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Pythagorean Picture Fuzzy Hyperideals in Semihypergroups

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Abstract

In this article, we introduce Pythagorean picture fuzzy sets (PPFS) in semihypergroups. We define Pythagorean picture fizzy hyperideals and study some related properties. We characterize regular classes of semihypergroups through Pythagorean picture fuzzy sets hyperideals.

1 Introduction

Zadeh [13] introduced the idea of fuzzy sets, which is a function f from X to $[0,1]$. This concept not only brought a revolution in mathematics and logic, but also in science and technology. It is a very nice tool to handle uncertainties. After the introduction of this concept several authors applied it to other branches of mathematics, computer science, physics, chemistry and so on. In 1971, Rosenfeld [9] applied the fuzzy set theroy to the theroy of groups and defined the concept of fuzzy group. After that, many papers have been published in the field of fuzzy algebra, for instance, Kuroki [7] applied fuzzy set theory to the ideal theory of semigroups. Yager [12] studied Pythagorean membership grades in multicriteria decision making. Picture fuzzy set and Pythagorean fuzzy set is a generalization of the Zadeh's fuzzy

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set and the Antanassov's intuitionistic fuzzy set. The new concept could be useful for many computational intelligence problems. Basic operators of the picture fuzzy logic were studied by Cuong et al. [4] and also the basic operators of the Pythagorean fuzzy logic were studied by Yager et al. [11, 12]. Cuong [5] introduced Pythagorean picture fuzzy set (PPFS) which is a combination of picture fuzzy set with the Yager's Pythagorean fuzzy set.

Hyperstructure theory was developed in 1934 when Marty [8] defined hypergroups. Nowadays, hyperstructures have a lot of applications to several domains of mathematics and computer science. In [6], Davvaz introduced the concept of fuzzy hyperideals in a semihypergroup. A lot of papers and several books have been written on hyperstructure theory, see [2], [10]. A recent book on hyperstructures [3] pointed out their applications in rough set theory, codes, cryptography, automata, probability, geometry, lattices, binary relations, graphs and hypergraphs. Zhan et al. [14] presented some results on probability n-ary hypergroups. Recently fuzzy set theory has been well developed in the context of hyperalgebraic structure theory. Elif Ozel et al. [1] studied δ-Primary hyperideals on commutative hyperrings. Several papers have been written on fuzzy sets in several algebraic hyperstructures.

In this paper, we study Pythagorean picture fuzzy sets in semihypergroups. We provide some properties of Pythagorean picture fuzzy hyperideals in a semihypergroup.

We have divided this paper in four sections. Section 2 contains preliminaries and related definitions. Section 3 is based on Pythagorean picture fuzzy hyperideals in semihypergroups where related definitions with some examples are given and main results have been shown. Section 4 conclude the whole work in brief.

2 Preliminaries and basic definitions

In this section, we will recall the basic terms and definitions from the hyperstructure theory.

Definition 2.1. *A map* $\circ : H \times H \to \mathcal{P}^*(H)$ *is called a hyperoperation or join operation on the set* H*, where* H *is a non-empty set and* $\mathcal{P}^*(H) = \mathcal{P}(H)\backslash\{\emptyset\}$ *denotes the set of all non-empty subsets of* H*.*

A hypergroupoid is a set H together with a (binary) hyperoperation.

Definition 2.2. *A hypergroupoid* $(H, \circ)$*, which is associative, that is* $x \circ (y \circ z) = (x \circ y) \circ z$*,* $\forall x, y, z \in H$*, is called a semihypergroup.*

Let A and B be two non-empty subsets of H. Then we define

$$A \circ B = \bigcup_{a \in A, b \in B} a \circ b, \qquad a \circ B = \{a\} \circ B,\ A \circ b = A \circ \{b\}.$$

Definition 2.3. *Let $(H, \circ)$ be a semihypergroup. A non-empty subset A of H is called a sub semihypergroup of H if $x \circ y \subseteq A$ for every $x, y \in A$.*

Definition 2.4. *A semihypergroup H is called commutative if for all $x, y \in H$, we have $x \circ y = y \circ x$.*

Definition 2.5. *A non-empty subset A of a semihypergroup H is a right (left) hyperideal of H if $A \circ H \subseteq A$ $(H \circ A \subseteq A)$, and is a hyperideal of H if it is both a right and a left hyperideal.*

Definition 2.6. *A sub semihypergroup B of a semihypergroup H is called a bi-hyperideal of H if $B \circ H \circ B \subseteq B$.*

Definition 2.7. *A non-empty subset I of a semihypergroup H is called an interior hyperideal of H if $H \circ I \circ H \subseteq I$.*

Definition 2.8. *A sub semihypergroup B of a semihypergroup H is called a $(1, 2)$-hyperideal of H if $B \circ H \circ B^2 \subseteq B$.*

We call a semihypergroup H a regular if for every $x \in H$, $x \in x \circ y \circ x$, for some $y \in H$. We call a semihypergroup H an intra-regular semihypergroup if for every $x \in H$, $x \in y \circ x \circ x \circ z$, for some $y, z \in H$. We call a semihypergroup H a completely regular semihypergroup if for every $a \in H$, $a \in a \circ a \circ x \circ a \circ a$, for some $x \in H$.

Definition 2.9. *[6] Let μ be a fuzzy subset of a semihypergroup H. Then μ is called*

(1) a fuzzy left hyperideal of H if

$$\mu(y) \leq \inf_{z \in x \circ y} \{\mu(z)\},\ \textit{for all } x, y \in H;$$

(2) a fuzzy right hyperideal of H if

$$\mu(x) \leq \inf_{z \in x \circ y} \{\mu(z)\},\ \textit{for all } x, y \in H;$$

(3) a fuzzy hyperideal or fuzzy two-sided hyperideal if it is both a fuzzy left hyperideal and fuzzy right hyperideal.

Definition 2.10. *[4] A picture fuzzy set A on a universe X is an object of the form*

$$\mathcal{A} = \{\langle x, \alpha_{\mathcal{A}}(x), \beta_{\mathcal{A}}(x), \delta_{\mathcal{A}}(x)\rangle : x \in X\}.$$

where $\alpha_{\mathcal{A}}(x), \beta_{\mathcal{A}}(x), \delta_{\mathcal{A}}(x)$ *are respectively called the degree of positive membership, the degree of neutral membership, the degree of negative membership of* x *in* A*, and the following conditions are satisfied*

$$0 \leq \alpha_{\mathcal{A}}(x), \beta_{\mathcal{A}}(x), \delta_{\mathcal{A}}(x) \leq 1 \quad \textit{and} \quad 0 \leq \alpha_{\mathcal{A}}(x) + \beta_{\mathcal{A}}(x) + \delta_{\mathcal{A}}(x) \leq 1.$$

Then, for all x *in* X, $\zeta_{\mathcal{A}}(x) = 1 - (\alpha_{\mathcal{A}}(x) + \beta_{\mathcal{A}}(x) + \delta_{\mathcal{A}}(x))$ *is called the degree of refusal membership of* x *in* A.

Definition 2.11. *[11] A Pythagorean fuzzy set A on a universe X is an object of the form*

$$\mathcal{A} = \{\langle x, \alpha_{\mathcal{A}}(x), \beta_{\mathcal{A}}(x)\rangle : x \in X\}.$$

where $\alpha_{\mathcal{A}}(x), \beta_{\mathcal{A}}(x)$ *are respectively called the degree of membership and the degree non-membership of* x *in* A*, and the following condition is satisfied*

$$0 \leq (\alpha_{\mathcal{A}}(x))^2 + (\beta_{\mathcal{A}}(x))^2 \leq 1.$$

3 Pythagorean picture fuzzy hyperideals in semihypergroups

Cuong [5] introduced the concept of Pythagorean picture fuzzy set (PPFS) which is a combination of picture fuzzy set with the Yager's Pythagorean fuzzy set. Pythagorean picture fuzzy set defined on a non-empty set X as objects having the form:

$$\mathcal{P} = \{\langle x, \alpha_{\mathcal{P}}(x), \beta_{\mathcal{P}}(x), \delta_{\mathcal{P}}(x)\rangle : x \in X\}.$$

where $\alpha_{\mathcal{A}}(x), \beta_{\mathcal{A}}(x), \delta_{\mathcal{A}}(x)$ are respectively called the degree of positive membership, the degree of neutral membership, the degree of negative membership of x in A, and the following conditions are satisfied

$$0 \leq \alpha_{\mathcal{A}}(x), \beta_{\mathcal{A}}(x), \delta_{\mathcal{A}}(x) \leq 1 \quad \text{and} \quad 0 \leq (\alpha_{\mathcal{A}}(x))^2 + (\beta_{\mathcal{A}}(x))^2 + (\delta_{\mathcal{A}}(x))^2 \leq 1.$$

For the sake of simplicity PPFS is denoted by $\mathcal{P} = (\alpha_{\mathcal{P}}, \beta_{\mathcal{P}}, \delta_{\mathcal{P}})$.

Definition 3.1. *Let $\mathcal{P} = (\alpha_{\mathcal{P}}, \beta_{\mathcal{P}}, \delta_{\mathcal{P}})$ and $\mathcal{Q} = (\alpha_{\mathcal{Q}}, \beta_{\mathcal{Q}}, \delta_{\mathcal{Q}})$ be two PPF subsets of a semihypergroup H. Then for all $x \in H$, their intersection $\mathcal{P} \cap \mathcal{Q}$ is defined by*

$$\mathcal{P} \cap \mathcal{Q} = \{\langle x, \alpha_{\mathcal{P}\cap\mathcal{Q}}(x), \beta_{\mathcal{P}\cap\mathcal{Q}}(x), \delta_{\mathcal{P}\cap\mathcal{Q}}(x)\rangle : x \in H\},$$

where

$$\begin{aligned} \alpha_{\mathcal{P}\cap\mathcal{Q}}(x) &= (\alpha_{\mathcal{P}} \wedge \alpha_{\mathcal{Q}})(x) = \alpha_{\mathcal{P}}(x) \wedge \alpha_{\mathcal{Q}}(x) \\ \beta_{\mathcal{P}\cap\mathcal{Q}}(x) &= (\beta_{\mathcal{P}} \wedge \beta_{\mathcal{Q}})(x) = \beta_{\mathcal{P}}(x) \wedge \beta_{\mathcal{Q}}(x) \\ \text{and } \delta_{\mathcal{P}\cap\mathcal{Q}}(x) &= (\delta_{\mathcal{P}} \vee \delta_{\mathcal{Q}})(x) = \delta_{\mathcal{P}}(x) \vee \delta_{\mathcal{Q}}(x). \end{aligned}$$

Their union $\mathcal{P} \cup \mathcal{Q}$ is defined by

$$\mathcal{P} \cup \mathcal{Q} = \{\langle x, \alpha_{\mathcal{P}\cup\mathcal{Q}}(x), \beta_{\mathcal{P}\cup\mathcal{Q}}(x), \delta_{\mathcal{P}\cup\mathcal{Q}}(x)\rangle : x \in H\},$$

where

$$\begin{aligned} \alpha_{\mathcal{P}\cup\mathcal{Q}}(x) &= (\alpha_{\mathcal{P}} \vee \alpha_{\mathcal{Q}})(x) = \alpha_{\mathcal{P}}(x) \vee \alpha_{\mathcal{Q}}(x) \\ \beta_{\mathcal{P}\cup\mathcal{Q}}(x) &= (\beta_{\mathcal{P}} \vee \beta_{\mathcal{Q}})(x) = \beta_{\mathcal{P}}(x) \vee \beta_{\mathcal{Q}}(x) \\ \text{and } \delta_{\mathcal{P}\cup\mathcal{Q}}(x) &= (\delta_{\mathcal{P}} \wedge \delta_{\mathcal{Q}})(x) = \delta_{\mathcal{P}}(x) \wedge \delta_{\mathcal{Q}}(x). \end{aligned}$$

Definition 3.2. *Let $\mathcal{P} = (\alpha_{\mathcal{P}}, \beta_{\mathcal{P}}, \delta_{\mathcal{P}})$ and $\mathcal{Q} = (\alpha_{\mathcal{Q}}, \beta_{\mathcal{Q}}, \delta_{\mathcal{Q}})$ be two PPF subsets of a semihypergroup H. Then their product $\mathcal{P} \circ \mathcal{Q}$ is defined by*

$$\mathcal{P} \circ \mathcal{Q} = \{\langle x, \alpha_{\mathcal{P}\circ\mathcal{Q}}(x), \beta_{\mathcal{P}\circ\mathcal{Q}}(x), \delta_{\mathcal{P}\circ\mathcal{Q}}(x)\rangle : x \in H\},$$

where

$$\alpha_{\mathcal{P}\circ\mathcal{Q}} : H \longrightarrow [0,1] | x \longmapsto \alpha_{\mathcal{P}\circ\mathcal{Q}}(x) := \begin{cases} \sup\limits_{x \in y \circ z} \{\alpha_{\mathcal{P}}(y) \wedge \alpha_{\mathcal{Q}}(z)\} & \text{if } x \in y \circ z \\ 0 & \text{otherwise,} \end{cases}$$

$$\beta_{\mathcal{P}\circ\mathcal{Q}} : H \longrightarrow [0,1] | x \longmapsto \beta_{\mathcal{P}\circ\mathcal{Q}}(x) := \begin{cases} \sup\limits_{x \in y \circ z} \{\beta_{\mathcal{P}}(y) \wedge \beta_{\mathcal{Q}}(z)\} & \text{if } x \in y \circ z \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\delta_{\mathcal{P}\circ\mathcal{Q}} : H \longrightarrow [0,1] | x \longmapsto \delta_{\mathcal{P}\circ\mathcal{Q}}(x) := \begin{cases} \inf\limits_{x \in y \circ z} \{\delta_{\mathcal{P}}(y) \vee \delta_{\mathcal{Q}}(z)\} & \text{if } x \in y \circ z \\ 1 & \text{otherwise,} \end{cases}$$

for some $x, y, z \in H$.

Definition 3.3. *Let H be a semihypergroup. A PPF subset $\mathcal{P} = (\alpha_{\mathcal{P}}, \beta_{\mathcal{P}}, \delta_{\mathcal{P}})$ of H is called*

(1) a PPF sub-semihypergroup of H if

$$\begin{aligned} \inf_{x\in y\circ z} \alpha_{\mathcal{P}}(x) &\geq \min\{\alpha_{\mathcal{P}}(y), \alpha_{\mathcal{P}}(z)\}, \\ \inf_{x\in y\circ z} \beta_{\mathcal{P}}(x) &\geq \min\{\beta_{\mathcal{P}}(y), \beta_{\mathcal{P}}(z)\} \\ \text{and} \quad \sup_{x\in y\circ z} \delta_{\mathcal{P}}(x) &\leq \max\{\delta_{\mathcal{P}}(y), \delta_{\mathcal{P}}(z)\}, \end{aligned}$$

for all $x, y, z \in H$.

(2) a PPF left hyperideal of H if

$$\inf_{x\in y\circ z} \alpha_{\mathcal{P}}(x) \geq \alpha_{\mathcal{P}}(z), \quad \inf_{x\in y\circ z} \beta_{\mathcal{P}}(x) \geq \beta_{\mathcal{P}}(z) \quad \text{and} \quad \sup_{x\in y\circ z} \delta_{\mathcal{P}}(x) \leq \delta_{\mathcal{P}}(z),$$

for all $x, y, z \in H$.

(3) a PPF right hyperideal of H if

$$\inf_{x\in y\circ z} \alpha_{\mathcal{P}}(x) \geq \alpha_{\mathcal{P}}(y), \quad \inf_{x\in y\circ z} \beta_{\mathcal{P}}(x) \geq \beta_{\mathcal{P}}(y) \quad \text{and} \quad \sup_{x\in y\circ z} \delta_{\mathcal{P}}(x) \leq \delta_{\mathcal{P}}(y),$$

for all $x, y, z \in H$.

(4) a PPF hyperideal of H if

$$\begin{aligned} \inf_{x\in y\circ z} \alpha_{\mathcal{P}}(x) &\geq \max\{\alpha_{\mathcal{P}}(y), \alpha_{\mathcal{P}}(z)\} \\ \inf_{x\in y\circ z} \beta_{\mathcal{P}}(x) &\geq \max\{\beta_{\mathcal{P}}(y), \beta_{\mathcal{P}}(z)\} \\ \text{and} \quad \sup_{x\in y\circ z} \delta_{\mathcal{P}}(x) &\leq \min\{\delta_{\mathcal{P}}(y), \delta_{\mathcal{P}}(z)\}, \end{aligned}$$

for all $x, y, z \in H$.

Definition 3.4. *Let H be a semihypergroup. A PPF sub-semihypergroup $\mathcal{P} = (\alpha_{\mathcal{P}}, \beta_{\mathcal{P}}, \delta_{\mathcal{P}})$ of H is called a PPF bi-hyperideal of H if*

$$\begin{aligned} \inf_{a\in x\circ y\circ z} \alpha_{\mathcal{P}}(a) &\geq \min\{\alpha_{\mathcal{P}}(x), \alpha_{\mathcal{P}}(z)\} \\ \inf_{a\in x\circ y\circ z} \beta_{\mathcal{P}}(a) &\geq \min\{\beta_{\mathcal{P}}(x), \beta_{\mathcal{P}}(z)\} \\ \text{and} \quad \sup_{a\in x\circ y\circ z} \delta_{\mathcal{P}}(a) &\leq \max\{\delta_{\mathcal{P}}(x), \delta_{\mathcal{P}}(z)\}, \end{aligned}$$

for all $x, y, z \in H$.

Example 3.5. *Let $H = \{a, b, c, d, e\}$ with the binary hyperoperation "$\circ$" defined below:*

$\circ$	a	b	c	d	e
a	$\{a, b\}$	$\{b, e\}$	c	$\{c, d\}$	e
b	$\{b, e\}$	e	c	$\{c, d\}$	e
c	c	c	c	c	c
d	$\{c, d\}$	$\{c, d\}$	c	d	$\{c, d\}$
e	e	e	c	$\{c, d\}$	e

Clearly $(H, \circ)$ is a semihypergroup. Now we define a PPF set $\mathcal{P} = (\alpha_{\mathcal{P}}, \beta_{\mathcal{P}}, \delta_{\mathcal{P}})$ on H as:

$\mathcal{P}$	$\alpha_{\mathcal{P}}$	$\beta_{\mathcal{P}}$	$\delta_{\mathcal{P}}$
a	0.1	0.2	0.8
b	0.1	0.2	0.8
c	0.7	0.7	0.1
d	0.5	0.6	0.6
e	0.4	0.3	0.8

Then by routine calculations, $\mathcal{P} = (\alpha_{\mathcal{P}}, \beta_{\mathcal{P}}, \delta_{\mathcal{P}})$ is a PPF bi-hyperideal of H. Also $0 \leq (\alpha_{\mathcal{P}}(x))^2 + (\beta_{\mathcal{P}}(x))^2 + (\delta_{\mathcal{P}}(x))^2 \leq 1$.

Definition 3.6. *Let H be a semihypergroup. A PPF subset $\mathcal{P} = (\alpha_{\mathcal{P}}, \beta_{\mathcal{P}}, \delta_{\mathcal{P}})$ of H is called a PPF interior hyperideal of H if*

$$\begin{aligned} \inf_{a \in x \circ y \circ z} \alpha_{\mathcal{P}}(a) &\geq \alpha_{\mathcal{P}}(y) \\ \inf_{a \in x \circ y \circ z} \beta_{\mathcal{P}}(a) &\geq \beta_{\mathcal{P}}(y) \\ \text{and} \sup_{a \in x \circ y \circ z} \delta_{\mathcal{P}}(a) &\leq \delta_{\mathcal{P}}(y), \end{aligned}$$

for all $x, y, z \in H$.

Example 3.7. *Let $H = \{a, b, c, d\}$ with the binary hyperoperation "$\circ$" defined below:*

$\circ$	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	$\{a, b\}$	a
d	a	a	$\{a, b\}$	$\{a, b\}$

Clearly $(H, \circ)$ is a semihypergroup. Now we define a PPF set $\mathcal{P} = (\alpha_{\mathcal{P}}, \beta_{\mathcal{P}}, \delta_{\mathcal{P}})$

on H *as:*

$\mathcal{P}$	$\alpha_{\mathcal{P}}$	$\beta_{\mathcal{P}}$	$\delta_{\mathcal{P}}$
a	0.7	0.6	0.3
b	0.4	0.5	0.7
c	0.3	0.4	0.8
d	0.2	0.3	0.9

Then by routine calculations, $\mathcal{P} = (\alpha_{\mathcal{P}}, \beta_{\mathcal{P}}, \delta_{\mathcal{P}})$ *is a PPF interior hyperideal of* H*. Also* $0 \leq (\alpha_{\mathcal{P}}(x))^2 + (\beta_{\mathcal{P}}(x))^2 + (\delta_{\mathcal{P}}(x))^2 \leq 1$.

Definition 3.8. *Let* H *be a semihypergroup. A PPF sub-semihypergroup* $\mathcal{P} = (\alpha_{\mathcal{P}}, \beta_{\mathcal{P}}, \delta_{\mathcal{P}})$ *of* H *is called a PPF* $(1,2)$*-hyperideal of* H *if*

$$\begin{aligned} \inf_{a \in x \circ w \circ (y \circ z)} \alpha_{\mathcal{P}}(a) &\geq \min\{\alpha_{\mathcal{P}}(x), \alpha_{\mathcal{P}}(y), \alpha_{\mathcal{P}}(z)\} \\ \inf_{a \in x \circ w \circ (y \circ z)} \beta_{\mathcal{P}}(a) &\geq \min\{\beta_{\mathcal{P}}(x), \beta_{\mathcal{P}}(y), \beta_{\mathcal{P}}(z)\} \\ \text{and} \quad \sup_{a \in x \circ w \circ (y \circ z)} \delta_{\mathcal{P}}(a) &\leq \max\{\delta_{\mathcal{P}}(x), \delta_{\mathcal{P}}(y), \delta_{\mathcal{P}}(z)\}, \end{aligned}$$

for all $w, x, y, z \in H$.

Definition 3.9. *Let* $\mathcal{P} = (\alpha_{\mathcal{P}}, \beta_{\mathcal{P}}, \delta_{\mathcal{P}})$ *be a PPF set and* $(s, t, u) \in [0, 1] \times [0, 1] \times [0, 1]$. *Define:*

(1) the sets $\mathcal{P}_s = \{x \in H \mid \alpha_{\mathcal{P}}(x) \geq s\}$, $\mathcal{P}_t = \{x \in H \mid \beta_{\mathcal{P}}(x) \geq t\}$ *and* $\mathcal{P}_u = \{x \in H \mid \delta_{\mathcal{P}}(x) \leq u\}$*, are called s-cut, t-cut and u-cut of* $\mathcal{P} = (\alpha_{\mathcal{P}}, \beta_{\mathcal{P}}, \delta_{\mathcal{P}})$*, respectively,*

(2) the sets $\mathcal{P}_s^{>} = \{x \in H \mid \alpha_{\mathcal{P}}(x) > s\}$, $\mathcal{P}_t^{>} = \{x \in H \mid \beta_{\mathcal{P}}(x) > t\}$ *and* $\mathcal{P}_u^{<} = \{x \in H \mid \delta_{\mathcal{P}}(x) < u\}$*, are called strong s-cut, strong t-cut and strong u-cut of* $\mathcal{P} = (\alpha_{\mathcal{P}}, \beta_{\mathcal{P}}, \delta_{\mathcal{P}})$*, respectively,*

(3) the set $H_{\mathcal{P}}^{(s,t,u)} = \{x \in H \mid \alpha_{\mathcal{P}}(x) \geq s,\ \beta_{\mathcal{P}}(x) \geq t,\ \delta_{\mathcal{P}}(x) \leq u\}$ *is called an* (s, t, u)*-level subset of* $\mathcal{P}$*,*

(4) the set ${}^{S}H_{\mathcal{P}}^{(s,t,u)} = \{x \in H \mid \alpha_{\mathcal{P}}(x) > s,\ \beta_{\mathcal{P}}(x) > t,\ \delta_{\mathcal{P}}(x) < u\}$ *is called a strong* (s, t, u)*-level subset of* $\mathcal{P}$*,*

(5) the set of all $(s, t, u) \in Im(\alpha_{\mathcal{P}}) \times Im(\beta_{\mathcal{P}}) \times Im(\delta_{\mathcal{P}})$ *is called the image of* $\mathcal{P} = (\alpha_{\mathcal{P}}, \beta_{\mathcal{P}}, \delta_{\mathcal{P}})$.

4 Main results

In this section we have given all results based on Pythagorean Picture Fuzzy Hyperideals in Semihypergroups.

Theorem 4.1. *Let $\mathcal{P} = (\alpha_{\mathcal{P}}, \beta_{\mathcal{P}}, \delta_{\mathcal{P}})$ be a PPF subset of H such that the least upper bound s_0,t_0 of $Im(\alpha_{\mathcal{P}}), Im(\beta_{\mathcal{P}})$ and the greatest lower bound u_0 of $Im(\delta_{\mathcal{P}})$ exist. Then the following conditions are equivalent:*

(1) $\mathcal{P}$ is a PPF sub-semihypergroup of H,

(2) For all $(s, t, u) \in Im(\alpha_{\mathcal{P}}) \times Im(\beta_{\mathcal{P}}) \times Im(\delta_{\mathcal{P}})$, the non-empty level subset $H_{\mathcal{P}}^{(s,t,u)}$ of $\mathcal{P}$ is a sub-semihypergroup of H.

(3) For all $(s, t, u) \in Im(\alpha_{\mathcal{P}}) \times Im(\beta_{\mathcal{P}}) \times Im(\delta_{\mathcal{P}}) \setminus (s_0, t_0, u_0)$, the non-empty strong level subset ${}^{S}H_{\mathcal{P}}^{(s,t,u)}$ of $\mathcal{P}$ is a sub-semihypergroup of H.

(4) For all $(s,\ t, u) \in [0,\ 1] \times [0,\ 1] \times [0,\ 1]$, the non-empty strong level subset ${}^{S}H_{\mathcal{P}}^{(s,t,u)}$ of $\mathcal{P}$ is a sub-semihypergroup of H.

(5) For all $(s,\ t, u) \in [0,\ 1] \times [0,\ 1] \times [0,\ 1]$, the non-empty level subset $H_{\mathcal{P}}^{(s,t,u)}$ of $\mathcal{P}$ is a sub-semihypergroup of H.

Proof. Let H be a semihypergroup.

($1 \to 4$) Let $\mathcal{P}$ be a PPF sub-semihypergroup of H, $(s,\ t, u) \in [0,\ 1] \times [0,\ 1]\times[0,\ 1]$ and $x, y \in {}^{S}H_{\mathcal{P}}^{(s,t,u)}$. Then we have $\alpha_{\mathcal{P}}(x), \alpha_{\mathcal{P}}(y) > s$, $\beta_{\mathcal{P}}(x), \beta_{\mathcal{P}}(y) > t$ and $\delta_{\mathcal{P}}(x), \delta_{\mathcal{P}}(y) < u$. Thus,

$$\begin{aligned} \min\{\alpha_{\mathcal{P}}(x), \alpha_{\mathcal{P}}(y)\} &> s \\ \min\{\beta_{\mathcal{P}}(x), \beta_{\mathcal{P}}(y)\} &> t \\ \text{and} \quad \max\{\delta_{\mathcal{P}}(x), \delta_{\mathcal{P}}(y)\} &< u. \end{aligned}$$

Since $\mathcal{P} = (\alpha_{\mathcal{P}}, \beta_{\mathcal{P}}, \delta_{\mathcal{P}})$ is a sub-semihypergroup of H, so $\inf\limits_{z\in x\circ y} \alpha_{\mathcal{P}}(z) > s$, $\inf\limits_{z\in x\circ y} \beta_{\mathcal{P}}(z) > t$ and $\sup\limits_{z\in x\circ y} \delta_{\mathcal{P}}(z) < u$. Thus $x \circ y \subseteq {}^{S}H_{\mathcal{P}}^{(s,t,u)}$. Hence ${}^{S}H_{\mathcal{P}}^{(s,t,u)}$ is a sub-semihypergroup of H.

($4 \to 3$) It is clear.

($3 \to 2$) Let $(s, t, u) \in Im(\alpha_{\mathcal{P}}) \times Im(\beta_{\mathcal{P}}) \times Im(\delta_{\mathcal{P}})$. Then $H_{\mathcal{P}}^{(s,t,u)}$ is non-empty. Since $H_{\mathcal{P}}^{(s,t,u)} = \bigcap\limits_{s>0,\ t>0,\ u<0}^{H} H_{\mathcal{P}}^{(s,t,u)}$, where $s \in Im(\alpha_{\mathcal{P}}) \setminus s_0$, $t \in Im(\beta_{\mathcal{P}}) \setminus t_0$ and $u \in Im(\delta_{\mathcal{P}}) \setminus u_0$. Then by (3) we get that $H_{\mathcal{P}}^{(s,t,u)}$ is a sub-semihypergroup of H.

($2 \to 5$) Let $(s,\ t, u) \in [0,\ 1] \times [0,\ 1] \times [0,\ 1]$ and $H_{\mathcal{P}}^{(s,t,u)}$ be non-empty. Suppose that $x, y \in H_{\mathcal{P}}^{(s,t,u)}$. Then we have $\alpha_{\mathcal{P}}(x), \alpha_{\mathcal{P}}(y) \geq s$, $\beta_{\mathcal{P}}(x), \beta_{\mathcal{P}}(y) \geq t$ and $\delta_{\mathcal{P}}(x), \delta_{\mathcal{P}}(y) \leq u$. Let $p = \min\{\alpha_{\mathcal{P}}(x), \alpha_{\mathcal{P}}(y)\}$, $q = \min\{\beta_{\mathcal{P}}(x), \beta_{\mathcal{P}}(y)\}$ and $r = \max\{\delta_{\mathcal{P}}(x), \delta_{\mathcal{P}}(y)\}$. It is clear that $p \geq s$, $q \geq t$ and $r \leq u$. Thus $x, y \in H_{\mathcal{P}}^{(s,t,u)}$ and $p \in Im(\alpha_{\mathcal{P}})$, $q \in Im(\beta_{\mathcal{P}})$ and $r \in Im(\delta_{\mathcal{P}})$, by (2) $H_{\mathcal{P}}^{(p,q,r)}$

is a sub-semihypergroup of H, hence $x \circ y \subseteq H_{\mathcal{P}}^{(p,q,r)}$. Then we have

$$\begin{aligned} \inf_{z \in x \circ y} \alpha_{\mathcal{P}}(z) &\geq p \geq s \\ \inf_{z \in x \circ y} \beta_{\mathcal{P}}(z) &\geq q \geq u \\ \text{and} \quad \sup_{z \in x \circ y} \delta_{\mathcal{P}}(z) &\leq r \leq u. \end{aligned}$$

Therefore $x \circ y \subseteq H_{\mathcal{P}}^{(s,t,u)}$. Then $H_{\mathcal{P}}^{(s,t,u)}$ is a sub-semihypergroup of H.

$(5 \to 1)$ Assume that the non-empty set $H_{\mathcal{P}}^{(s,t,u)}$ is a sub-semihypergroup of H, for any $(s, t, u) \in [0, 1] \times [0, 1] \times [0, 1]$. Let $x, y \in H$. Let us take $s = \min\{\alpha_{\mathcal{P}}(x), \alpha_{\mathcal{P}}(y)\}$, $t = \min\{\beta_{\mathcal{P}}(x), \beta_{\mathcal{P}}(y)\}$ and $u = \max\{\delta_{\mathcal{P}}(x), \delta_{\mathcal{P}}(y)\}$. Then $\alpha_{\mathcal{P}}(x), \alpha_{\mathcal{P}}(y) \geq s$, $\beta_{\mathcal{P}}(x), \beta_{\mathcal{P}}(y) \geq t$ and $\delta_{\mathcal{P}}(x), \delta_{\mathcal{P}}(y) \leq u$. Thus $x, y \in H_{\mathcal{P}}^{(s,t,u)}$. Since $H_{\mathcal{P}}^{(s,t,u)}$ is a sub-semihypergroup of H, so $x \circ y \subseteq H_{\mathcal{P}}^{(s,t,u)}$. Thus,

$$\begin{aligned} \inf_{z \in x \circ y} \alpha_{\mathcal{P}}(z) &\geq s = \min\{\alpha_{\mathcal{P}}(x), \alpha_{\mathcal{P}}(y)\} \\ \inf_{z \in x \circ y} \beta_{\mathcal{P}}(z) &\geq t = \min\{\beta_{\mathcal{P}}(x), \beta_{\mathcal{P}}(y)\} \\ \text{and} \quad \sup_{z \in x \circ y} \delta_{\mathcal{P}}(z) &\leq u = \max\{\delta_{\mathcal{P}}(x), \delta_{\mathcal{P}}(y)\}. \end{aligned}$$

Thus $\mathcal{P}$ is a PPF sub-semihypergroup of H. This completes the proof. □

Theorem 4.2. *Let $\mathcal{P} = (\alpha_{\mathcal{P}}, \beta_{\mathcal{P}}, \delta_{\mathcal{P}})$ be a PPF subset of H such that the least upper bound s_0,t_0 of $Im(\alpha_{\mathcal{P}}), Im(\beta_{\mathcal{P}})$ and the greatest lower bound u_0 of $Im(\delta_{\mathcal{P}})$ exist. Then the following conditions are equivalent:*

(1) $\mathcal{P}$ is a PPF hyperideal (resp., left hyperideal, right hyperideal, bi-hyperideal, interior hyperideal) of H,

(2) For all $(s, t, u) \in Im(\alpha_{\mathcal{P}}) \times Im(\beta_{\mathcal{P}}) \times Im(\delta_{\mathcal{P}})$, the non-empty level subset $H_{\mathcal{P}}^{(s,t,u)}$ of $\mathcal{P}$ is a hyperideal (resp., left hyperideal, right hyperideal, bi-hyperideal, interior hyperideal) of H.

(3) For all $(s, t, u) \in Im(\alpha_{\mathcal{P}}) \times Im(\beta_{\mathcal{P}}) \times Im(\delta_{\mathcal{P}}) \setminus (s_0, t_0, u_0)$, the non-empty strong level subset ${}^S H_{\mathcal{P}}^{(s,t,u)}$ of $\mathcal{P}$ is a hyperideal (resp., left hyperideal, right hyperideal, bi-hyperideal, interior hyperideal) of H.

(4) For all $(s, t, u) \in [0, 1] \times [0, 1] \times [0, 1]$, the non-empty strong level subset ${}^S H_{\mathcal{P}}^{(s,t,u)}$ of $\mathcal{P}$ is a hyperideal (resp., left hyperideal, right hyperideal, bi-hyperideal, interior hyperideal) of H.

(5) For all $(s, t, u) \in [0, 1] \times [0, 1] \times [0, 1]$, the non-empty level subset $H_{\mathcal{P}}^{(s,t,u)}$ of $\mathcal{P}$ is a hyperideal (resp., left hyperideal, right hyperideal, bi-hyperideal, interior hyperideal) of H.

Proof. The proof is similar to the proof of Theorem 4.1. □

Definition 4.3. *Let X be a non-empty set. For any $A \subseteq X$ and $(l, m, n) \in [0,\ 1] \times [0,\ 1] \times [0,\ 1]$, the PPF subset $A^{(l,m,n)} = (\alpha_{l_A}, \beta_{m_A}, \delta_{n_A})$ of X is defined by*

$$\alpha_{l_A} : H \longrightarrow (0,1]|A \longmapsto \alpha_{l_A}(x) := \begin{cases} l & \text{if } x \in A \\ 0 & \text{if } x \notin A, \end{cases}$$

$$\beta_{m_A} : H \longrightarrow (0,1]|A \longmapsto \beta_{m_A}(x) := \begin{cases} m & \text{if } x \in A \\ 0 & \text{if } x \notin A, \end{cases}$$

and

$$\delta_{n_A} : H \longrightarrow (0,1]|A \longmapsto \delta_{n_A}(x) := \begin{cases} n & \text{if } x \in A \\ 1 & \text{if } x \notin A, \end{cases}$$

for all $x \in X$. In particular, when $l = 1$, $m = 1$ and $n = 0$, $A^{(l,m,n)}$ is said to be the characteristic function of A, denoted by $\chi_A = (\alpha_{\chi_A}, \beta_{\chi_A}, \delta_{\chi_A})$. When $A = \{x\}$, $A^{(l,m,n)}$ is said to be a PPF point with support x and values l, m and n and is denoted by $x_{(l,m,n)}$.

Theorem 4.4. *Let A be a non-empty subset of a semihypergroup H. Then A is a sub-semihypergroup (resp., hyperideal, left hyperideal, right hyperideal, bi-hyperideal, interior hyperideal and $(1,2)$-hyperideal) of H if and only if $A^{(l,m,n)} = (\alpha_{l_A}, \beta_{m_A}, \delta_{n_A})$ is a PPF sub-semihypergroup (resp., hyperideal, left hyperideal, right hyperideal, bi-hyperideal, interior hyperideal and $(1,2)$-hyperideal) of H.*

Proof. Let A be a sub-semihypergroup of H. For any $x, y \in H$, we have the following cases:

Case (1) : If $x, y \in A$. Since A is a sub-semihypergroup of H, we have $x \circ y \subseteq A$. Then $\alpha_{l_A}(x) = l$ and $\alpha_{l_A}(y) = l$. Therefore $\inf\limits_{z \in x \circ z} \alpha_{l_A}(z) = l = \min\{\alpha_{l_A}(x), \alpha_{l_A}(y)\}$. Similarly, $\inf\limits_{z \in x \circ z} \beta_{m_A}(z) = m = \min\{\beta_{m_A}(x), \beta_{m_A}(y)\}$ and $\sup\limits_{z \in x \circ z} \delta_{n_A}(z) = n = \max\{\delta_{n_A}(x), \delta_{n_A}(y)\}$.

Case (2) : If $x, y \notin A$. Then $\alpha_{l_A}(x) = 0$ and $\alpha_{l_A}(y) = 0$. Therefore $\inf\limits_{z \in x \circ z} \alpha_{l_A}(z) \geq 0 = \min\{\alpha_{l_A}(x), \alpha_{l_A}(y)\}$. Similarly, $\inf\limits_{z \in x \circ z} \beta_{m_A}(z) \geq 0 = \min\{\beta_{m_A}(x), \beta_{m_A}(y)\}$ and $\sup\limits_{z \in x \circ z} \delta_{n_A}(z) \leq 1 = \max\{\delta_{n_A}(x), \delta_{n_A}(y)\}$.

Case (3) : If $x \in A$ or $y \in A$. Then $\inf\limits_{z \in x \circ z} \alpha_{l_A}(z) \geq 0 = \min\{\alpha_{l_A}(x), \alpha_{l_A}(y)\}$, $\inf\limits_{z \in x \circ z} \beta_{m_A}(z) \geq 0 = \min\{\beta_{m_A}(x), \beta_{m_A}(y)\}$ and $\sup\limits_{z \in x \circ z} \delta_{n_A}(z) \leq 1 = \max\{\delta_{n_A}(x), \delta_{n_A}(y)\}$.

Hence $A^{(l,m,n)} = (\alpha_{l_A}, \beta_{m_A}, \delta_{n_A})$ is a PPF sub-semihypergroup of H.

Conversely, suppose that $A^{(l,m,n)} = (\alpha_{l_A}, \beta_{m_A}, \delta_{n_A})$ is a PPF sub-semihypergroup of H. Let $x, y \in A$. Then we have

$$\begin{aligned}
\inf_{z \in x \circ z} \alpha_{l_A}(z) &\leq \min\{\alpha_{l_A}(x), \alpha_{l_A}(y)\} = l \wedge l = l \\
\inf_{z \in x \circ z} \alpha_{l_A}(z) &\geq l, \text{ but } \inf_{z \in x \circ z} \alpha_{l_A}(z) \leq l \\
\inf_{z \in x \circ z} \alpha_{l_A}(z) &= l,
\end{aligned}$$

similarly,

$$\begin{aligned}
\inf_{z \in x \circ z} \beta_{m_A}(z) &\leq \min\{\beta_{m_A}(x), \beta_{m_A}(y)\} = m \wedge m = m \\
\inf_{z \in x \circ z} \beta_{m_A}(z) &\geq m, \text{ but } \inf_{z \in x \circ z} \beta_{m_A}(z) \leq m \\
\inf_{z \in x \circ z} \beta_{m_A}(z) &= m,
\end{aligned}$$

and

$$\begin{aligned}
\sup_{z \in x \circ y} \delta_{n_A}(z) &\leq \max\{\delta_{n_A}(x), \delta_{n_A}(y)\} = n \vee n = n \\
\sup_{z \in x \circ y} \delta_{n_A}(z) &\leq n, \text{ but } \sup_{z \in x \circ y} \delta_{n_A}(z) \geq n \\
\sup_{z \in x \circ y} \delta_{n_A}(z) &= n.
\end{aligned}$$

Hence $x \circ y \subseteq A$. Therefore A is a sub-semihypergroup of H. The other cases can be seen in a similar way. □

Theorem 4.5. *A PPF subset $\mathcal{P} = (\alpha_{\mathcal{P}}, \beta_{\mathcal{P}}, \delta_{\mathcal{P}})$ of a semihypergroups H is a PPF*

(1) sub-semihypergroup of H if and only if $\mathcal{P} \circ \mathcal{P} \subseteq \mathcal{P}$,

(2) left hyperideal of H if and only if $\mathcal{H} \circ \mathcal{P} \subseteq \mathcal{P}$,

(3) right hyperideal of H if and only if $\mathcal{P} \circ \mathcal{H} \subseteq \mathcal{P}$,

(4) hyperideal of H if and only if $\mathcal{H} \circ \mathcal{P} \subseteq \mathcal{P}$ and $\mathcal{P} \circ \mathcal{H} \subseteq \mathcal{P}$,

(5) bi-hyperideal of H if and only if $\mathcal{P} \circ \mathcal{P} \subseteq \mathcal{P}$ and $\mathcal{P} \circ \mathcal{H} \circ \mathcal{P} \subseteq \mathcal{P}$,

(6) interior hyperideal of H if and only if $\mathcal{H} \circ \mathcal{P} \circ \mathcal{H} \subseteq \mathcal{P}$,

(7) $(1, 2)$-hyperideal of H if and only if $\mathcal{P} \circ \mathcal{P} \subseteq \mathcal{P}$ and $\mathcal{P} \circ \mathcal{H} \circ \mathcal{P} \circ \mathcal{P} \subseteq \mathcal{P}$,

where $\mathcal{H} = (\alpha_{\mathcal{H}}, \beta_{\mathcal{H}}, \delta_{\mathcal{H}})$, such that $\alpha_{\mathcal{H}}(x) = 1$, $\beta_{\mathcal{H}}(x) = 1$ and $\delta_{\mathcal{H}}(x) = 0$, for all $x \in H$.

Proof. (1) Let $\mathcal{P} = (\alpha_{\mathcal{P}}, \beta_{\mathcal{P}}, \delta_{\mathcal{P}})$ be a PPF sub-semihypergroup of H and

$z \in H$. Let us suppose that $z \in x \circ y$ for $x, y \in H$. Then

$$\begin{aligned}\alpha_{\mathcal{P}\circ\mathcal{P}}(z) &= \bigvee_{z\in x\circ y}\{\alpha_{\mathcal{P}}(x)\wedge\alpha_{\mathcal{P}}(y)\} \leq \bigvee_{z\in x\circ y}\left\{\inf_{z\in x\circ y}\alpha_{\mathcal{P}}(z)\right\}\\ &\leq \bigvee_{z\in x\circ y}\{\alpha_{\mathcal{P}}(x\circ y)\} = \alpha_{\mathcal{P}}(z),\end{aligned}$$

$$\begin{aligned}\beta_{\mathcal{P}\circ\mathcal{P}}(z) &= \bigvee_{z\in x\circ y}\{\beta_{\mathcal{P}}(x)\wedge\beta_{\mathcal{P}}(y)\} \leq \bigvee_{z\in x\circ y}\left\{\inf_{z\in x\circ y}\beta_{\mathcal{P}}(z)\right\}\\ &\leq \bigvee_{z\in x\circ y}\{\beta_{\mathcal{P}}(x\circ y)\} = \beta_{\mathcal{P}}(z),\end{aligned}$$

and

$$\begin{aligned}\delta_{\mathcal{P}\circ\mathcal{P}}(z) &= \bigwedge_{z\in x\circ y}\{\delta_{\mathcal{P}}(x)\vee\delta_{\mathcal{P}}(y)\} \geq \bigwedge_{z\in x\circ y}\left\{\sup_{z\in x\circ y}\delta_{\mathcal{P}}(z)\right\}\\ &\geq \bigwedge_{z\in x\circ y}\{\delta_{\mathcal{P}}(x\circ y)\} = \delta_{\mathcal{P}}(z).\end{aligned}$$

Therefore $\alpha_{\mathcal{P}\circ\mathcal{P}} \subseteq \alpha_{\mathcal{P}}$, $\beta_{\mathcal{P}\circ\mathcal{P}} \subseteq \beta_{\mathcal{P}}$ and $\delta_{\mathcal{P}\circ\mathcal{P}} \supseteq \delta_{\mathcal{P}}$. If there do not exist any $x, y \in H$ such that $z \in x \circ y$, then

$$\alpha_{\mathcal{P}\circ\mathcal{P}}(z) = 0 \leq \alpha_{\mathcal{P}}(z),\ \beta_{\mathcal{P}\circ\mathcal{P}}(z) = 0 \leq \beta_{\mathcal{P}}(z) \text{ and } \delta_{\mathcal{P}\circ\mathcal{P}}(z) = 1 \geq \delta_{\mathcal{P}}(z).$$

Hence for all cases $\mathcal{P}\circ\mathcal{P} \subseteq \mathcal{P}$.

Conversely, let us assume that $\mathcal{P}\circ\mathcal{P} \subseteq \mathcal{P}$ holds for all PPF subsets of H. Let $x, y \in H$. Then, we have

$$\alpha_{\mathcal{P}\circ\mathcal{P}}(z) \leq \alpha_{\mathcal{P}}(z),\ \beta_{\mathcal{P}\circ\mathcal{P}}(z) \leq \beta_{\mathcal{P}}(z) \text{ and } \delta_{\mathcal{P}\circ\mathcal{P}}(z) \geq \delta_{\mathcal{P}}(z).$$

If there exist $p, q \in H$ such that $x \circ y \subseteq p \circ q$, then

$$\begin{aligned}\alpha_{\mathcal{P}}(x\circ y) &\geq \alpha_{\mathcal{P}\circ\mathcal{P}}(x\circ y) = \bigvee_{x\circ y\subseteq p\beta q}\{\alpha_{\mathcal{P}}(p)\wedge\alpha_{\mathcal{P}}(q)\}\\ &\geq \alpha_{\mathcal{P}}(x)\wedge\alpha_{\mathcal{P}}(y).\\ \inf_{z\in x\circ y}\alpha_{\mathcal{P}}(z) &\geq \min\{\alpha_{\mathcal{P}}(x), \alpha_{\mathcal{P}}(y)\},\end{aligned}$$

$$\begin{aligned}\beta_{\mathcal{P}}(x\circ y) &\geq \beta_{\mathcal{P}\circ\mathcal{P}}(x\circ y) = \bigvee_{x\circ y\subseteq p\beta q}\{\beta_{\mathcal{P}}(p)\wedge\beta_{\mathcal{P}}(q)\}\\ &\geq \beta_{\mathcal{P}}(x)\wedge\beta_{\mathcal{P}}(y).\\ \inf_{z\in x\circ y}\beta_{\mathcal{P}}(z) &\geq \min\{\beta_{\mathcal{P}}(x), \beta_{\mathcal{P}}(y)\},\end{aligned}$$

and

$$\begin{aligned}\delta_{\mathcal{P}}(x \circ y) &\leq \delta_{\mathcal{P}\circ\mathcal{P}}(x \circ y) = \bigwedge_{x\circ y \subseteq p\beta q} \{\delta_{\mathcal{P}}(p) \vee \delta_{\mathcal{P}}(q)\} \\ &\leq \delta_{\mathcal{P}}(x) \vee \delta_{\mathcal{P}}(y). \\ \sup_{z\in x\circ y} \delta_{\mathcal{P}}(z) &\leq \max\{\delta_{\mathcal{P}}(x), \delta_{\mathcal{P}}(y)\}.\end{aligned}$$

This means that $\mathcal{P} = (\alpha_{\mathcal{P}}, \beta_{\mathcal{P}}, \delta_{\mathcal{P}})$ is a PPF sub-semihypergroup of H. The other cases can be seen in a similar way. □

Theorem 4.6. *If $\{\mathcal{P}_i\}_{i\in I}$ is a family of PPF sub-semihypergroups (resp., left hyperideals, right hyperideals, hyperideals, bi-hyperideals, interior hyperideals and $(1,2)$-hyperideals) of H. Then $\bigcap_{i\in I} \mathcal{P}_i$ is a PPF sub-semihypergroup (resp., left hyperideal, right hyperideal, hyperideal, bi-hyperideal, interior hyperideal and $(1,2)$-hyperideal) of H, where $\bigcap_{i\in I} \mathcal{P}_i = (\bigwedge_{i\in I} \alpha_{\mathcal{P}_i}, \bigwedge_{i\in I} \beta_{\mathcal{P}_i}, \bigvee_{i\in I} \delta_{\mathcal{P}_i})$ and*

$$\begin{aligned}\bigwedge_{i\in I} \alpha_{\mathcal{P}_i} &: H \longrightarrow [0,1]|A \longmapsto \bigwedge_{i\in I} \alpha_{\mathcal{P}_i}(x) := \inf_{i\in I} \{\alpha_{\mathcal{P}_i}(x) : x \in H\}, \\ \bigwedge_{i\in I} \beta_{\mathcal{P}_i} &: H \longrightarrow [0,1]|A \longmapsto \bigwedge_{i\in I} \beta_{\mathcal{P}_i}(x) := \inf_{i\in I} \{\beta_{\mathcal{P}_i}(x) : x \in H\},\end{aligned}$$

and

$$\bigvee_{i\in I} \delta_{\mathcal{P}_i} : H \longrightarrow [0,1]|A \longmapsto \bigvee_{i\in I} \delta_{\mathcal{P}_i}(x) := \sup_{i\in I} \{\delta_{\mathcal{P}_i}(x) : x \in H\}.$$

Proof. Consider $\{\mathcal{P}_i\}_{i\in I}$ is a family of PPF sub-semihypergroups of H. Let $x, y, z \in H$. Then for every $z \in x \circ y$, we have

$$\begin{aligned}\inf_{z\in x\circ y} \left\{\bigwedge_{i\in I} \alpha_{\mathcal{P}_i}(z)\right\} &= \bigwedge_{i\in I} \left\{\inf_{z\in x\circ y} \{\alpha_{\mathcal{P}_i}(z)\}\right\} \\ &\geq \bigwedge_{i\in I} \{\min\{\alpha_{\mathcal{P}_i}(x), \alpha_{\mathcal{P}_i}(y)\}\} \\ &= \min\left\{\bigwedge_{i\in I} \alpha_{\mathcal{P}_i}(x), \bigwedge_{i\in I} \alpha_{\mathcal{P}_i}(y)\right\},\end{aligned}$$

$$\begin{aligned}
\inf_{z\in x\circ y}\left\{\bigwedge_{i\in I}\beta_{\mathcal{P}_i}(z)\right\} &= \bigwedge_{i\in I}\left\{\inf_{z\in x\circ y}\{\beta_{\mathcal{P}_i}(z)\}\right\}\\
&\geq \bigwedge_{i\in I}\{\min\{\beta_{\mathcal{P}_i}(x),\beta_{\mathcal{P}_i}(y)\}\}\\
&= \min\left\{\bigwedge_{i\in I}\beta_{\mathcal{P}_i}(x),\bigwedge_{i\in I}\beta_{\mathcal{P}_i}(y)\right\},
\end{aligned}$$

and

$$\begin{aligned}
\sup_{z\in x\circ y}\left\{\bigvee_{i\in I}\delta_{\mathcal{P}_i}(z)\right\} &= \bigvee_{i\in I}\left\{\sup_{z\in x\circ y}\{\delta_{\mathcal{P}_i}(z)\}\right\}\\
&\leq \bigvee_{i\in I}\{\max\{\delta_{\mathcal{P}_i}(x),\delta_{\mathcal{P}_i}(y)\}\}\\
&= \max\left\{\bigvee_{i\in I}\delta_{\mathcal{P}_i}(x),\bigvee_{i\in I}\delta_{\mathcal{P}_i}(y)\right\}.
\end{aligned}$$

Hence this shows that $\bigcap_{i\in I}\mathcal{P}_i$ is a PPF sub-semihypergroup of H. The other cases can be seen in a similar way. □

Proposition 4.7. *Let $\mathcal{P}=(\alpha_{\mathcal{P}},\beta_{\mathcal{P}},\delta_{\mathcal{P}})$ be a PPF bi-hyperideal and $\mathcal{Q}=(\alpha_{\mathcal{Q}},\beta_{\mathcal{Q}},\delta_{\mathcal{Q}})$ a PPF sub-semihypergroup of H. Then, $\mathcal{P}\cap\mathcal{Q}$ is a PPF bi-hyperideal of H.*

Proof. The proof is straightforward. □

Proposition 4.8. *Let H be an idempotent semihypergroup and $\mathcal{P}=(\alpha_{\mathcal{P}},\beta_{\mathcal{P}},\delta_{\mathcal{P}})$, $\mathcal{Q}=(\alpha_{\mathcal{Q}},\beta_{\mathcal{Q}},\delta_{\mathcal{Q}})$ be two PPF sets in H. Then $\mathcal{P}\circ\mathcal{Q}\supseteq\mathcal{P}\cap\mathcal{Q}$.*

Proof. Let $a\in H$. Since H is idempotent, then there exist an element $x\in H$ such that $a\in a\circ a$. Then,

$$\begin{aligned}
\alpha_{\mathcal{P}\circ\mathcal{Q}}(a) &= \sup_{a\in a\circ a}\{\alpha_{\mathcal{P}}(a)\wedge\alpha_{\mathcal{Q}}(a)\}\\
&\geq \alpha_{\mathcal{P}}(a)\wedge\alpha_{\mathcal{Q}}(a)=\alpha_{\mathcal{P}\cap\mathcal{Q}}(a),
\end{aligned}$$

$$\begin{aligned}
\beta_{\mathcal{P}\circ\mathcal{Q}}(a) &= \sup_{a\in a\circ a}\{\beta_{\mathcal{P}}(a)\wedge\beta_{\mathcal{Q}}(a)\}\\
&\geq \beta_{\mathcal{P}}(a)\wedge\beta_{\mathcal{Q}}(a)=\beta_{\mathcal{P}\cap\mathcal{Q}}(a),
\end{aligned}$$

and

$$\begin{aligned}\delta_{\mathcal{P}\circ\mathcal{Q}}(a) &= \inf_{a\in a\circ a}\{\delta_{\mathcal{P}}(a)\vee\delta_{\mathcal{Q}}(a)\} \\ &\le \delta_{\mathcal{P}}(a)\vee\delta_{\mathcal{Q}}(a)=\delta_{\mathcal{P}\cap\mathcal{Q}}(a),\end{aligned}$$

Hence $\mathcal{P}\circ\mathcal{Q}\supseteq\mathcal{P}\cap\mathcal{Q}$. □

Theorem 4.9. *Let H be a semihypergroup. The following statements are equivalent:*

(i) H is regular

(ii) $\mathcal{P}\circ\mathcal{Q}=\mathcal{P}\cap\mathcal{Q}$, where $\mathcal{P}=(\alpha_{\mathcal{P}},\beta_{\mathcal{P}},\delta_{\mathcal{P}})$ is a PPF right hyperideal of H and $\mathcal{Q}=(\alpha_{\mathcal{Q}},\beta_{\mathcal{Q}},\delta_{\mathcal{Q}})$ is a PPF left hyperideal of H.

Proof. The proof is straightforward. □

Theorem 4.10. *Let H be a regular semihypergroup. Then the following statements hold:*

(1) $\mathcal{P}\cap\mathcal{Q}\subseteq\mathcal{P}\circ\mathcal{Q}$, for every PPF bi-hyperideal $\mathcal{Q}$ and PPF right hyperideal $\mathcal{P}$ of H,

(2) $\mathcal{P}\cap\mathcal{Q}\subseteq\mathcal{P}\circ\mathcal{Q}$, for every PPF bi-hyperideal $\mathcal{P}$ and PPF left hyperideal $\mathcal{Q}$ of H.

(3) $\mathcal{P}\cap\mathcal{Q}\cap\mathcal{R}\subseteq\mathcal{P}\circ\mathcal{Q}\circ\mathcal{R}$, for every PPF right hyperideal $\mathcal{P}$, PPF bi-hyperideal $\mathcal{Q}$ and PPF left hyperideal $\mathcal{R}$ of H respectively.

Proof. (1) Let H be regular semihypergroup and $a\in H$. Then there exist $x\in H$ such that $a\in a\circ x\circ a$. Then

$$\begin{aligned}\alpha_{\mathcal{P}\circ\mathcal{Q}}(a) &= \sup_{a\in(a\circ x)\circ a}\min\left\{\inf_{t\in a\circ x}\alpha_{\mathcal{P}}(t),\alpha_{\mathcal{Q}}(a)\right\} \\ &\ge \min\left\{\inf_{t\in a\circ x}\alpha_{\mathcal{P}}(t),\alpha_{\mathcal{Q}}(a)\right\} \\ &\ge \min\{\alpha_{\mathcal{P}}(a),\alpha_{\mathcal{Q}}(a)\} \\ &= (\alpha_{\mathcal{P}}\wedge\alpha_{\mathcal{Q}})(a) \\ &= \alpha_{\mathcal{P}\cap\mathcal{Q}}(a),\end{aligned}$$

$$\begin{aligned}\beta_{\mathcal{P}\circ\mathcal{Q}}(a) &= \sup_{a\in(a\circ x)\circ a}\min\left\{\inf_{t\in a\circ x}\beta_{\mathcal{P}}(t),\beta_{\mathcal{Q}}(a)\right\} \\ &\ge \min\left\{\inf_{t\in a\circ x}\beta_{\mathcal{P}}(t),\beta_{\mathcal{Q}}(a)\right\} \\ &\ge \min\{\beta_{\mathcal{P}}(a),\beta_{\mathcal{Q}}(a)\} \\ &= (\beta_{\mathcal{P}}\wedge\beta_{\mathcal{Q}})(a) \\ &= \beta_{\mathcal{P}\cap\mathcal{Q}}(a),\end{aligned}$$

and

$$\begin{aligned}
\delta_{\mathcal{P}\circ\mathcal{Q}}(a) &= \inf_{a\in(a\circ x)\circ a} \max\left\{\sup_{t\in a\circ x} \delta_{\mathcal{P}}(t), \delta_{\mathcal{Q}}(a)\right\} \\
&\leq \max\left\{\sup_{t\in a\circ x} \delta_{\mathcal{P}}(t), \delta_{\mathcal{Q}}(a)\right\} \\
&\leq \max\{\delta_{\mathcal{P}}(a), \delta_{\mathcal{Q}}(a)\} \\
&= (\delta_{\mathcal{P}} \vee \delta_{\mathcal{Q}})(a) \\
&= \delta_{\mathcal{P}\cap\mathcal{Q}}(a).
\end{aligned}$$

Hence $\mathcal{P} \cap \mathcal{Q} \subseteq \mathcal{P} \circ \mathcal{Q}$.

(2) The proof is similar to (1).

(3) Let us suppose that H is regular and $a \in H$. Then there exist $x \in H$ and $\circ, \circ \in \circ$ such that $a \in a \circ x \circ a$. Then

$$\begin{aligned}
\alpha_{\mathcal{P}\circ\mathcal{Q}\circ\mathcal{R}}(a) &= \sup_{a\in a\circ x\circ a} \min\left\{\inf_{t\in a\circ x} \alpha_{\mathcal{P}}(t), \alpha_{\mathcal{Q}\circ\mathcal{R}}(a)\right\} \\
&\geq \min\left\{\inf_{t\in a\circ x} \alpha_{\mathcal{P}}(t), \alpha_{\mathcal{Q}\circ\mathcal{R}}(a)\right\} \\
&\geq \min\left\{\alpha_{\mathcal{P}}(a), \sup_{a\in a\circ x\circ a} \min\left\{\alpha_{\mathcal{Q}}(a), \inf_{h\in x\circ a} \alpha_{\mathcal{R}}(h)\right\}\right\} \\
&\geq \min\left\{\alpha_{\mathcal{P}}(a), \alpha_{\mathcal{Q}}(a), \inf_{h\in x\circ a} \alpha_{\mathcal{R}}(h)\right\} \\
&\geq \min\{\alpha_{\mathcal{P}}(a), \alpha_{\mathcal{Q}}(a), \alpha_{\mathcal{R}}(a)\} \\
&= (\alpha_{\mathcal{P}} \wedge \alpha_{\mathcal{Q}} \wedge \alpha_{\mathcal{R}})(a) \\
&= \alpha_{\mathcal{P}\cap\mathcal{Q}\cap\mathcal{R}}(a),
\end{aligned}$$

$$\begin{aligned}
\beta_{\mathcal{P}\circ\mathcal{Q}\circ\mathcal{R}}(a) &= \sup_{a\in a\circ x\circ a} \min\left\{\inf_{t\in a\circ x} \beta_{\mathcal{P}}(t), \beta_{\mathcal{Q}\circ\mathcal{R}}(a)\right\} \\
&\geq \min\left\{\inf_{t\in a\circ x} \beta_{\mathcal{P}}(t), \beta_{\mathcal{Q}\circ\mathcal{R}}(a)\right\} \\
&\geq \min\left\{\beta_{\mathcal{P}}(a), \sup_{a\in a\circ x\circ a} \min\left\{\beta_{\mathcal{Q}}(a), \inf_{h\in x\circ a} \beta_{\mathcal{R}}(h)\right\}\right\} \\
&\geq \min\left\{\beta_{\mathcal{P}}(a), \beta_{\mathcal{Q}}(a), \inf_{h\in x\circ a} \beta_{\mathcal{R}}(h)\right\} \\
&\geq \min\{\beta_{\mathcal{P}}(a), \beta_{\mathcal{Q}}(a), \beta_{\mathcal{R}}(a)\} \\
&= (\beta_{\mathcal{P}} \wedge \beta_{\mathcal{Q}} \wedge \beta_{\mathcal{R}})(a) \\
&= \beta_{\mathcal{P}\cap\mathcal{Q}\cap\mathcal{R}}(a),
\end{aligned}$$

and

$$\begin{aligned}
\delta_{\mathcal{P}\circ\mathcal{Q}\circ\mathcal{R}}(a) &= \inf_{a\in a\circ x\circ a} \max\left\{\sup_{t\in a\circ x} \delta_{\mathcal{P}}(t), \delta_{\mathcal{Q}\circ\mathcal{R}}(a)\right\} \\
&\le \max\left\{\sup_{t\in a\circ x} \delta_{\mathcal{P}}(t), \delta_{\mathcal{Q}\circ\mathcal{R}}(a)\right\} \\
&\le \max\left\{\delta_{\mathcal{P}}(a), \inf_{a\in a\circ x\circ a} \max\left\{\delta_{\mathcal{Q}}(a), \sup_{h\in x\circ a} \delta_{\mathcal{R}}(h)\right\}\right\} \\
&\le \max\left\{\delta_{\mathcal{P}}(a), \delta_{\mathcal{Q}}(a), \sup_{h\in x\circ a} \delta_{\mathcal{R}}(h)\right\} \\
&\le \max\{\delta_{\mathcal{P}}(a), \delta_{\mathcal{Q}}(a), \delta_{\mathcal{R}}(a)\} \\
&= (\delta_{\mathcal{P}} \vee \delta_{\mathcal{Q}} \vee \delta_{\mathcal{R}})(a) \\
&= \delta_{\mathcal{P}\cap\mathcal{Q}\cap\mathcal{R}}(a).
\end{aligned}$$

Hence $\mathcal{P}\cap\mathcal{Q}\cap\mathcal{R} \subseteq \mathcal{P}\circ\mathcal{Q}\circ\mathcal{R}$. □

Theorem 4.11. *Let H be a semihypergroup. Then every PPF bi-hyperideal of H is a PPF $(1,2)$-hyperideal of H.*

Proof. Let $\mathcal{P} = (\alpha_{\mathcal{P}}, \beta_{\mathcal{P}}, \delta_{\mathcal{P}})$ be a PPF bi-hyperideal of H and let $w, x, y, z \in H$. Then for all $a \in x\circ w\circ(y\circ z)$, we have

$$\begin{aligned}
\inf_{a\in x\circ w\circ(y\circ z)} \alpha_{\mathcal{P}}(a) &= \inf_{a\in(x\circ w\circ y)\circ z} \alpha_{\mathcal{P}}(a) \\
&= \inf_{c\in x\circ w\circ y} \inf_{a\in c\circ z} \alpha_{\mathcal{P}}(a) \\
&\ge \inf_{c\in x\circ w\circ y} \{\min\{\alpha_{\mathcal{P}}(c), \alpha_{\mathcal{P}}(z)\}\}, \text{ for every } c\in x\circ w\circ y \\
&= \min\left\{\inf_{c\in x\circ w\circ y} \alpha_{\mathcal{P}}(c), \alpha_{\mathcal{P}}(z)\right\} \\
&\ge \min\{\min\{\alpha_{\mathcal{P}}(x), \alpha_{\mathcal{P}}(y)\}, \alpha_{\mathcal{P}}(z)\} \\
&= \min\{\alpha_{\mathcal{P}}(x), \alpha_{\mathcal{P}}(y), \alpha_{\mathcal{P}}(z)\},
\end{aligned}$$

$$\begin{aligned}
\inf_{a\in x\circ w\circ(y\circ z)}\beta_{\mathcal{P}}(a) &= \inf_{a\in(x\circ w\circ y)\circ z}\beta_{\mathcal{P}}(a)\\
&= \inf_{c\in x\circ w\circ y}\inf_{a\in c\circ z}\beta_{\mathcal{P}}(a)\\
&\geq \inf_{c\in x\circ w\circ y}\{\min\{\beta_{\mathcal{P}}(c),\beta_{\mathcal{P}}(z)\}\},\ \text{for every } c\in x\circ w\circ y\\
&= \min\left\{\inf_{c\in x\circ w\circ y}\beta_{\mathcal{P}}(c),\beta_{\mathcal{P}}(z)\right\}\\
&\geq \min\{\min\{\beta_{\mathcal{P}}(x),\beta_{\mathcal{P}}(y)\},\beta_{\mathcal{P}}(z)\}\\
&= \min\{\beta_{\mathcal{P}}(x),\beta_{\mathcal{P}}(y),\beta_{\mathcal{P}}(z)\},
\end{aligned}$$

and

$$\begin{aligned}
\sup_{a\in x\circ w\circ(y\circ z)}\delta_{\mathcal{P}}(a) &= \sup_{a\in(x\circ w\circ y)\circ z}\delta_{\mathcal{P}}(a)\\
&= \sup_{c\in x\circ w\circ y}\sup_{a\in c\circ z}\delta_{\mathcal{P}}(a)\\
&\leq \sup_{c\in x\circ w\circ y}\{\max\{\delta_{\mathcal{P}}(c),\delta_{\mathcal{P}}(z)\}\},\ \text{for every } c\in x\circ w\circ y\\
&= \max\left\{\sup_{c\in x\circ w\circ y}\delta_{\mathcal{P}}(c),\delta_{\mathcal{P}}(z)\right\}\\
&\leq \max\{\max\{\delta_{\mathcal{P}}(x),\delta_{\mathcal{P}}(y)\},\delta_{\mathcal{P}}(z)\}\\
&= \max\{\delta_{\mathcal{P}}(x),\delta_{\mathcal{P}}(y),\delta_{\mathcal{P}}(z)\},
\end{aligned}$$

Hence $\mathcal{P}=(\alpha_{\mathcal{P}},\beta_{\mathcal{P}},\delta_{\mathcal{P}})$ is a PPF $(1,2)$-hyperideal of H. □

Theorem 4.12. *Let H be a regular semihypergroup. Then every PPF $(1,2)$-hyperideal of H is a PPF bi-hyperideal of H.*

Proof. Let us assume that a semihypergroup H is regular and let $\mathcal{P}=(\alpha_{\mathcal{P}},\beta_{\mathcal{P}},\delta_{\mathcal{P}})$ be a PPF $(1,2)$-hyperideal of H. Let $w,x,y\in H$. Since H is regular, we have for every $w\in x\circ y\subseteq(x\circ a\circ x)\circ y=x\circ(a\circ x)\circ y$ for some $a\in H$. Thus for every $c\in a\circ x$, $w\in x\circ c\circ y$, we have

$$\begin{aligned}
\inf_{w\in x\circ c\circ y\subseteq x\circ(a\circ x)\circ y}\alpha_{\mathcal{P}}(w) &\geq \min\{\alpha_{\mathcal{P}}(x),\alpha_{\mathcal{P}}(x),\alpha_{\mathcal{P}}(y)\}\\
&= \min\{\alpha_{\mathcal{P}}(x),\alpha_{\mathcal{P}}(y)\},
\end{aligned}$$

$$\begin{aligned}
\inf_{w\in x\circ c\circ y\subseteq x\circ(a\circ x)\circ y}\beta_{\mathcal{P}}(w) &\geq \min\{\beta_{\mathcal{P}}(x),\beta_{\mathcal{P}}(x),\beta_{\mathcal{P}}(y)\}\\
&= \min\{\beta_{\mathcal{P}}(x),\beta_{\mathcal{P}}(y)\},
\end{aligned}$$

and

$$\begin{aligned}\sup_{w \in x \circ c \circ y \subseteq x \circ (a \circ x) \circ y} \delta_{\mathcal{P}}(w) &\leq \max\{\delta_{\mathcal{P}}(x), \delta_{\mathcal{P}}(x), \delta_{\mathcal{P}}(y)\} \\ &= \max\{\delta_{\mathcal{P}}(x), \delta_{\mathcal{P}}(y)\},\end{aligned}$$

Hence $\mathcal{P} = (\alpha_{\mathcal{P}}, \beta_{\mathcal{P}}, \delta_{\mathcal{P}})$ is a PPF bi-hyperideal of H. □

Theorem 4.13. *Let H be a completely regular (resp., intra-regular) semihypergroup and $\mathcal{P} = (\alpha_{\mathcal{P}}, \beta_{\mathcal{P}}, \delta_{\mathcal{P}})$ a PPF bi-hyperideal (resp., hyperideal) of H. Then for every $r \in a \circ a$, we have $\mathcal{P}(a) = \mathcal{P}(r)$ for all $a \in H$.*

Proof. The proof is straightforward. □

5 Conclusions

In this paper, we studied Pythagorean picture fuzzy sets in semihypergroups and provided some properties of Pythagorean picture fuzzy hyperideals in a semihypergroup. We show characterized sub-semihypergroup (resp., hyperideal, left hyperideal, right hyperideal, bi-hyperideal, interior hyperideal and $(1, 2)$-hyperideal) of a non-empty subset H of a semihypergroups. Different cases in different ideals (resp., hyperideal, left hyperideal, right hyperideal, bi-hyperideal, interior hyperideal and $(1, 2)$-hyperideal) have been shown.
This concept can be applied in different branches including fuzzy logic, multicriteria decision making and computational intelligence problems by using other algebraic structures.

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