International Journal of Mathematics and
Computer Science, **16**(2021), no. 4, 1791–1802

Existence of solutions to fourth order problems with sign-changing weights

Hessa Alharbi, Makkia Dammak

Department of Mathematics
College of Science
Taibah University
Al-Madinah Al-Munawarah, Saudi Arabia

email: H-badrani@hotmail.com, makkia.dammak@gmail.com

(Received April 30, 2021, Accepted June 17, 2021)

Abstract

In this article, we investigate the existence of solutions for the indefinite nonlinear fourth order elliptic problem with Navier boundary conditions:

$$\Delta^2 v - \Delta v + V_\lambda(x)v = f(x,v) \quad \text{in} \quad \Omega, v = \Delta v = 0 \quad \text{on} \quad \partial\Omega,$$

where Ω is a bounded open domain in $\mathbb{R}^N$, $N \geq 2$, $V_\lambda(x)$ is sign-changing weight function. Also, the reaction source term f is not necessary positive. We will prove that for λ large enough, there exists a nontrivial solution. Our method is a variational one. The most delicate point is to choose the appropriate Sobolev space and the suitable norm.

1 Introduction

In this note, we study the existence of solutions to the following partial differential equation:

$$\begin{cases} \Delta^2 v - \Delta v + V_\lambda(x)v = f(x,v) & \text{in} \quad \Omega, \\ v = \Delta v = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.1}$$

Key words and phrases: Biharmonic equation, indefinite nonlinearities, variational method, sign-changing weight.
AMS (MOS) Subject Classifications: 35J20, 35J60, 35J57, 35J70.
ISSN 1814-0432, 2021, http://ijmcs.future-in-tech.net

where Ω is an open smooth bounded subset of $\mathbb{R}^N$, $N \geq 2$, $f(x,t)$ and V are continuous functions. The function $V_\lambda(x)$ is defined as

$$V_\lambda(x) = \lambda V^+(x) - V^-(x),$$

where V^+ and V^- are nonnegative functions given by $V^\pm = \max(\pm V, 0)$ and $\lambda > 0$ is a positive parameter.

In the recent two decades, fourth order partial differential equations with weight functions are considered as interesting problems with applications in thin film theory and surface diffusion on solids. In addition, they appear in the study of the flow in Hele-Shaw cells, interface dynamics and the micro-electro-mechanical systems (see for example [7, 10, 14] and the references therein).

When Ω is a regular bounded domain in $\mathbb{R}^4$, we have the following equation involving fourth order elliptic operator and with exponential nonlinearity:

$$\begin{cases} \Delta^2 v = \rho e^v & \text{in } \Omega, \\ v = \Delta v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

appears in the classification of surfaces in Conformal Geometry and in the study of the Q-curvature on 4-dimensional Riemannian manifolds [2, 3, 4, 5]. Also, many authors have investigated the existence of solution for the problem

$$\begin{cases} \Delta^2 v = g(x,v) & \text{in } \Omega \\ v = \Delta v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where Ω subset of $\mathbb{R}^N$, $N \geq 2$ and g is superlinear at ∞, [8, 15, 16].
Also, with superlinar nonlinearity, in [11, 12, 13], the authors studied the fourth order problem

$$\begin{cases} \Delta^2 v - \Delta v = g(x,v) & \text{in } \Omega \\ v = \Delta v = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.4}$$

In this paper, we will consider the problem (1.2) when the nonlinearity is not necessarily positive and the potential (weight) V, or more exactly, V_λ change sign. In what follows, we suppose that the function $V(x)$ satisfies the following hypothesis:

(V1) The function V is in $C^0(\overline{\Omega}, \mathbb{R})$.

(V2) $V^- \not\equiv 0$.

(V3) There exists a constant $\mu_0 > 1$ such that

$$\mu_1(\lambda) = \inf_{v \in H^2(\Omega)\setminus\{0\}} \frac{\int_\Omega [|\Delta v|^2 + |\nabla v|^2 + \lambda V^+(x) v^2 dx]}{\int_\Omega V^-(x) v^2 dx} \geq \mu_0, \quad \text{for all } \lambda > 0.$$

Let

$$2_* = \begin{cases} \frac{2N}{N-4} & if \quad N > 4 \\ \infty & if \quad N \leq 4. \end{cases} \tag{1.5}$$

Furthermore, we suppose that the nonlinearity $f(x,t)$ satisfies:

(F1) $f \in C^0(\Omega \times \mathbb{R}, \mathbb{R})$ and there exist $a_1 > 0$, $p \in (2, 2_*)$ such that

$$|f(x,t)| \leq a_1(1 + |t|^{p-1}),$$

for all $(x,t) \in \Omega \times \mathbb{R}$.

(F2) $\lim_{t \to 0} \dfrac{f(x,t)}{t} = 0$ uniformly for $x \in \Omega$.

(F3) There exists $q \in (2,p)$ such that $\lim_{|t| \to \infty} \dfrac{F(x,t)}{|t|^q} = \infty$ uniformly for $x \in \Omega$.

(F4) There exist $\tau > 2$ and $K > 0$ such that

$$0 < \tau F(x,t) \leq t f(x,t)$$

for all $|t| > K$ and uniformly in $x \in \Omega$.

We will prove the following result:

Theorem 1.1. *Assume $(V1) - (V3)$ and $(F1) - (F4)$. Then, there exists a constant $\lambda^* > 0$ such that problem (1.1) has nontrivial solution for all $\lambda > \lambda^*$.*

In the present work, a constant C may change from line to another.

2 Variational setting

In this section, we fix some notations and recall some definitions and basic properties of the functional associated to the equation (1.1).
We recall that

$$\|v\|_q = \Big(\int_\Omega |v|^q dx \Big)^{\frac{1}{q}}$$

is the standard norm in the Lebesgue space $L^q(\Omega)$, for $1 \leq q < \infty$.
Consider the Sobolev space $H^2(\Omega) = W^{2,2}(\Omega)$ endowed with the standard inner product:

$$\langle w, v\rangle_{H^2(\Omega)} = \int_\Omega [\Delta w \Delta v + \nabla w.\nabla v + wv]dx,$$

and the induced norm:

$$\|v\|_{H^2(\Omega)} = \Big(\int_\Omega [|\Delta v|^2 + |\nabla v|^2 + v^2]dx\Big)^{\frac{1}{2}}.$$

For a function $V(x) : \Omega \to \mathbb{R}$, we will consider the weighted Sobolev space defined by

$$H^2(\Omega, V^+) = \{v \in H^2(\Omega)/ \int_\Omega V^+(x)v^2dx < \infty\}.$$

In the space $H^2(\Omega, V^+)$, we have the inner product:

$$\langle w, v\rangle = \int_\Omega [\Delta w \Delta v + \nabla w.\nabla v + V^+(x)wv]dx, \tag{2.6}$$

and so, it is equipped with the norm

$$\|w\| = \Big(\int_\Omega [|\Delta w|^2 + |\nabla w|^2 + V^+(x)w^2]dx\Big)^{\frac{1}{2}}.$$

If the function V is continuous on $\overline{\Omega}$, the embedding $H^2(\Omega, V^+) \hookrightarrow H^2(\Omega)$ is continuous.
For $\lambda > 0$, we consider the following inner product on the space $H^2(\Omega, V^+)$

$$\langle w, v\rangle_\lambda = \int_\Omega [\Delta w \Delta v + \nabla w.\nabla v + \lambda V^+(x)wv]dx \tag{2.7}$$

and the associated norm

$$\|w\|_\lambda = \Big(\int_\Omega [|\Delta w|^2 + |\nabla w|^2 + \lambda V^+(x)w^2]dx\Big)^{\frac{1}{2}}.$$

Weighted Sobolev spaces have been studied for a long time as the embedding theory and we can refer to [6, 9] for details.
Through this paper, we will suppose that $\lambda \geq 1$ and so $\|v\| \leq \|v\|_\lambda$ and the embedding

$$(H^2(\Omega, V^+), \|v\|_\lambda) \hookrightarrow L^q(\Omega)$$

is continuous for $q \in [2, 2_*]$ and compact if $q \in [2, 2_*)$.
For the weighted space $H^2(\Omega, V^+)$, we have also the following important remark.
Remark: Let $v \in H^2(\Omega, V^+)$, we have

$$\int_\Omega [|\Delta v|^2 + |\nabla v|^2 + V_\lambda(x)v^2]dx = \|v\|_\lambda - \int_\Omega V^-(x)v^2 dx.$$

From the condition (V3), we get

$$\int_\Omega [|\Delta v|^2 + |\nabla v|^2 + V_\lambda(x)v^2]dx \geq \|v\|_\lambda - \frac{1}{\mu_0}\|v\|_\lambda.$$

That is,

$$\int_\Omega [|\Delta v|^2 + |\nabla v|^2 + V_\lambda(x)v^2]dx \geq \frac{\mu_0 - 1}{\mu_0}\|v\|_\lambda. \tag{2.8}$$

Now, let

$$\mathcal{H} = H^2(\Omega, V^+) \cap H_0^1(\Omega)$$

and recall the following definition.

Definition 2.1. *Let $v \in \mathcal{H}$. v is said a solution of the equation (1.1) if*

$$\int_\Omega [\Delta v \Delta \varphi + \nabla v.\nabla \varphi + V_\lambda(x) v \varphi]\ dx = \int_\Omega f(x, v)\varphi\ dx, \qquad \forall \varphi \in \mathcal{H} \tag{2.9}$$

We will use variational method to prove the existence of solutions to the problem (1.1). Let I_λ the functional defined on $\mathcal{H}$ by

$$I_\lambda(v) = \frac{1}{2}\int_\Omega [|\Delta v|^2 + |\nabla v|^2 + V_\lambda(x)v^2]dx - \int_\Omega F(x, v)dx \tag{2.10}$$

where

$$F(x, t) = \int_0^t f(x, s)ds.$$

The functional I_λ can be written as

$$I_\lambda(v) = \frac{1}{2}\|v\|_\lambda^2 - \frac{1}{2}\int_\Omega V^-(x)v^2 dx - \int_\Omega F(x, v)dx, \tag{2.11}$$

and so

$$I_\lambda(v) \leq \frac{1}{2}\|v\|_\lambda^2 - \int_\Omega F(x, v)dx. \tag{2.12}$$

On the other hand, from $(V3)$ and (2.8) we get

$$I_\lambda(v) \geq \frac{\mu_0 - 1}{2\mu_0}\|v\|_\lambda^2 - \int_\Omega F(x,v)\, dx. \tag{2.13}$$

So, from $(F1)$ we conclude that the functional I_λ is well defined, of class C^1 and for all $v,\ \phi \in \mathcal{H}$, we have

$$\langle I'_\lambda(v), \varphi\rangle = \int_\Omega [\Delta v \Delta\varphi + \nabla v \nabla.\varphi + V_\lambda(x) v\ \varphi]\, dx - \int_\Omega f(x,v)\varphi\, dx. \tag{2.14}$$

It is clear that a nonzero critical point of I_λ is a nontrivial solution of the problem (1.1). So in the next paragraph, we will prove the existence of nonzero critical point of I_λ by using the following theorem due to Ambrosetti-Rabionowitz.

Theorem 2.2. *[1] Consider a functional I defined on a Banach space H satisfying $I \in C^1(H,\mathbb{R})$ and $I(0) = 0$. Suppose that:*

(i) There exist $\rho,\quad \alpha > 0$ such that $I(v) \geq \alpha$, for all $v \in \partial B(0,\rho)$;

(ii) There exists $x_1 \in H \setminus B(0,\rho)$ such that $I(x_1) < 0$;

(iii) The Palais-Smaile (PS) condition: every sequence (v_n) in H such that $I(v_n)$ converges in $\mathbb{R}$ and $I'(v_n)$ converges to 0 in H', the dual space of H, the sequence (v_n) has a convergent subsequence.

Then I has a critical point v and $c = I(v) \geq \alpha > 0$.

3 Proof of the main result

First, we prove that the functional I_λ has a mountain pass geometry.

Lemma 3.1. *Suppose that the function f satisfies $(F1) - (F2)$ and the potential V verifies $(V1) - (V3)$. Then for all $\lambda > 1$, there exist $\rho,\ \alpha > 0$ such that*

$$I_\lambda(v) \geq \alpha \quad \text{for all} \quad v \in \mathcal{H} \quad \text{with} \quad \|v\| = \rho.$$

Proof. For $\varepsilon > 0$, from $(F2)$ there exists δ_0 such that

$$F(x,t) \leq \varepsilon |t|^2, \quad \text{for all} \quad |t| \leq \delta_0.$$

It follows from $(F1)$ the existence of a constant $A = A(\varepsilon) \geq 0$ such that

$$F(x,t) \leq \varepsilon A|t|^p, \quad \text{for all} \quad |t| \geq \delta_0,$$

when p is in $(2, 2_*)$. Therefore,

$$F(x,t) \leq \varepsilon|t|^2 + A|t|^p, \quad \forall t \in \mathbb{R}. \tag{3.15}$$

So, by using (2.8), we obtain

$$I_\lambda(v) \geq \frac{\mu_0 - 1}{2\mu_0}\|v\|_\lambda^2 - \varepsilon\|v\|_2^2 - A\|v\|_p^p.$$

By continuous embedding result, we have $\|v\|_2^2 \leq C_1\|v\|_\lambda^2$ and $\|v\|_p^p \leq C_2\|v\|_\lambda^2$ for some constants $C_1, \quad C_2 > 0$. Therefore, we have

$$I_\lambda(v) \geq \frac{\mu_0 - 1}{2\mu_0}\|v\|_\lambda^2 - \varepsilon C_1\|v\|_\lambda^2 + AC_2\|v\|_\lambda^p. \tag{3.16}$$

By taking $0 < \varepsilon < \frac{\mu_0 - 1}{2\mu_0}c_1^{-1}$ and choosing $\|v\|_\lambda = \rho$ small enough, the Lemma 3.1 follows, since $2 < p$. □

To complete the geometric requirements of the Theorem 2.2, we prove the following Lemma.

Lemma 3.2. *Assume that the function f satisfies $(F1) - (F2)$ and the potential V verifies $(V1) - (V3)$. Then, for all $\lambda > 1$, for all $e \in H^2(\Omega, V^+)$ and $e \neq 0$, we have*

$$I(te) \to -\infty \quad \textit{as} \quad t \to \infty.$$

Proof. Let $A > 0$, from the condition $(F3)$, there exists $T > 1$ such that

$$F(x,t) \geq A|t|^q, \quad \forall |t| > T \quad \text{and} \quad x \in \Omega. \tag{3.17}$$

It follows from $(F2)$ and $(F1)$ that there exists a constant C_1 such that

$$\frac{|f(x,t)|}{|t|} \leq C_1, \quad \forall |t| \leq T \quad \text{and} \quad x \in \Omega. \tag{3.18}$$

By the main value theorem and (3.18), we obtain

$$|F(x,t)| \leq C_1|t|^2, \quad \forall |t| \leq T \quad \text{and} \quad x \in \Omega. \tag{3.19}$$

Now, let $C_2 > 0$, such that

$$C_2 > C_1 + AT^{q-2}.$$

Then, for all $|t| \leq T$, we have

$$C_2 > C_1 + A|t|^{q-2}$$

and so,

$$C_2|t|^2 \geq C_1|t|^2 + A|t|^q, \quad \forall |t| \leq T.$$

Therefore, from (3.19), we obtain

$$F(x,t) \geq A|t|^q - C_2|t|^2, \quad \forall |t| \leq T \;\text{ and }\; x \in \Omega. \tag{3.20}$$

It follows from (3.17) and (3.20) that

$$F(x,t) \geq A|t|^q - C_2|t|^2, \quad \forall t \in \mathbb{R} \;\text{ and }\; x \in \Omega. \tag{3.21}$$

For $t > 0$ and e a nonzero element of $\mathcal{H}$, we have from (2.12)

$$I_\lambda(te) \leq \frac{t^2}{2}\|e\|_\lambda^2 - At^q\|e\|_q^q + C_2t^2\|e\|_2^2.$$

Since $2 < q$, we conclude that $\lim\limits_{t\to\infty} I_\lambda(te) = -\infty$. This completes the proof of Lemma 3.2. □

Proof of the Theorem 1.1

In order to prove the existence of nonzero critical point for I_λ, by Lemma 3.1 and Lemma 3.2, we have only to prove that I_λ satisfies the (PS) condition. Let (v_n) be a (PS) sequence of I_λ at level $d \in \mathbb{R}$; that is,

$$I_\lambda(v_n) \to d \quad \text{ as } \; n \to \infty \tag{3.22}$$

$$\|I'_\lambda(v_n)\|_* \to 0 \quad \text{ as } \; n \to \infty. \tag{3.23}$$

We are going to prove that the sequence (v_n) has a convergent subsequence in $\mathcal{H}$ in two steps.

Step 1. In this step, we show that (v_n) is bounded in $\mathcal{H}$.
For this, suppose that (v_n) is not bounded. So, up to subsequence,
$\|v_n\|_\lambda \to \infty$ as $n \to \infty$.
Let

$$w_n = \frac{v_n}{\|v_n\|_\lambda}.$$

Hence, there exists $w \in \mathcal{H}$ such that, up to subsequence $w_n \rightharpoonup w$ in $\mathcal{H}$, $w_n \to w$ in $L^2(\Omega)$ and $w_n \to w$ a.e. in Ω.
We claim that $w \equiv 0$ a.e. in Ω. Indeed, set

$$\Omega_0 = \{x \in \Omega; \quad w(x) \neq 0\}.$$

From $(F3)$ and $(F4)$, for $A > 0$, there exist $M_1 > 0$ such that

$$f(x,t)t \geq \tau A|t|^q, \quad \forall |t| > M_1 \ \text{ and } \ x \in \Omega.$$

Now, from $(F2)$ and as in the proof of Lemma 3.2, there exists a constant $C > 0$ such that

$$f(x,t)t \geq \tau A|t|^q - C|t|^2, \quad \forall t \in \mathbb{R} \ \text{ and } \ x \in \Omega.$$

So

$$\int_\Omega \frac{f(x,v_n)v_n}{\|v\|_\lambda^q}\,dx \geq \tau A\|w_n\|_q^q - C\frac{\|w_n\|_2^2}{\|v\|_\lambda^{q-2}}. \tag{3.24}$$

On the other hand, from (3.23) we get

$$\langle I'_\lambda(v_n), v_n\rangle \to 0 \quad \text{ as } \ n \to \infty,$$

and then

$$\frac{\int_\Omega[|\Delta v_n|^2 + |\nabla v_n|^2 + V_\lambda(x)v_n^2]\,dx}{\|v\|_\lambda^q} - \int_\Omega \frac{f(x,v_n)v_n}{\|v\|_\lambda^q}\,dx \to 0. \tag{3.25}$$

Or

$$\begin{aligned}\frac{\int_\Omega[|\Delta v_n|^2 + |\nabla v_n|^2 + V_\lambda(x)v_n^2]\,dx}{\|v\|_\lambda^q} &\leq \frac{\int_\Omega[|\Delta v_n|^2 + |\nabla v_n|^2 + \lambda V^+(x)v_n^2]\,dx}{\|v\|_\lambda^q}\\ &\leq \frac{1}{\|v\|_\lambda^{q-2}}.\end{aligned}$$

Therefore, from $(F4)$ and (3.25), we obtain

$$\int_\Omega \frac{f(x,v_n)v_n}{\|v\|_\lambda^q} \to 0 \quad \text{ as } \ n \to \infty. \tag{3.26}$$

So, if we pass to the limit in (3.24), one has

$$0 \geq \tau A\|w\|_q^q > 0$$

and then

$$\|w\|_q^q = \int_{\Omega_0} |w|^q \, dx = 0;$$

that is, meas$(\Omega_0) = 0$ and the claim follows. As consequence, $|v_n(x)| \to \infty$ for a.e. $x \in \Omega$.
If we consider $I_\lambda(v_n) - \frac{1}{\tau}\langle I'_\lambda(v_n), v_n\rangle$, we have

$$I_\lambda(v_n) - \frac{1}{\tau}\langle I'_\lambda(v_n), v_n\rangle = (\frac{1}{2} - \frac{1}{\tau}) \int_\Omega [|\Delta v_n|^2 + |\nabla v_n|^2 + V_\lambda(x) v_n^2] dx$$
$$+ \int_\Omega [\frac{1}{\tau} f(x, v_n) v_n - F(x, v_n)] \, dx.$$

It follows from (2.13), (3.22) and (3.23) that

$$d + o(1) \geq (\frac{1}{2} - \frac{1}{\tau}) \frac{\mu_0 - 1}{\mu_0} \|v_n\|_\lambda^2 + \int_\Omega [\frac{1}{\tau} f(x, v_n) v_n - F(x, v_n)] \, dx.$$

Now, for n large enough, by using $(F4)$, we have

$$d + o(1) \geq (\frac{1}{2} - \frac{1}{\tau}) \frac{\mu_0 - 1}{\mu_0} \|v_n\|_\lambda^2$$

by passing to the limit, we get a contradiction.

Step 2. In this second step, we show that (v_n) is is relatively compact in $\mathcal{H}$. From the step 1, the sequence (v_n) is bounded in $\mathcal{H}$. Up to subsequence, we can suppose that $v_n \rightharpoonup v$ weakly in $\mathcal{H}$ and $v_n \to v$ strongly in $L^r(\Omega)$, for each $1 \leq r < 2_* = \frac{2N}{N-4}$. By the condition (3.23), we have

$$\Delta^2 v_n - \Delta v_n + V_\lambda(x) v_n - f(x, v_n) \to 0 \ \text{ in } \ \mathcal{H}' \ \text{ as } \ n \to \infty. \tag{3.27}$$

From the condition $(F1)$, the continuous embedding of $\mathcal{H}$ in $L^{2_*}(\Omega)$ the sequence $f(x, v_n(x))$ converge to $f(x, v)$ in $L^p(\Omega)$ for $p = \frac{2N}{N+4}$.
Since $p = (2_*)'$ the conjugate of 2_*, we have

$$f(x, v_n(x)) \to f(x, v) \ \text{ in } \ \mathcal{H}' \ \text{ as } \ n \to \infty.$$

So,

$$\Delta^2 v_n - \Delta v_n + V_\lambda(x) v_n \to f(x, v) \ \text{ in } \ \mathcal{H}' \ \text{ as } \ n \to \infty. \tag{3.28}$$

But the operator $L : \mathcal{H} \to \mathcal{H}'$ given by

$$\langle Lv, w\rangle = \int_\Omega [\Delta v \Delta w + \nabla v.\nabla w + V_\lambda(x)vw]dx$$

is invertible and then,

$$v_n \to L^{-1}[f(x, v)].$$

From the uniqueness of the limit when it exists, we get $v_n \to v$ in the space $\mathcal{H}$. This finish the proof of the Theorem 1.1.

References

[1] A. Ambrosetti, P. H. Rabionowitz, *Dual variational methods in critical points theory and applications*, J. Funct. Anal., **14,** (1973), 349–381.

[2] S. Baraket, M. Dammak, T. Ouni, F. Pacard, *Singular limits for a* 4*-dimensional semilinear elliptic problem with exponential nonlinearity,* Ann. I. H. Poincaré - AN 24, (2007), 875–895.

[3] S. Y. A. Chang, P. Yang, *Fourth order equations in conformal geometry,* in: Global Analysis and Harmonic Analysis, (Marseille-Luminy, 1999), in: Smin. Congr., **4,** Soc. Math. France, Paris (2000), 155–165.

[4] S. Y. A. Chang, P. Yang, *On a fourth order curvature invariant,* in: T. Branson (Ed.), Spectral Problems in Geometry and Arithmetic, in: Contemporary Mathematics, **237,** Amer. Math. Soc., (1999), 9–28.

[5] M. Dammak, T. Ouni, *Singular limits for* 4*-dimensional semi linear elliptic problem with exponential nonlinearity adding a singular source term given by Dirac masses,* Differential and Integral Equations, **21,** Nos. 11-12, (2008), 1019–1036.

[6] P. Drabek, A. Kufner, F. Nicolosi, *Quasilinear Elliptic Equations with Degenerations and Singularities*, Walter de Gruyter, Berlin, 1997.

[7] A. Ferrero, G. Warnault, *On a solutions of second and fourth order elliptic with power type nonlinearities,* Nonlinear Anal. T. M. A., **70,** no. 8, (2009), 2889–2902.

[8] A. Harrabi, *Fourth-order Elliptic Equations,* Advanced Nonlinear Studies, **14,** (2014), 593–604.

[9] A. Kufner, *Weighted Sobolev spaces,* John Wiley and Sons Ltd, 1985.

[10] M. C. Lai, H. C. Liu, *Fast direct solver for the biharmonic equation on a disk and its application to incompressible flows,* Appl. Math. Comput., (2005), 679–695.

[11] A. C. Lazer, P.J. McKenna, *Global bifurcation and a theorem of Tarantello,* J. Math. Anal. Appl., **181,** (1994), 648–655.

[12] A.M. Micheletti, A. Pistoia *Multiplicity results for a fourth-order semilinear elliptic problem,* Nonlinear Anal., **31,** (1998), 895–903.

[13] A. M. Micheletti, A. Pistoia *Nontrivial solutions for some fourth order semilinear elliptic problems,* Nonlinear Anal., **34,** (1998), 509–523.

[14] T. G. Myers, *Thin films with high surface tension,* SIAM Review, **40,** no. 3, (1998), 441–462.

[15] M. Ramos, P. Rodrigues, *On a fourth order superlinear elliptic problem,* Electron. J. Diff. Eqns, Conf. 06, (2001), 243–255.

[16] R. Soranzo, *A priori estimates and existence of positive solutions of superlinear polyharmonic equation,* Dynamic System and Appl., **3,** (1994), 465–488.

[17] Q. Xiao, H. Liu, Z. Ouyang, *Existence and concentration of a nonlinear biharmonic equation with sign-changing potentials and indefinite nonlinearity,* Advances in Difference Equations 2018:384 ,(2018), https://doi.org/10.1186/s13662-018-1782-9.

[18] J. Zhou, X. Wu, *Sign-changing solutions for some fourth-order nonlinear elliptic problems,* J. Math. Anal. Appl., 342 (2008), 542–558.