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Numerical evaluation on the solution of the porous medium equation by 4-Point Newton-Explicit Group

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Abstract

The porous medium equation is known as one of the nonlinear partial differential equations that are used to describe many physical phenomena involving fluid flow, mass and heat transfer, and diffusion of gas particles. The solution to this equation is important to understand the phenomena better. With the numerical method and advanced computing tools nowadays, solving nonlinear partial differential equations like the porous medium equation has gained interest from several researchers. Motivated by numerous efforts in solving porous medium equations, we propose a 4-point Newton-Explicit

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Group method and evaluate its efficiency to solve the one-dimensional problem of the porous medium equation. The formulation of the finite difference scheme, the derivation of the 4-point Newton-Explicit Group, and the numerical evaluation of the proposed method are presented. From the numerical study, we show that the 4-point Newton-Explicit Group method has a promising improvement in terms of efficiency when compared to the Newton-Gauss-Seidel and the 2-point Newton-Explicit Group methods.

1 Introduction

The porous medium equation or in short PME is known as one of the nonlinear partial differential equations that are used to describe many physical phenomena involving the fluid flow, the mass and heat transfer, and the diffusion of gas particles. The application of PME has been discovered since the early 20th century. As mentioned in [1], one of the earliest applications of PME started in the year 1903 in which Boussinesq made a significant contribution through his study in groundwater infiltration. Then, around 1930, Leibenzon and Muskat, independently, have modeled the phenomenon of an isentropic gas flows through a porous medium. Other than that, Zeldovich and his several co-workers have developed the equation of thermal-radiation transfer in plasma since 1950.

The solution of PME is important to understand the phenomena better. Several numerical methods have been proposed to solve PME. For instance, [2] have proposed the moving mesh finite element method. They developed their method by minimizing energy based on the equidistribution and alignment conditions. They also controlled the mesh adaptation through a matrix-valued function. Then, [3] have proposed a non-standard finite difference method on a quasi-uniform mesh to solve PME. In their work, the one-dimensional PME is reduced to a second-order nonlinear ordinary differential equation before the non-standard finite difference method is applied for the numerical solutions. Other than that, [4] developed two numerical methods by using an energetic variational approach for PME. Their methods preserve the energy dissipation law.

With the numerical method and advanced computing tool nowadays, solving PME has gained interest from many researchers and has also become the motivation of this paper. Hence, this paper aims to propose a 4-point Newton-Explicit Group method for the numerical solution of the one-dimensional PME. This numerical method can be abbreviated as 4-point NEG and the numerical evaluation is conducted by focusing its efficiency to solve several PME problems.

In this numerical method, the implicit finite difference method will be employed to formulate the approximation equation for the one-dimensional PME. And then, the Newton method is applied to transform the approximation equation into the corresponding linear approximation before the Explicit Group is used to iterate the solutions. Explicit Group or EG is a computational complexity reduction technique that has been introduced by [5]. This technique makes use of several groups of a fixed number of mesh points to reduce the computational complexity while iteration in solving a system of linear equations. The emergence of the EG technique together with widely known iterative methods such as Successive Over Relaxation and its modified version have shown promising numerical results in solving linear systems, see [6], [7] and [8].

In this paper, the proposed numerical method for the solution of the one-dimensional PME is discussed in the following sections. Section 2 shows the formulation of the implicit finite difference approximation equation, the application of the Newton method, and the derivation of Newton-Gauss-Seidel (NGS), 2-point NEG, and 4-point NEG iterative methods. Section 3 presents the numerical experiment of NGS, 2-point, and 4-point NEG in solving four selected PME problems. The numerical evaluation based on the percentage of reduction in the number of iterations and the computation time compared to the benchmark, the NGS iterative method is also illustrated. Section 4 concludes the findings of this paper.

2 Methodology

Given a general form of initial-boundary value problem of one-dimensional PME as follows [9],

$$\frac{\partial}{\partial t}u(x,t) = \alpha\frac{\partial}{\partial x}((u(x,t))^m\frac{\partial}{\partial x}u(x,t)), \tag{2.1}$$

$$u(x,0) = g_0(x), 0 \leq x \leq L, \tag{2.2}$$

$$u(a,t) = g_a(t), u(b,t) = g_b(t), t \geq 0, \tag{2.3}$$

where α is an arbitrary constant and m can be any integer. The solution of equation 2.1, $u(x,t)$, is a scalar function and the restriction $u(x,t) \geq 0$ is considered mathematically convenient. Moreover, the diffusion term $(u(x,t))^m$ degenerates wherever $u(x,t) = 0$ and equation 2.1 is parabolic wherever $u(x,t) > 0$. In physics, the arbitrary constant used is $\alpha = (\rho\mu)/(\kappa v_0)$, where ρ, μ, κ and v_0 represent the porosity of a medium, the viscosity of a fluid, the

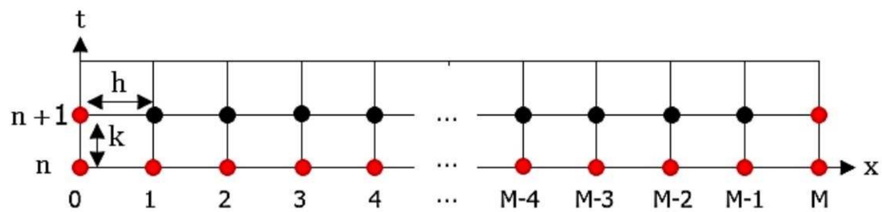

Figure 1: Mesh points distribution with red dots as the specified initial-boundary values and black dots as the unknown values

permeability of pores and the reference pressure, respectively. These parameters are assumed to be positive in many practical physics experiments and, therefore, $\alpha > 0$.

To compute the approximate solution of equation 2.1 subjects to equation 2.2 and 2.3, we consider a rectangular domain with mesh points $U(ih, nk) = U_{i,n}$, uniformly distributed with a fixed size of M subintervals in space with distance $h = L/M$ and T subintervals in time with distance $k = 1/T$.

To facilitate the formulation of the finite difference approximation to equation 2.1, we consider the mesh points distribution as in Figure 1. For the right-hand side of equation 2.1, we apply the chain rule of calculus to obtain

$$\alpha((u(x,t))^m \frac{\partial^2}{\partial x^2} u(x,t) + m(u(x,t))^{m-1}(\frac{\partial}{\partial x} u(x,t))^2). \tag{2.4}$$

Substituting equation 2.4 into equation 2.1 and using the implicit finite difference scheme, we obtain the nonlinear approximation equation to equation 2.1 as follows.

$$\begin{aligned} & U_{i,n+1} - pU_{i,n+1}^m U_{i+1,n+1} + 2pU_{i,n+1}^{m+1} - pU_{i,n+1}^m U_{i-1,n+1} \\ & - qmU_{i,n+1}^{m-1} U_{i+1,n+1}^2 + 2qmU_{i,n+1}^{m-1} U_{i+1,n+1} U_{i-1,n+1} - qmU_{i,n+1}^{m-1} U_{i-1,n+1}^2 = U_{i,n}, \end{aligned} \tag{2.5}$$

where $p = \alpha k/h^2, q = p/4$, $i = 1, 2, \ldots, M-1$ and $n = 0, 1, 2, \ldots, T$.

Before equation 2.5 is used to form the system of equations, we apply the Newton method to linearize equation 2.5 as follows. First, we rearrange equation 2.5 into a nonlinear function of

$$\begin{aligned} f_{i,n+1} = {} & U_{i,n+1} - pU_{i,n+1}^m U_{i+1,n+1} + 2pU_{i,n+1}^{m+1} - pU_{i,n+1}^m U_{i-1,n+1} \\ & - qmU_{i,n+1}^{m-1} U_{i+1,n+1}^2 + 2qmU_{i,n+1}^{m-1} U_{i+1,n+1} U_{i-1,n+1} - qmU_{i,n+1}^{m-1} U_{i-1,n+1}^2 - U_{i,n}. \end{aligned} \tag{2.6}$$

The system of nonlinear equations based on equation 2.6 can be formed into

$$F_{n+1} = 0, \tag{2.7}$$

where $F_{n+1} = (f_{1,n+1} f_{2,n+1} \ldots f_{M-1,n+1})^{Tr}$. Here, Tr stands for transpose. Then, by using the procedure of linearization via Newton method, the corresponding system of linear equations from the system of equation 2.7 has the form of

$$J_f C_{n+1} = -F_{n+1}, \tag{2.8}$$

where the tridiagonal Jacobian matrix is defined as

$$J_f = \frac{\partial(f_{1,n+1}, f_{2,n+1}, \ldots, f_{M-1,n+1})}{\partial(U_{1,n+1}, U_{2,n+1}, \ldots, U_{M-1,n+1})}, \tag{2.9}$$

and the corrector is defined as

$$C_{n+1} = \underset{\sim}{U}_{n+1}^{(l+1)} - \underset{\sim}{U}_{n+1}^{(l)}, \tag{2.10}$$

with the index of iteration l and $\underset{\sim}{U}_{n+1}^{(l)} = (U_{1,n+1}, U_{2,n+1}, \ldots, U_{M-1,n+1})^{(l)}$.

By rewriting the system of equation 2.8 into

$$AC_{n+1} = -F_{n+1}, \tag{2.11}$$

with the coefficient matrix A to be further decomposed into [10],

$$A = D - W - V. \tag{2.12}$$

The NGS iterative method for solving equation 2.1 can be formulated into

$$c_{i,n+1}^{(l+1)} = (d_i - w_i)^{-1}(v_i c_{i,n+1}^{(l)} - f_{i,n+1}), \tag{2.13}$$

for $i = 1, 2, ..., M-1$.

When a system of two linear equations based on the system of equation 2.11 is considered, the 2-point NEG iterative method can be obtained as follows.

$$\begin{bmatrix} c_{i,n+1} \\ c_{i+1,n+1} \end{bmatrix}^{(l+1)} = \begin{bmatrix} a_{i,i} & a_{i,i+1} \\ a_{i+1,i} & a_{i+1,i+1} \end{bmatrix}^{-1} \begin{bmatrix} -f_{i,n+1} - a_{i,i-1} c_{i-1,n+1}^{(l)} \\ -f_{i,n+1} - a_{i+1,i+2} c_{i+2,n+1}^{(l)} \end{bmatrix}, \tag{2.14}$$

for $i = 1, 3, ..., M-1$. To extend these iterative methods into 4-point which we call it as the 4-point NEG, we may get

$$\begin{bmatrix} c_{i,n+1} \\ c_{i+1,n+1} \\ c_{i+2,n+1} \\ c_{i+3,n+1} \end{bmatrix}^{(l+1)} = \begin{bmatrix} a_{i,i} & a_{i,i+1} & 0 & 0 \\ a_{i+1,i} & a_{i+1,i+1} & a_{i+1,i+2} & 0 \\ 0 & a_{i+2,i+1} & a_{i+2,i+2} & a_{i+2,i+3} \\ 0 & 0 & a_{i+3,i+2} & a_{i+3,i+3} \end{bmatrix}^{-1} \begin{bmatrix} s_i \\ s_{i+1} \\ s_{i+2} \\ s_{i+3} \end{bmatrix}, \tag{2.15}$$

for $i = 1, 5, ..., M-1$, and the values of the right-hand side column matrix are

$$s_i = -f_{i,n+1} - a_{i,i-1}c^{(l)}_{i-1,n+1}, s_{i+1} = -f_{i+1,n+1}, s_{i+2} = -f_{i+2,n+1},$$
$$s_{i+3} = -f_{i+3,n+1} - a_{i+3,i+4}c^{(l)}_{i+4,n+1}. \quad (2.16)$$

To implement the 4-point NEG iterative method, we develop the following algorithm for numerical computation.

Algorithm 1: 4-point NEG iterative method
Step 1. For $n = 0, 1, 2, ..., T$, set the initial guess $\underset{\sim}{U}^{(0)}_{n+1} = 1.0$, the corrector $C^{(0)}_{n+1} = 0$, and the tolerance error $\varepsilon = 1 \times 10^{-10}$,
Step 2. Iterate equation 2.15,
Step 3. Check if $|C^{(l+1)}_{n+1} - C^{(l)}_{n+1}| \leq \varepsilon$. If yes, go to Step 4. Otherwise, return to Step 2,
Step 4. Compute the approximate solutions using $\underset{\sim}{U}^{(l+1)}_{n+1} = \underset{\sim}{U}^{(l)}_{n+1} + C^{(l)}_{n+1}$,
Step 5. Check if $|F(\underset{\sim}{U}^{(l+1)}_{n+1}) - F(\underset{\sim}{U}^{(l)}_{n+1})| \leq \varepsilon$. If yes, $n+1$. Otherwise, return to Step 2,
Step 6. Stop and display approximate solutions.
To implement the NGS and 2-point NEG iterative methods. Step 2. can be switched into equation 2.13 and 2.14 respectively.

3 Numerical Experiment

To evaluate the efficiency of the 4-point NEG iterative method, we have selected four one-dimensional PME problems to be tested. For more details about the PME problems used, see in [9] and [11]. In this numerical experiment, the efficiency of the 4-point NEG iterative method in solving the four PME problems is compared to 2-point NEG iterative method and NGS is set to be the benchmark. To compare the efficiency among the three iterative methods, two criteria are observed: the number of iterations (l) and the computation time which is measured in seconds. The accuracy of the 2-point and 4-point NEG iterative methods is also observed based on the absolute error compared to the benchmark (error). Table 1 shows the four selected PME problems and then all numerical results are tabulated from Table 2 to Table 5.

4 Conclusion

From the numerical results shown from Table 2 to Table 5, we found that the proposed 4-point NEG iterative method require a much lesser number of iterations and computation time in computing the approximate solutions for the four PME problems when compared to the NGS and 2-point NEG iterative methods. By comparing to the benchmark which is NGS, the 4-point NEG has successfully reduced the number of iterations by about 70.09%-72.06% and the computation time by about 51.33%-85.71%. Moreover, the three iterative methods are good in terms of accuracy as the absolute errors are almost equivalent to different values of M. It can be concluded that the 4-point NEG iterative method can be a good alternative technique for solving PME and other types of the nonlinear partial differential equation.

Table 1: Selected PME problems

Problem	α	m	Exact solution
1	1	1	$u(x,t) = x + t$
2	1/2	-1	$u(x,t) = (0.6x - 0.18t + 1.3)^{-1}$
3	1	2	$u(x,t) = (x+1)(2(4-t)^{1/2})^{-1}$
4	1/2	-2	$u(x,t) = (0.7x - 0.1225t + 1.35)^{-1/2}$

Table 2: Numerical result for Problem 1

M	Method	l	Seconds	Error
64	NGS	3835	2.38	2.76E-08
	2-point NEG	2037	0.85	1.26E-08
	4-point NEG	1109	0.34	5.88E-09
128	NGS	13678	7.50	1.22E-07
	2-point NEG	7249	2.72	5.63E-08
	4-point NEG	3899	1.65	2.64E-08
256	NGS	48395	38.58	5.33E-07
	2-point NEG	25717	18.61	2.56E-07
	4-pointNEG	13799	11.20	1.10E-07
512	NGS	169693	252.94	2.10E-06
	2-point NEG	90637	127.28	1.07E-06
	4-point NEG	48666	77.31	4.99E-07
1024	NGS	587031	1712.49	7.62E-06
	2-point NEG	316030	877.83	4.03E-06
	4-point NEG	170300	557.86	2.08E-06

Table 3: Numerical result for Problem 2

M	Method	l	Seconds	Error
64	NGS	1720	1.13	2.03E-05
	2-point NEG	922	0.63	2.03E-05
	4-point NEG	504	0.55	2.03E-05
128	NGS	6034	4.06	2.02E-05
	2-point NEG	3213	2.18	2.03E-05
	4-point NEG	1718	1.73	2.03E-05
256	NGS	20907	27.03	2.00E-05
	2-point NEG	11173	13.84	2.01E-05
	4-pointNEG	5976	11.25	2.02E-05
512	NGS	71385	287.34	1.93E-05
	2-point NEG	38439	125.48	1.98E-05
	4-point NEG	20701	97.75	2.00E-05
1024	NGS	239975	1741.01	1.72E-05
	2-point NEG	130425	664.72	1.86E-05
	4-point NEG	70888	571.03	1.94E-05

Table 4: Numerical result for Problem 3

M	Method	l	Seconds	Error
64	NGS	1344	1.17	8.39E-05
	2-point NEG	721	0.67	8.39E-05
	4-point NEG	402	0.38	8.39E-05
128	NGS	4824	2.84	8.39E-05
	2-point NEG	2549	1.55	8.39E-05
	4-point NEG	1361	1.00	8.39E-05
256	NGS	17308	20.03	8.39E-05
	2-point NEG	9139	10.37	8.39E-05
	4-pointNEG	4836	6.77	8.39E-05
512	NGS	61658	270.11	8.40E-05
	2-point NEG	32688	158.35	8.39E-05
	4-point NEG	17333	46.85	8.39E-05
1024	NGS	218147	2008.35	8.43E-05
	2-point NEG	115936	995.57	8.41E-05
	4-point NEG	61779	342.02	8.40E-05

Table 5: Numerical result for Problem 4

M	Method	l	Seconds	Error
64	NGS	2015	1.26	2.88E-06
	2-point NEG	1083	0.65	2.89E-06
	4-point NEG	592	0.53	2.89E-06
128	NGS	7082	4.90	2.90E-06
	2-point NEG	3780	2.44	2.93E-06
	4-point NEG	2033	1.85	2.94E-06
256	NGS	24325	45.42	2.71E-06
	2-point NEG	13033	22.56	2.85E-06
	4-pointNEG	7007	15.11	2.92E-06
512	NGS	81729	354.79	1.86E-06
	2-point NEG	44101	172.65	2.42E-06
	4-point NEG	23769	112.83	2.73E-06
1024	NGS	265698	2293.23	3.33E-06
	2-point NEG	144614	1113.58	1.01E-06
	4-point NEG	79057	733.85	1.89E-06

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