphi}(B, S)$. Hence, $c \in \overline{\rho}(A, S) \cap \overline{\varphi}(B, S)$. Thus

$$\overline{\rho \cap \varphi}(A \cap B, S) \subseteq \overline{\rho}(A, S) \cap \overline{\varphi}(B, S).$$

(2) Since $\rho \cap \varphi \subseteq \rho$ and $\rho \cap \varphi \subseteq \varphi$,

$$\underline{\rho}(A, S) \subseteq \underline{\rho \cap \varphi}(A \cap B, S) \quad \text{and} \quad \underline{\varphi}(B, S) \subseteq \underline{\rho \cap \varphi}(A \cap B, S)$$

$$\Longrightarrow \underline{\rho}(A, S) \cap \underline{\varphi}(B, S) \subseteq \underline{\rho \cap \varphi}(A \cap B, S).$$

This completes the proof.

Theorem 2.8. *Let* (X, ρ) *be an approximation space. Then*
(1) *for every* $S \subseteq X$, $\underline{\rho}(A, S)$ *and* $\overline{\rho}(A, S)$ *are definable sets,*
(2) *for every* $x \in X$, A_x *is definable set.*

Proof.
(1) By Proposition 2.5 (1) and (3), we have

$$\underline{\rho}(A, \underline{\rho}(A, S)) = \underline{\rho}(A, S) = \overline{\rho}(A, \underline{\rho}(A, S)).$$

Hence $\underline{\rho}(A, S)$ is definable.
On the other hand, by Proposition 2.5 (2) and (4), we have

$$\overline{\rho}(A, \overline{\rho}(A, S)) = \overline{\rho}(A, S) = \underline{\rho}(A, \overline{\rho}(A, S)).$$

Therefore $\overline{\rho}(A, S)$ is a definable set.
(2) By Proposition 2.5 (7), the proof is clear.

Definition 2.9. *A nonempty subset* S *of* X *is called an upper (resp. a lower) rough JU-subalgebra of* X *if the upper (resp. nonempty lower) approximation of* S *is a JU-subalgebra of* X. *If* S *is both an upper and a lower rough JU-subalgebra of* X, *then we say that* S *is a rough JU-subalgebra of* X.

Theorem 2.10. *Let* ρ *be an congruence relation on* X *related to a JU-ideal* A *of* X. *If* S *is a JU-subalgebra of* X, *then*
(1) $\overline{\rho}(A, S)$ *is a JU-subalgebra of* X.
(2) $\underline{\rho}(A, S)$ *is a JU-subalgebra of* X.

Proof.
(1) Let $x, y \in \overline{\rho}(A, S)$. Then

$$A_x \cap S \neq \emptyset \text{ and } A_y \cap S \neq \emptyset,$$

and so there exist $a, b \in S$ such that $a \in A_x$ and $b \in A_y$. It follows that $(a, x) \in \rho$ and $(b, y) \in \rho$. Since ρ is a congruence relation on X, $(a \diamond b, x \diamond y) \in \rho$. Hence $a \diamond b \in A_{x \diamond y}$. Since S is a JU-subalgebra of X, $a \diamond b \in S$, and therefore $a \diamond b \in A_{x \diamond y} \cap S$; that is, $A_{x \diamond y} \cap S \neq \emptyset$. This shows that $x \diamond y \in \overline{\rho}(A, S)$. Consequently, $\overline{\rho}(A, S)$ is a JU-subalgebra of X.
(2) Let $x, y \in \underline{\rho}(A, S)$. Then $A_x \subseteq S$ and $A_y \subseteq S$. Since S is a JU-subalgebra of X,

$$A_{x \diamond y} = A_x \diamond A_y \subseteq S$$

so that $x \diamond y \in \underline{\rho}(A, S)$. Hence $\underline{\rho}(A, S)$ is a JU-subalgebra of X.

The following example shows that the converse of Theorem 2.10(1) is not true.

Example 2.11. *Let $X = \{0, a, b, c, d\}$ be a JU-algebra with the Cayley's table as follows:*

$\diamond$	0	a	b	c	d
0	0	a	b	c	d
a	0	0	b	b	d
b	0	0	0	a	d
c	0	0	0	0	d
d	0	a	a	a	0

Let $A = \{0, a, b\}$ be a JU-ideal of X ($A \lhd X$) and let ρ be an equivalence relation on X related to A. Then $A_0 = A_a = A_b = A$, $A_c = \{c\}$, and $A_d = \{d\}$. Note that, $S = \{a, 3\}$ is not a JU-subalgebra of X, but $\overline{\rho}(A, S) = \{0, a, b, c\}$ is JU-subalgebra of X.

Definition 2.12. *A nonempty subset S of X is called an upper (resp. a lower) rough JU-ideal of X if the upper (resp. nonempty lower) approximation of S is a JU-ideal of X. If S is both an upper and a lower rough JU-ideal of X, then we say that S is a rough JU-ideal of X.*

Theorem 2.13. *Let ρ be an congruence relation on X related to a JU-ideal A of X. If S is a JU-ideal of X containing A, then*
(1) $\overline{\rho}(A, S)$ is a JU-ideal of X.
(2) $\underline{\rho}(A, S)$ is a JU-ideal of X.

Proof.
(1) Let S be a JU-ideal of X containing A. Obviously, $0 \in \overline{\rho}(A, S)$. Let $x, y, z \in X$ be such that $y \in \overline{\rho}(A, S)$ and $x \diamond (y \diamond z) \in \overline{\rho}(A, S)$. Then

$$A_y \cap S \neq \emptyset \text{ and } A_{x\diamond(y\diamond z)} \cap S \neq \emptyset,$$

and so there exist $a, b \in S$ such that $a \in A_y$ and $b \in A_{x\diamond(y\diamond z)}$. Hence $(a, y) \in \rho$ and $(b, (x\diamond(y\diamond z))) \in \rho$, which implies that $y\diamond a \in A \subseteq S$ and $(x\diamond(y\diamond z))\diamond b \in A \subseteq S$. Since $a, b \in S$ and S is a JU-ideal,

$$y \in S \text{ and } x \diamond (y \diamond z) \in S,$$

From Definition 1.5(2), it follows that $x \diamond z \in S$. Note that, $x \diamond z \in A_{x\diamond z}$. Thus $x \diamond z \in A_{x\diamond z} \cap S$; that is, $A_{x\diamond z} \cap S \neq \emptyset$. Hence $x \diamond z \in \overline{\rho}(A, S)$ and therefore $\overline{\rho}(A, S)$ is a JU-ideal of X.

(2) Let S be a JU-ideal of X containing A. Let $x \in A_0$. Then $x \in A \subseteq S$, and so $A_0 \subseteq S$. Hence $0 \in \underline{\rho}(A, S)$. Let $x, y, z \in X$ be such that $y \in \underline{\rho}(A, S)$ and $x \diamond (y \diamond z) \in \underline{\rho}(A, S)$. Then

$$A_y \in S \text{ and } A_x \diamond (A_y \diamond A_z) = A_{x \diamond (y \diamond z)} \subseteq S.$$

Let $w \in A_{x \diamond z} = A_x \diamond A_z$. Then $w = A_x \diamond A_z$ for some $a \in A_x$ and $c \in A_z$. From $a \in A_x$ and $c \in A_z$, we have $(a, x) \in \rho$ and $(c, z) \in \rho$. Taking $b \in A_y$, we get $(b, y) \in \rho$. Since ρ is a congruence relation,

$$(a \diamond (b \diamond c), x \diamond (y \diamond z)) \in \rho \text{ and so } a \diamond (b \diamond c) \in A_{x \diamond (y \diamond z)} \subseteq S.$$

Since S is a JU-ideal of X, from Definition 1.5(2) it follows that $w \in a \diamond c \in S$. So $A_{x \diamond z} \subseteq S$. Hence $x \diamond z \in \underline{\rho}(A, S)$ and therefore $\underline{\rho}(A, S)$ is a JU-ideal of X.

Theorem 2.14. *Let ρ be an congruence relation on X related to a JU-ideal A of X. If S is a weak JU-ideal of X containing A, then*
(1) $\overline{\rho}(A, S)$ *is a weak JU-ideal of* X.
(2) $\underline{\rho}(A, S)$ *is a weak JU-ideal of* X.

Proof.
(1) Let S be a weak JU-ideal of X containing A. Obviously, $0 \in \overline{\rho}(A, S)$. Let $x, y \in X$ be such that $y \in \overline{\rho}(A, S)$ and $y \diamond x \in \overline{\rho}(A, S)$. Then

$$A_y \cap S \neq \emptyset \text{ and } A_{y \diamond x} \cap S \neq \emptyset,$$

and so there exist $a, b \in S$ such that $a \in A_y$ and $b \in A_{y \diamond x}$. Hence $(a, y) \in \rho$ and $(b, (y \diamond x)) \in \rho$, which implies

$$y \diamond a \in A \subseteq S \text{ and } (y \diamond x) \diamond b \in A \subseteq S.$$

Since $a, b \in S$ and S is a weak JU-ideal, $y \in S$ and $y \diamond x \in S$. From Definition 1.7(1) it follows that $x \in S$. Note that, $x \in A_x$. Thus $x \in A_x \cap S$; that is, $A_x \cap S \neq \emptyset$. Hence $x \in \overline{\rho}(A, S)$ and therefore $\overline{\rho}(A, S)$ is a weak JU-ideal of X.

(2) Let S be a weak JU-ideal of X containing A. Let $x \in A_0$. Then $x \in A \subseteq S$, and so $A_0 \subseteq S$. Hence $0 \in \underline{\rho}(A, S)$. Let $x, y \in X$ be such that $y \in \underline{\rho}(A, S)$ and $y \diamond x \in \underline{\rho}(A, S)$. Then

$$A_y \in S \text{ and } A_y \diamond A_x = A_{y \diamond x} \subseteq S.$$

Let $w \in A_x$. Then $w = A_x$ for some $a \in A_x$. From $a \in A_x$, we have $(a, x) \in \rho$. Taking $b \in A_y$, we get $(b, y) \in \rho$. Since ρ is a congruence relation,

$$(b \diamond a, y \diamond x) \in \rho \text{ and } b \diamond a \in A_{y \diamond x} \subseteq S.$$

Since S is a weak JU-ideal of X, from Definition 1.7(1) it follows that $w = a \in S$. So $A_x \subseteq S$. Hence $x \in \underline{\rho}(A, S)$ and therefore $\underline{\rho}(A, S)$ is a weak JU-ideal of X.

Theorem 2.15. *Let ρ be an congruence relation on X related to a JU-ideal A of X. If S is a strong JU-ideal of X containing A, then*
(1) *$\overline{\rho}(A, S)$ is a strong JU-ideal of X.*
(2) *$\underline{\rho}(A, S)$ is a strong JU-ideal of X.*

Proof.
(1) Let $x, y \in X$ be such that

$$(y \diamond x) \cap \overline{\rho}(A, S) \neq \emptyset \text{ and } y \in \overline{\rho}(A, S).$$

Then $A_y \cap S \neq \emptyset$ and so there exist $z \in X$ such that $z = y \diamond x$ and $z \in \overline{\rho}(A, S)$. Hence $A_z \cap S \neq \emptyset$ and so there exist $c, d \in X$ such that

$$c \in A_z \cap S \text{ and } d \in A_y \cap S.$$

Hence $c\rho z$ and $d\rho y$ where $c, d \in S$. Thus $z \diamond c \in A \subseteq S$ and $y \diamond d \in A \subseteq S$. Since S is a strong JU-ideal and $c, d \in S$, $z \in S$ and $y \in S$. Thus we have proved that $(y \diamond x) \cap A \neq \emptyset$ and $y \in A$. Since S is a strong JU-ideal, $x \in S$ and so $A_x \cap S \neq \emptyset$ which means that $\overline{\rho}(A, S)$ is a strong JU-ideal of S.
(2) Let $x, y \in X$ be such that

$$(y \diamond x) \cap \underline{\rho}(A, S) \neq \emptyset \text{ and } y \in \underline{\rho}(A, S).$$

Let $a \in A_x$ and $b \in A_y$. Then $a\rho x$ and $b\rho y$. Since ρ is a congruence relation on X, $b \diamond a\rho y \diamond x$. Since $(y \diamond x) \cap \underline{\rho}(A, S) \neq \emptyset$, there exist $t \in X$ such that $t \in y \diamond x$ and $t \in \underline{\rho}(A, S)$. Now, $t \in b \diamond a\rho y \diamond x$ implies that there exist $z \in b \diamond a$ such that $z\rho t$ and so $A_t = A_z \subseteq S$. Hence $z \in S$ and so $(b \diamond a) \cap S \neq \emptyset$. On the other hand, we have $b \in A_y \subseteq S$. Since S is a strong JU-ideal of X, $a \in S$ which implies $A_x \subseteq S$. This means that $x \in \underline{\rho}(A, S)$. Therefore, $\underline{\rho}(A, S)$ is a strong JU-ideal of S.

3 Conclusion.

Roughness is one of the important methods to tackle the uncertainty and vagueness in information system. In the theory of rough sets in algebraic structures, the main concepts are an equivalence relation and the classes formed by them. They form the main theme for the construction of lower

and upper approximations which play a significant role in computer science in database management system, in big data mining, as well as any other data handling techniques. In recent years, roughness has been applied successfully in a number of challenging fields such as pure and applied algebras, engineering, medical sciences and soft computing method in biology and computer technologies.
In this paper, we related JU-algebras (JU-ideals) with roughness through definitions, examples and results based on lower and upper approximations. This study may provide some new directions in different JU-structures including soft JU-algebras, hyper JU-algebras and fuzzy JU-algebras.

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