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Tree languages with fixed variables and their algebraic structures

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Abstract

Tree languages, sets of terms, have been more widely applied in automata theory and related fields. In this paper, a specific class of tree languages, which induced by terms of a fixed variable, is introduced. Under non-deterministic superposition operations of tree languages, the power clone of tree languages with fixed variables is constructed and some of their properties are investigated. These lead us to propose certain semigroups consisting of the set of tree languages with fixed variables and different binary associative operations.

1 Introduction and preliminaries

Basically, a term of type τ is a formal expression combining variables and compositions of operation symbols in a sequence τ of arities.

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Let $X_n = \{x_1, \ldots, x_n\}$, for $n \in \{1, 2, \ldots\}$, be a set whose elements are called *variables* and $X = \{x_1, \ldots, x_n, \ldots\}$. To define terms, we use a set $\{f_i \mid i \in I\}$ of operation symbols, indexed by the set I. The type is the sequence $\tau = (n_i)_{i \in I}$ of the natural number arities of the symbols f_i. Formally, an *n-ary term of type* τ is inductively defined as follows:

(1) Every variable $x_i \in X_n$ is an n-ary term of type τ.

(2) If $t_1, \ldots, t_{n_i}$ are n-ary terms of type τ and f_i has arity n_i, then the composition $f_i(t_1, \ldots, t_{n_i})$ is also an n-ary term of type τ.

For a set X_n, by $W_\tau(X_n)$ we mean the set of all n-ary terms of type τ. On the other hand, $W_\tau(X)$ denotes the set of all terms of type τ. The set of all variables that appear in t is denoted by $var(t)$.

We now illustrate some examples of terms. Let us consider the type $\tau = (3, 2)$ with one ternary operation symbol g and one binary operation symbol f. Then

$$x_1, x_2, f(g(x_1, x_1, x_2), f(x_2, x_1)), f(f(x_1, x_2), f(x_2, x_1)) \in W_{(3,2)}(X_2),$$

$$x_1, x_2, x_3, g(x_3, x_3, x_2), f(f(x_2, x_3), x_2), f(f(x_3, x_2), g(x_1, x_1, x_2)) \in W_{(3,2)}(X_3).$$

Naturally, a new term can be obtained after substituting the variables occuring in an original term with the other terms. This can be described by the many-sorted superpostion operation S^n_m, $n, m \geq 1$,

$$S^n_m : W_\tau(X_n) \times (W_\tau(X)_m)^n \to W_\tau(X)_m$$

defined inductively by the following steps:
For $t \in W_\tau(X_n), t_1, \ldots, t_n \in W_\tau(X_m)$

(1) If $t = x_i$; $1 \leq i \leq n$, then $S^n_m(x_i, t_1, \ldots, t_n) := t_i$.

(2) If $t = f_i(s_1, \ldots, s_{n_i})$, then
$S^n_m(t, t_1, \ldots, t_n) := f_i(S^n_m(s_1, t_1, \ldots, t_n), \ldots, S^n_m(s_{n_i}, t_1, \ldots, t_n))$.

Then the algebraic structure

$$clone(\tau) = ((W_\tau(X_n))_{n \geq 1}, (S^n_m)_{n,m \geq 1}, (x_i)_{i \leq n}),$$

which is called the *clone of all terms of type* τ, is constructed. More background and current trends of terms can be found in [2, 13, 15, 16, 22].

Let $P(W_\tau(X))$ be the power set of all the set of all terms of type τ. Every element of $P(W_\tau(X))$ is a set of terms, called *tree languages*. Semigroups of tree languages were studied in [7]. For applications of tree languages in automata theory, we suggest [10, 11]. Generally, the superposition operation on $P(W_\tau(X))$ was first presented in [5]. Let n, m be natural numbers, $A \in P(W_\tau(X_n)), B_1, \ldots, B_n \in P(W_\tau(X_m))$. Then the superposition operation

$$\widehat{S}^n_m : P(W_\tau(X_n)) \times P(W_\tau(X_m))^n \to P(W_\tau(X_m))$$

is inductively defined by the following steps:

(1) If $A = \{x_i\}$; $1 \leq i \leq n$, then $\widehat{S}^n_m(\{x_i\}, B_1, \ldots, B_n) := B_i$.

(2) If $A = \{f_i(t_1, \ldots, t_{n_i})\}$, and suppose that $\widehat{S}^n_m(\{t_k\}, B_1, \ldots, B_n)$ for all $k = 1, \ldots, n_i$ are already defined, then $\widehat{S}^n_m(\{f_i(t_1, \ldots, t_{n_i})\}, B_1, \ldots, B_n) := \{f_i(r_1, \ldots, r_{n_i}) \mid r_k \in \widehat{S}^n_m(\{t_k\}, B_1, \ldots, B_n), 1 \leq k \leq n_i\}$.

(3) If A is an arbitrary non-singleton subset of $W_\tau(X)$, then $\widehat{S}^n_m(A, B_1, \ldots, B_n) := \bigcup\limits_{a \in A} \widehat{S}^n_m(\{a\}, B_1, \ldots, B_n)$.
If one of the sets $A, B_1, \ldots, B_n$ is empty, then we define $\widehat{S}^n_m(A, B_1, \ldots, B_n) := \emptyset$.

The power Menger algebra of terms, denoted by $P - clone(\tau)$, is a many-sorted algebra which consists of a sequence $(P(W_\tau(X_n)))_{n\geq 1}$ of the power set of n-ary terms of type τ for every $n \geq 1$, a sequence of the superposition operation $(\widehat{S}^n_m)_{n,m\geq 1}$, and a sequence of a singleton of all variables $(\{x_i\})_{i\leq n}$; i.e.,

$$P - clone(\tau) = ((P(W_\tau(X_n)))_{n\geq 1}, (\widehat{S}^n_m)_{n,m\geq 1}, (\{x_i\})_{i\leq n}).$$

Several algebraic properties of $P - clone(\tau)$ were studied by many researchers [4]. For an overview of many-sorted algebra, we refer the reader to [1, 3, 10, 18]. An important result is that $P - clone(\tau)$ satisfies the clone axioms:
For every natural number $m, n, p \geq 1$:

(C1) $\widehat{S}^n_m(\widehat{S}^p_n(A, B_1, \ldots, B_p), C_1, \ldots, C_n)$
$= \widehat{S}^p_m(A, \widehat{S}^n_m(B_1, C_1, \ldots, C_n), \ldots, \widehat{S}^n_m(B_p, C_1, \ldots, C_n))$
whenever $A \subseteq W_\tau(X_p), B_1, \ldots, B_p \subseteq W_\tau(X_n)$, $C_1, \ldots, C_n \subseteq W_\tau(X_m)$.

(C2) $\widehat{S}^n_m(\{x_i\}, B_1, \ldots, B_n) = B_i$ whenever $B_1, \ldots, B_n \subseteq W_\tau(X_m)$ for all $i \in \{1, \ldots, n\}$.

(C3) $\widehat{S}^n_n(A, \{x_1\}, \ldots, \{x_n\}) = A$.

(C1) is called the *superassociative law*. Notice that $P-clone(\tau)$ is an example of Menger systems. For further details on Menger systems and related topics, we refer the reader to [9, 23]. Additionally, a restriction of Menger systems in one-sorted is called Menger algebras (see [8, 14, 27] for recent developments in this area). The study of semigroups of tree languages that were generated by $P-clone(\tau)$ appeared in [7]. A special class of tree languages, called *linear tree languages*, was obtained by many authors (see for example [12, 20]).

Recently, a special class of terms of type τ, called *terms of a fixed variable*, was introduced by Wattanatripop and Changphas [26]. We now recall the concept of n-ary terms of a fixed variable of type τ.

(1) Every $x_j \in X_n$ is an n-ary term of a fixed variable of type τ.

(2) If $t_1, \ldots, t_{n_i}$ are n-ary terms of a fixed variable of type τ with $var(t_l) = var(t_k)$ for every $1 \leq l < k \leq n_i$, then $f_i(t_1, \ldots, t_{n_i})$ is an n-ary term of a fixed variable of type τ.

The symbol $W_\tau^{fv}(X_n)$ stands for the set of all an n-ary terms of a fixed variable of type τ over X_n.

For example, let us consider type $\tau = (2)$ with one binary operation symbol f. Then

$$x_1, x_2, f(x_1, x_1), f(x_2, x_2), f(f(x_1, x_1), x_1), f(f(x_2, x_2), f(x_2, x_2)) \in W_{(2)}^{fv}(X_2),$$

$$x_1, x_2, x_3, f(x_1, x_1), f(f(x_2, x_2), x_2), f(f(x_3, x_3), f(x_3, x_3)) \in W_{(2)}^{fv}(X_3).$$

On the other hand,

$$f(x_1, x_2), f(f(x_1, x_1), x_2), f(f(x_1, x_1), f(x_2, x_2)) \notin W_{(2)}^{fv}(X_2),$$

$$f(x_1, x_3), f(x_3, x_1), f(f(x_1, x_2), x_3), f(f(x_1, x_3), f(x_3, x_3)) \notin W_{(2)}^{fv}(X_3),$$

For further study on terms of a fixed variable, we refer the reader to [21, 24].

On the set $W_\tau^{fv}(X_n)$, superposition of terms were applied and so the many-sorted algebra of those terms, called clone of terms of a fixed variable, was constructed. The results indicated that clone of terms of a fixed variable is a subalgebra of clone of all terms of type τ. It is widely accepted that hypersubstitutions always play a key role in the study of hyperidentities [6, 17, 25]. As a result, the semigroup of fv-hypersubstitutions of type τ consisting of the set $Hyp^{fv}(\tau)$ of all mappings σ which takes any operation symbols to terms of a fixed variable of the same arity and a binary associative operation $\circ_h$ was proposed.

In this paper, our objective is to continue the study of terms of a fixed variable by constructing tree languages induced by such terms. In Section 2, under the superposition operations of tree languages, the power clone of terms of a fixed variable is constructed and some of their algebraic properties are investigated. In Section 3, the concept of mappings which from the set of operation symbols to the set of all tree languages with fixed variables is presented. This leads us to form the semigroup of these mappings based on some preparations. Finally, we give a nice relationship between the semigroup of fv-hypersubstitutions which was given in [26] and our main results through an embedding theorem.

2 Power clone of tree languages with fixed variables

Based on the set $W_\tau^{fv}(X_n)$ of all n-ary terms of a fixed variable of type τ, the powerset $P(W_\tau^{fv}(X_n))$ can be constructed in a canonical way. Each element of $P(W_\tau^{fv}(X_n))$ is called a tree language of n-ary terms of fixed variables, or a tree language of terms of fixed variables. For example, let us consider the type (2) with one binary operation symbol g and variables from X_3. The following sets are subsets of $W_{(2)}^{fv}(X_3)$; i.e., some elements of $P(W_{(2)}^{fv}(X_3))$ are:

$$\emptyset, \{x_1\}, \{x_2\}, \{x_3\}, \{x_3, g(x_1, x_1)\}, \{x_3, g(x_2, x_2), g(x_3, g(x_3, x_3))\}.$$

The following theorem shows that the sequence of $P(W_\tau^{fv}(X_n))$ is closed under the superposition operation $\widehat{S}_m^n$.

Theorem 2.1. *For any natural numbers n and m, we have*

$$\widehat{S}_m^n(A, B_1, \ldots, B_n) \in P(W_\tau^{fv}(X_m))$$

for all $A \subseteq W_\tau^{fv}(X_n)$ and $B_1, \ldots, B_n \subseteq W_\tau^{fv}(X_m)$.

Proof. Let $A \subseteq W_\tau^{fv}(X_n)$ and $B_1, \ldots, B_n \subseteq W_\tau^{fv}(X_m)$. If one of the sets $A, B_1, \ldots, B_n$ in the domain of the superposition operation $\widehat{S}_m^n$ is empty, then we have an empty set and thus there is nothing to prove. Assume now that all of the sets $A, B_1, \ldots, B_n$ are not empty. We give a proof on a set A. If A is a singleton set of the n-ary term of a fixed variable t, then we consider the following two cases: t is a variable from X_n and $t = f_i(t_1, \ldots, t_{n_i})$.
For the first case, we have $\widehat{S}_m^n(A, B_1, \ldots, B_n) = \widehat{S}_m^n(\{x_i\}, B_1, \ldots, B_n) = B_i \in$

$P(W_\tau^{fv}(X_m))$.
In the second case, we show that $\widehat{S}_m^n(\{f_i(t_1,\ldots,t_{n_i})\},B_1,\ldots,B_n)$ belong to $P(W_\tau^{fv}(X_m))$. By the definition of $\widehat{S}_m^n$, it is $\{f_i(r_1,\ldots,r_{n_i}) \mid r_k \in \widehat{S}_m^n(\{t_k\},B_1,\ldots,B_n), 1 \leq k \leq n_i\}$. Hence, for every $1 \leq k \leq n_i$, we have to show that $\widehat{S}_m^n(\{t_k\},B_1,\ldots,B_n) \in P(W_\tau^{fv}(X_m))$. By hypothesis, we have $\{f_i(t_1,\ldots,t_{n_i})\} \subseteq W_\tau^{fv}(X_n)$. This means that $var(\{f_i(t_1,\ldots,t_{n_i})\}) = \{x_j\}$, for some $j \in \{1,\ldots,n\}$. As a result, $var(\widehat{S}_m^n(\{t_k\},B_1,\ldots,B_n)) = \{x_j\}$, for some $j \in \{1,\ldots,n\}$; that is, this language has one variable only. Consequently, $|var(\widehat{S}_m^n(\{f_i(t_1,\ldots,t_{n_i})\},B_1,\ldots,B_n))| = 1$.
If $|A| > 1$, then $\widehat{S}_m^n(A,B_1,\ldots,B_n) = \bigcup\limits_{t\in A} \widehat{S}_m^n(\{t\},B_1,\ldots,B_n)$ by the definition of $\widehat{S}_m^n$. Using the singleton case, the proof is now complete. □

Applying Theorem 2.1, we have many-sorted mappings on the set of tree languages of fixed variables

$$\widehat{S}_m^n : P(W_\tau^{fv}(X_n)) \times P(W_\tau^{fv}(X_m))^n \to P(W_\tau^{fv}(X_m))$$

for $n, m \geq 1$. Moreover, we obtain the many-sort algebra

$$P - clone^{fv}(\tau) = ((P(W_\tau^{fv}(X_n)))_{n\geq 1}, (\widehat{S}_m^n)_{n,m\geq 1}, (\{x_i\})_{i\leq n}),$$

which is called the *power clone of tree languages with fixed variables*.

The next theorem is a straightforward outcome from Theorem 2.1 and the fact that $P(W_\tau^{fv}(X_n))$ is smaller than $P(W_\tau(X_n))$.

Theorem 2.2. *The many-sort algebra $P - clone^{fv}(\tau)$ satisfies (C1)-(C3).*

3 The semigroup of nondeterministic hypersubstitutions with fixed variables

At the beginning of this section, we recall some fundamental ideas about nondeterministic hypersubstitutions. A non-deterministic hypersubstitution of type τ is a mapping

$$\sigma_{nd} : \{f_i \mid i \in I\} \to P(W_\tau(X)).$$

The set of all such mappings is denoted by $Hyp_{nd}(\tau)$.

For example, let consider a type $\tau = (3, 2)$ with one ternary operation symbol f and one binary operation symbol g. If we set $\sigma_{nd}(f) =$

$\{x_3, f(x_2, g(x_1, x_3), x_3)\}$ and $\sigma_{nd}(g) = \{x_1, g(x_2, x_1)\}$, then $\sigma_{nd} \in Hyp_{nd}(3, 2)$. On the other hand, $\alpha_{nd} \notin Hyp_{nd}(3, 2)$ if $\alpha_{nd}(f) = \{x_1, f(x_2, g(x_1, x_3), x_3)\}$ and $\alpha_{nd}(g) = \{x_3, x_2, g(x_3, x_3)\}$.

Normally, every σ_{nd} induces a mapping $\widehat{\sigma}_{nd}$ which sends a tree language; i.e., sets of terms into itself by the following inductive way:

(1) $\widehat{\sigma}_{nd}[\emptyset] := \emptyset$,

(2) $\widehat{\sigma}_{nd}[\{x_i\}] := \{x_i\}$, where x_i is a variable from X,

(3) $\widehat{\sigma}_{nd}[\{f_i(t_1, \ldots, t_{n_i})\}] := \widehat{S}^{n_i}_m(\sigma_{nd}(f_i), \widehat{\sigma}_{nd}[\{t_1\}], \ldots, \widehat{\sigma}_{nd}[\{t_{n_i}\}])$ if $\widehat{\sigma}_{nd}[\{t_k\}], 1 \leq k \leq n_i$ are already known,

(4) $\widehat{\sigma}_{nd}[B] := \bigcup_{b \in B} \widehat{\sigma}_{nd}[\{b\}]$ if B is an arbitrary nonsingleton subset of $W_\tau(X)$.

It is well known that every extension $\widehat{\sigma}_{nd}$ of non-deterministic hypersubstitution σ_{nd} is an endomorphism of the many-sorted algebra $P - clone(\tau)$. Applying an extension $\widehat{\sigma}_{nd}$ [5], the binary operation

$$\circ_{nd} : Hyp_{nd}(\tau) \times Hyp_{nd}(\tau) \to Hyp_{nd}(\tau)$$

was introduced by setting $\sigma_{nd} \circ_{nd} \alpha_{nd} := \widehat{\sigma}_{nd} \circ \alpha_{nd}$. Moreover, the non-deterministic hypersubstitution σ^{id}_{nd} was defined to be an identity element, where $\sigma^{id}_{nd}(f_i) := \{f_i(x_1, \ldots, x_{n_i})\}$ for all $i \in I$. As a consequence, the structure $(Hyp_{nd}(\tau), \circ_{nd}, \sigma^{id}_{nd})$ forms a monoid. Other constructions of non-deterministic hypersubstitutions in some particular classes were collected in [19, 20].

We now introduce the concept of mappings which take any operation symbols to sets of terms of a fixed variable.

Definition 3.1. *A non-deterministic hypersubstitution σ_{nd} of type τ is said to be non-deterministic hypersubstitution with fixed variables of type τ if $\sigma_{nd}(f_i) \in P(W^{fv}_\tau(X))$ of the corresponding arity. $Hyp^{fv}_{nd}(\tau)$ denotes the set of all non-deterministic hypersubstitutions with fixed variables of type τ.*

To prove that a binary operation $\circ_{nd}$ on $Hyp_{nd}(\tau)$ can be applied to $Hyp^{fv}_{nd}(\tau)$, we need the following lemma.

Lemma 3.2. *The extension of a non-deterministic hypersubstitution with fixed variables of type τ maps tree languages with fixed variables to tree languages with fixed variables.*

Proof. Let σ_{nd} be a mapping in $Hyp^{fv}_{nd}(\tau)$. We show that $\widehat{\sigma}_{nd} : P(W^{fv}_{\tau}(X)) \rightarrow P(W^{fv}_{\tau}(X))$ holds. To do this, let $A \in P(W^{fv}_{\tau}(X))$. The theorem is clear if A is empty. Suppose now that A is not empty. If $A = \{x_i\}$, then $\widehat{\sigma}_{nd}[\{x_i\}] = \{x_i\} \in P(W^{fv}_{\tau}(X))$. Assume that $f_i(t_1, \ldots, t_{n_i})$ is a term of a fixed variable and $A = \{f_i(t_1, \ldots, t_{n_i})\}$ such that $\widehat{\sigma}_{nd}[\{t_k\}] \in P(W^{fv}_{\tau}(X))$, for all $1 \leq k \leq n_i$. Using Theorem 2.1, since $\sigma_{nd}(f_i) \in P(W^{fv}_{\tau}(X))$, we have $\widehat{\sigma}_{nd}[\{f_i(t_1, \ldots, t_{n_i})\}] = \widehat{S}^{n_i}_{m}(\sigma_{nd}(f_i), \widehat{\sigma}_{nd}[\{t_1\}], \ldots, \widehat{\sigma}_{nd}[\{t_{n_i}\}])$ is a tree language with fixed variables. Finally, if B is a non-singleton subset of $W^{fv}_{\tau}(X)$, the proof follows because of the union property of sets. ☐

We provide the following example of Lemma 3.2.

Example 3.3. *Let $\tau = (2)$ with a binary operation symbol f and $\sigma_{nd} \in Hyp^{fv}(2)$ which is given by $\sigma_{nd}(f) = \{x_2, f(x_2, x_2)\}$. If a tree language with fixed variables $B = \{x_2, f(x_2, f(x_2, x_2))\} \in P(W^{fv}_{(2)}(x_2))$, then $\widehat{\sigma}_{nd}[B] =$ $\{x_2, f(x_2, x_2), f(x_2, f(x_2, x_2)), f(f(x_2, x_2), x_2), f(f(x_2, x_2), f(x_2, x_2))\} \in P(W^{fv}_{(2)}(x_2))$.*

As a consequence, we have

Theorem 3.4. *$(Hyp^{fv}_{nd}(\tau), \circ_{nd})$ is a subsemigroup of $(Hyp_{nd}(\tau), \circ_{nd})$.*

Proof. If $\sigma_{nd}, \alpha_{nd} \in Hyp^{fv}_{nd}(\tau)$, then we have to show that $\sigma_{nd} \circ_{nd} \alpha_{nd} \in Hyp^{fv}_{nd}(\tau)$. In fact, we obtain $(\sigma_{nd} \circ_{nd} \alpha_{nd})(f_i) = \widehat{\sigma}_{nd}[\alpha_{nd}(f_i)]$. Since $\alpha_{nd}(f_i)$ is a tree language with fixed variables and $\widehat{\sigma}_{nd}$ is a non-deterministic hypersubstitution with fixed variables of type τ, by Lemma 3.2 $\widehat{\sigma}_{nd}[\alpha_{nd}(f_i)]$ is a tree language with fixed variables. ☐

Now, we give two more examples of non-deterministic hypersubstitutions with fixed variables.

Example 3.5. *Let $\tau = (4, 2)$ be a type with a quaternary operation symbol h and a binary operation symbol g. Let σ be a non-deterministic hypersubstitution which takes h to $\{h(x_4, x_4, x_4, x_4)\}$ and g to $\{x_2, g(x_2, x_2), g(x_2, g(x_2, x_2))\}$. Then $\sigma \in Hyp^{fv}_{nd}(4, 2)$.*

Example 3.6. *Let $\tau = (3, 2, 1)$ be a type with a ternary operation symbol h, a binary operation symbol g, and a unary operation symbol f. Let β be the non-deterministic hypersubstitution which takes h to $\{x_3, h(g(x_3, x_2), x_3, x_1)\}$, g to $\{h(x_2, x_2, x_1), g(x_2, x_2)\}$, and f to $\{f(f(f(x_1)))\}$. Clearly, $\beta \in Hyp^{fv}_{nd}(3, 2, 1)$.*

We prove an important property which gives a nice relationship between $\widehat{\sigma}_{nd}$ and $P - clone^{fv}(\tau)$.

Theorem 3.7. *The extension of non-deterministic hypersubstitutions with fixed variables of type τ is an endomorphism of the many-sorted algebra $P-clone^{fv}(\tau)$.*

Proof. Since $P-clone^{fv}(\tau)$ is the many-sort algebra, we consider $\widehat{\sigma}_{nd}$ as a sequence $(\widehat{\sigma}_{nd,n})_{n\geq 1}$, where $\widehat{\sigma}_{nd,n}$ sends sets of n-ary terms of fixed variable to sets of n-ary terms of fixed variable. In order to prove that $(\widehat{\sigma}_{nd,n})_{n\geq 1}$ is an endomorphism of $P-clone^{fv}(\tau)$, we have to show that the equation

$$\widehat{\sigma}_{nd,m}[\widehat{S}^n_m(A,B_1,\ldots,B_n)]=\widehat{S}^n_m(\widehat{\sigma}_{nd,n}[A],\widehat{\sigma}_{nd,m}[B_1],\ldots,\widehat{\sigma}_{nd,m}[B_n])$$

for all $A\in P(W^{fv}_\tau(X))$ and $B_1,\ldots,B_n\in P(W^{fv}_\tau(X_m))$ is satisfied. If $A=\emptyset$ or $B_j=\emptyset$, for some $j\in\{1,\ldots,n\}$, then by the definitions of $\widehat{S}^n_m$ and $\widehat{\sigma}_{nd,n}$ we have

$$\widehat{\sigma}_{nd,m}[\widehat{S}^n_m(A,B_1,\ldots,B_n)]=\widehat{\sigma}_{nd,m}[\emptyset]=\emptyset=\widehat{S}^n_m(\widehat{\sigma}_{nd,n}[A],\widehat{\sigma}_{nd,m}[B_1],\ldots_n,\widehat{\sigma}_{nd,m}[B_n]).$$

Now we give a proof on a set A. If $A=\{t\}$, where t is a term of fixed variable, then we consider two cases. For $t=x_i$, $1\leq i\leq n$, we have
$\widehat{\sigma}_{nd,m}[\widehat{S}^n_m(A,B_1,\ldots,B_n)]=\widehat{\sigma}_{nd,m}[\widehat{S}^n_m(\{x_i\},B_1,\ldots,B_n)]$
$=\widehat{\sigma}_{nd,m}[B_i]=\widehat{S}^n_m(\{x_i\},\widehat{\sigma}_{nd,m}[B_1],\ldots_n,\widehat{\sigma}_{nd,m}[B_n])$
$=\widehat{S}^n_m(\widehat{\sigma}_{nd,n}[\{x_i\}],\widehat{\sigma}_{nd,m}[B_1],\ldots_n,\widehat{\sigma}_{nd,m}[B_n])$.
Assume now that $t=f_i(t_1,\ldots,t_{n_i})$ and

$$\widehat{\sigma}_{nd,m}[\widehat{S}^n_m(\{t_k\},B_1,\ldots,B_n)]=\widehat{S}^n_m(\widehat{\sigma}_{nd,n}[\{t_k\}],\widehat{\sigma}_{nd,m}[B_1],\ldots,\widehat{\sigma}_{nd,m}[B_n])$$

is satisfied, for all $1\leq k\leq n_i$. By Theorem 2.2, we obtain
$\widehat{\sigma}_{nd,m}[\widehat{S}^n_m(\{f_i(t_1,\ldots,t_{n_i})\},B_1,\ldots,B_n)]$

$$\begin{aligned}
&= \widehat{\sigma}_{nd,m}[\{f_i(r_1,\ldots,r_{n_i})\mid r_k\in\widehat{S}^n_m(\{t_k\},B_1,\ldots,B_n),1\leq k,\leq n_i\}]\\
&= \bigcup\widehat{\sigma}_{nd,m}[\{f_i(r_1\in\widehat{S}^n_m(\{t_1\},B_1,\ldots,B_n),\ldots,r_{n_i}\in\widehat{S}^n_m(\{t_{n_i}\},B_1,\ldots,B_n)\}]\\
&= \bigcup\widehat{S}^{n_i}_m(\sigma_{nd,n_i}(f_i),\widehat{\sigma}_{nd,m}[\{r_1\in\widehat{S}^n_m(\{t_1\},B_1,\ldots,B_n)\}],\ldots,\\
&\qquad \widehat{\sigma}_{nd,m}[\{r_{n_i}\in\widehat{S}^n_m(\{t_{n_i}\},B_1,\ldots,B_n)\}])\\
&= \widehat{S}^{n_i}_m(\sigma_{nd,n_i}(f_i),\widehat{\sigma}_{nd,m}[\widehat{S}^n_m(\{t_1\},B_1,\ldots,B_n)],\ldots,\widehat{\sigma}_{nd,m}[\widehat{S}^n_m(\{t_{n_i}\},B_1,\ldots,B_n)])\\
&= \widehat{S}^{n_i}_m(\sigma_{nd,n_i}(f_i),\widehat{S}^n_m(\widehat{\sigma}_{nd,n}[\{t_1\}],\widehat{\sigma}_{nd,m}[B_1],\ldots,\widehat{\sigma}_{nd,m}[B_n]),\ldots,\\
&\qquad \widehat{S}^n_m(\widehat{\sigma}_{nd,n}[\{t_{n_i}\}],\widehat{\sigma}_{nd,m}[B_1],\ldots,\widehat{\sigma}_{nd,m}[B_n]))\\
&= \widehat{S}^n_m(\widehat{S}^{n_i}_{nd,n}(\widehat{\sigma}_{n_i}(f_i),\widehat{\sigma}_{nd,n}[t_1],\ldots,\widehat{\sigma}_{nd,m}[t_{n_i}]),\widehat{\sigma}_{nd,m}[B_1],\ldots,\widehat{\sigma}_{nd,m}[B_n])\\
&= \widehat{S}^n_m(\widehat{\sigma}_{nd,n}[\{f_i(t_1,\ldots,t_{n_i})\}],\widehat{\sigma}_{nd,m}[B_1],\ldots,\widehat{\sigma}_{nd,m}[B_n]).
\end{aligned}$$

Let A be arbitrary nonsingleton subset of $W^{fv}_\tau(X_n)$. Then
$\widehat{\sigma}_{nd,m}[\widehat{S}^n_m(A,B_1,\ldots,B_n)]$

$$\begin{aligned}
&= \widehat{\sigma}_{nd,m}[\bigcup_{a\in A}\widehat{S}^n_m(\{a\},B_1,\ldots,B_n)]\\
&= \bigcup_{a\in A}\widehat{\sigma}_{nd,m}[\widehat{S}^n_m(\{a\},B_1,\ldots,B_n)]\\
&= \bigcup_{a\in A}\widehat{S}^n_m(\widehat{\sigma}_{nd,n}[\{a\}],\widehat{\sigma}_{nd,m}[B_1],\ldots,\widehat{\sigma}_{nd,m}[B_n])\\
&= \bigcup_{a\in A}(\bigcup_{t\in\widehat{\sigma}_{nd,n}[\{a\}]}\widehat{S}^n_m(\{t\},\widehat{\sigma}_{nd,m}[B_1],\ldots,\widehat{\sigma}_{nd,m}[B_n]))\\
&= \bigcup_{t\in\widehat{\sigma}_{nd,n}[A]}\widehat{S}^n_m(\{t\},\widehat{\sigma}_{nd,m}[B_1],\ldots,\widehat{\sigma}_{nd,m}[B_n])\\
&= \widehat{S}^n_m(\widehat{\sigma}_{nd,n}[A],\widehat{\sigma}_{nd,m}[B_1],\ldots,\widehat{\sigma}_{nd,m}[B_n]).
\end{aligned}$$

The proof is complete. □

Finally, we study a strong connection between the semigroup $(Hyp^{fv}(\tau),\circ_h)$ [26] and our semigroup $(Hyp^{fv}_{nd}(\tau),\circ_{nd})$ via a construction of monomorphisms. For this, some technical lemmas are needed.

Lemma 3.8. *For any terms of a fixed variable $t\in W^{fv}_\tau(X_n), s_1,\ldots,s_n\in W^{fv}_\tau(X_m)$, we have*

$$\{S^n_m(t,s_1,\ldots,s_n)\}=\widehat{S}^n_m(\{t\},\{s_1\},\ldots,\{s_n\}).$$

Proof. We give a proof by induction on the complexity of the term t. If t is a variable from X_n, then the proof is easy. Suppose now that $t=f_i(t_1,\ldots,t_{n_i})$ and

$$\{S^n_m(t_k,s_1,\ldots,s_n)\}=\widehat{S}^n_m(\{t_k\},\{s_1\},\ldots,\{s_n\}),$$

for every $k=1,\ldots,n_i$. By the definitions of many-sorted superpositions S^n_m and $\widehat{S}^n_m$, we obtain

$$\begin{aligned}
&\{S^n_m(t,s_1,\ldots,s_n)\}\\
&= \{S^n_m(f_i(t_1,\ldots,t_{n_i}),s_1,\ldots,s_n)\}\\
&= \{f_i(S^n_m(t_1,s_1,\ldots,s_n),\ldots,S^n_m(t_{n_i},s_1,\ldots,s_n))\}\\
&= \{f_i(r_1,\ldots,r_{n_i})\mid r_k\in\{S^n_m(t_k,s_1,\ldots,s_n)\},1\le k\le n_i\}\\
&= \{f_i(r_1,\ldots,r_{n_i})\mid r_k\in\widehat{S}^n_m(\{t_k\},\{s_1\},\ldots,\{s_n\}),1\le k\le n_i\}\\
&= \widehat{S}^n_m(\{f_i(t_1,\ldots,t_{n_i})\},\{s_1\},\ldots,\{s_{n_i}\})\\
&= \widehat{S}^n_m(\{t\},\{s_1\},\ldots,\{s_{n_i}\}).
\end{aligned}$$

This completes the proof. □

Lemma 3.9. *For any terms of a fixed variable t, a hypersubstitution of fixed variables of type τ σ, and a non-deterministic hypersubstitution with fixed variables σ_{nd},*

$$\{\widehat{\sigma}[t]\}=\widehat{\sigma}_{nd}[\{t\}].$$

Proof. This lemma can be proved by induction on the complexity of the term t. If t is a variable x_i, then $\{\widehat{\sigma}[t]\} = \{\widehat{\sigma}[x_i]\} = \{x_i\} = \widehat{\sigma}_{nd}[\{x_i\}] = \widehat{\sigma}_{nd}[\{t\}]$. Assume that $t = f_i(t_1, \ldots, t_{n_i})$ and the lemma is satisfied for t_k for every $1 \leq k \leq n_i$. By Lemma 3.8, we have $\{\widehat{\sigma}[t]\} = \{\widehat{\sigma}[f_i(t_1, \ldots, t_{n_i})]\} = \{S^{n_i}_m(\sigma(f_i), \widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_{n_i}])\} = \widehat{S}^{n_i}_m(\{\sigma(f_i)\}, \{\widehat{\sigma}[t_1]\}, \ldots, \{\widehat{\sigma}[t_{n_i}]\})$
$= \widehat{S}^{n_i}_m(\sigma_{nd}(f_i), \widehat{\sigma}_{nd}[\{t_1\}], \ldots, \widehat{\sigma}_{nd}[\{t_{n_i}\}]) = \widehat{\sigma}_{nd}[\{f_i(t_1, \ldots, t_{n_i})\}] = \widehat{\sigma}_{nd}[\{t\}]$. Therefore, the equation $\{\widehat{\sigma}[t]\} = \widehat{\sigma}_{nd}[\{t\}]$ holds. $\square$

Consequently, we have

Theorem 3.10. *$(Hyp^{fv}(\tau), \circ_h)$ can be embedded into $(Hyp^{fv}_{nd}(\tau), \circ_{nd})$.*

Proof. Let f_i be an n_i-ary operation symbol and let σ be a hypersubstitution of fixed variables of type τ. Then the mapping $\overline{\sigma_{nd}} : \{f_i \mid i \in I\} \rightarrow P(W^{fv}_\tau(X))$, defined by

$$\overline{\sigma_{nd}}(f_i) = \{\sigma(f_i)\},$$

is a mappping in $Hyp^{fv}_{nd}(\tau)$. This means that $\overline{\sigma_{nd}}$ is a non-deterministic hypersubstitution with fixed variables of type τ. To prove the existence of a monomorphism from $Hyp^{fv}(\tau)$ to $Hyp^{fv}_{nd}(\tau)$, we define a mapping $\beta : Hyp^{fv}(\tau) \rightarrow Hyp^{fv}_{nd}(\tau)$ by

$$\beta(\sigma) = \overline{\sigma_{nd}},$$

for all $\sigma \in Hyp^{fv}(\tau)$. It is clear that the mapping β is injective. Next, let σ, α be two mappings in $Hyp^{fv}(\tau)$. By Lemma 3.9, we have $\beta(\sigma \circ_h \alpha) = \overline{(\sigma \circ_h \alpha)_{nd}} = \overline{\sigma_{nd}} \circ_{nd} \overline{\alpha_{nd}} = \beta(\sigma) \circ_{nd} \beta(\alpha)$. This shows that β is a homomorphism. $\square$

4 Conclusion

The study of clone of tree languages began with the investigation of structural properties of the superposition operation on sets of terms. This paper surveyed algebraic structures of a specific kind of tree languages arising from terms of fixed variables. We obtained the many-sorted algebra of tree languages with fixed variables and gave related properties. Applying hypersubstitutions of fixed variables, we generalized the notion of non-deterministic hypersubstitutions with fixed variables. Several interesting properties of its extension were investigated. Finally, we proved that there is a monomorphism from the semigroup of hypersubstitutions with fixed variables to the semigroup of non-deterministic hypersubstitutions with fixed variables.

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