International Journal of Mathematics and
Computer Science, **16**(2021), no. 4, 1249–1257

Generalized soft bi-ideals over semigroups

A. A. Ramadan[1], **E. Hamouda**[2], **A. Seif**[2]

[1] Department of Mathematics
Faculty of Science
Beni-Suef University
Beni-Suef, Egypt

[2] Department of Basic Science
Faculty of Technology and Education
Beni-Suef University
Beni-Suef, Egypt

email: ehamouda70@gmail.com

(Received January 31, 2021, Accepted March 2, 2021)

Abstract

In this paper, we present the concept of a generalized soft bi-ideal over a semigroup S. Moreover, we investigate relations among soft ideals over S. Furthermore, we characterize regular and intra-regular semigroups in terms of generalized soft bi-ideals.

1 Introduction and basic definitions

A semigroup is an algebraic structure consisting of a nonempty set S with an associative binary operation on it. A subset $\phi \neq A \subseteq S$ is called a subsemigroup of S if $A^2 \subseteq A$, a left (right) ideal of S if $SA \subseteq A$ ($AS \subseteq A$) and a two-sided ideal (or simply ideal) of S if it is both a left and a right ideal of S. A nonempty subset $B \subseteq S$ is called a generalized bi-ideal of S if $BSB \subseteq B$ [5]. Some significant applications of semigroup theory exist in many fields like finite state machines, automata, and coding theory. Therefore, semigroups and related structures were investigated in fuzzy settings [8]

Key words and phrases: Soft semigroups, soft ideals, generalized soft bi-ideals.
AMS (MOS) Subject Classifications: 16Y30, 06D72.
ISSN 1814-0432, 2021, http://ijmcs.future-in-tech.net

and recently in (fuzzy soft) soft settings [1, 3, 4, 6, 10, 11, 12]. The concepts of soft semigroups and soft ideals were introduced in [2, 3] as a collection of subsemigroups (ideals) of a semigroup. As a result, we define the concept of a generalized soft bi- ideal over a semigroup S. We investigate relations between soft ideals and generalized soft bi-ideal and characterize regular and intra-regular semigroups in terms of generalized soft bi-ideals.

Definition 1.1. [7] Let E be a set of parameters, $P(S)$ the power set of S and $A \subseteq E$. The pair (F, A) is called a soft set over S, where F is a mapping $F : A \longrightarrow P(S)$.

Definition 1.2. [3] Let (F, A) and (G, B) be soft sets over S. Then (G, B) is called a soft subset of (F, A), denoted by $(G, B) \sqsubseteq (F, A)$, if $B \subseteq A$ and $G(b) \subseteq F(b)$ for all $b \in B$. These two soft sets (F, A) and (G, B) are equal iff $(G, B) \sqsubseteq (F, A)$ and $(F, A) \sqsubseteq (G, B)$.

Definition 1.3. [3] Let (F, A) and (G, B) be soft sets over S. The restricted intersection of (F, A) and (G, B), $(F, A) \sqcap (G, B)$, is a soft set (H, C) defined as $H(c) = F(c) \cap G(c)$ for all $c \in C = A \cap B$.

Definition 1.4. [3] Let (F, A) and (G, B) be soft sets over S. The restricted union of (F, A) and (G, B), denoted by $(F, A) \sqcup (G, B)$, is a soft set (H, C) defined as $H(c) = F(c) \cup G(c)$ for all $c \in C = A \cap B$.

Definition 1.5. [3] If (F, A) and (G, B) are soft sets over $(S, \star)$, then $(F, A) \star (G, B)$ is a soft set $(H, A \times B)$, where $H(a, b) = F(a) * G(b)$, for all $(a, b) \in A \times B$.

2 Generalized soft bi-ideals

In this section, we define the generalized soft bi-ideals over a semigroup S and investigate relations among soft ideals.

Definition 2.1. [3] Let (F, A) and (G, B) be soft sets over S. The soft product of (F, A) and (G, B), $(F, A) \diamond (G, B)$, is defined as the soft set (FG, C) where $FG(c) = F(c)G(c)$ for all $c \in C = A \cap B$.

Definition 2.2. [3] A soft set (F, A) over S is called a soft semigroup if $(F, A) \diamond (F, A) \subseteq (F, A)$.

Proposition 2.3. [3] A soft set (F, A) is a soft semigroup (ideal) over S if and only if, $\forall a \in A$, $F(a) \neq \phi$ is a subsemigroup (an ideal) of S.

Definition 2.4. *Let $(S, .)$ be a semigroup. A soft set (F, A) over S is called a generalized soft bi-ideal over S if*

$$(F, A) \diamond (S, A) \diamond (F, A) \subseteq (F, A)$$

.

We observe that every soft bi-ideal over S is a generalized soft bi-ideal over S. The following example shows that the opposite direction does not hold in general.

Example 2.5. *Let $S = \{a, b, c, d\}$ be a semigroup with the following table:*

	x	y	z	w
x	x	x	x	x
y	x	x	x	x
z	x	x	y	x
w	x	x	y	y

Let $A = \{a, b\}$ and let (F, A) be a soft set over S defined by

$$F(a) = \{x, y\}, \quad F(b) = \{x, z\}.$$

Then (F, A) is not soft semigroup since

$$F(a)F(a) = \{x, w\}\{x, w\} = \{x, y\} \nsubseteq F(a).$$

Hence (F, A) is not soft bi-ideal over S. On the other hand, we have

$$F(a)SF(a) = \{x\} \subseteq F(a)$$

$$F(b)SF(b) = \{x\} \subseteq F(b);$$

that is, $(F, A) \diamond (S, A) \diamond (F, A) \subseteq (F, A)$. Therefore, (F, A) is a generalized soft bi-ideal over S.

Theorem 2.6. *Let (F, A) be a soft set over S. Then the following are equivalent:*

1. (F, A) is a generalized soft bi-ideal over S,
2. For all $a \in A$, $F(a) \neq \phi$ is a generalized bi-ideal of S.

Proof. Assume that (F, A) is a generalized soft bi-ideal over S. Then

$$(F, A) \circ (S, A) \circ (F, A) \subseteq (F, A).$$

It follows that $F(a)SF(a) \subseteq F(a)$ for all $a \in A$ which means that $F(a)$ is a generalized bi-ideal of S, whenever $F(a) \neq \phi$.

Conversely, if $F(a) = \phi$ for all $a \in A$, then $F(a)SF(a) = \phi \subseteq F(a)$ for otherwise $F(a) \neq \phi$ is a generalized bi-ideal over S and so $F(a)SF(a) \subseteq F(a)$. Hence

$$(F, A) \circ (S, A) \circ (F, A) \subseteq (F, A).$$

This shows that (F, A) is a generalized soft bi-ideal over S.

Lemma 2.7. *Let (F, A) be a soft quasi-ideal over S. Then (F, A) is a generalized soft bi-ideal over S.*

Proof. Assume that (F, A) is a soft quasi-ideal over S. By Proposition 4.2 in [3], (F, A) is a generalized soft bi-ideal over S.

Proposition 2.8. *Let (F, A) and (G, B) be generalized soft bi-ideals over S. Then $(F, A) \sqcap (G, B)$ is a generalized soft bi-ideal over S.*

Proof. Assume that $(F, A) \sqcap (G, B) = (H, A \cap B)$ is a nonempty soft set over S. By hypotheses, $F(c)$ and $G(c)$ are generalized bi-ideals of S for all $c \in A \cap B$. Let $x, y \in H(c)$ and $z \in S$. Then $x, y \in F(c)$ and $x, y \in G(c)$. Hence,

$$xzy \in H(c)SH(c) \subseteq F(c)SF(c) \subseteq F(c)$$

and

$$xzy \in H(c)SH(c) \subseteq G(c)SG(c) \subseteq G(c).$$

Thus $xzy \in F(c) \cap G(c) = H(c)$; that is, $H(c)SH(c) \subseteq H(c)$ for all $c \in A \cap B$. Therefore, $(F, A) \sqcap (G, B)$ is a generalized soft bi-ideal over S.

Lemma 2.9. *Let $(S, \star)$ be a semigroup and let (F, A) and (G, B) be soft sets over S. Then $(F, A) \star (G, B)$ is a soft bi-ideal over S if (G, B) is a soft generalized bi-ideal over S.*

Proof. By definition 1.5, $H(a, b) = F(a) \star G(b)$, for all $(a, b) \in A \times B$. Assume that $H(a, b) \neq \phi$ and $G(b)$ is a generalized bi-ideal of S, for all $b \in B$. Then

$$F(a) \star G(b) \star F(a) \star G(b) \subseteq F(a) \star G(b) \star S \star G(b) \subseteq F(a) \star G(b).$$

Thus $F(a) \star G(b)$ is a subsemigroup of S; that is, $(F, A) \star (G, B)$ is a soft semigroup over S. Also, we have

$$\begin{aligned} F(a) \star G(b) \star S \star F(a) \star G(b) &\subseteq F(a) \star G(b) \star S \star S \star G(b) \\ &\subseteq F(a) \star G(b) \star S * G(b) \subseteq F(a) \star G(b) \end{aligned}$$

This implies that $F(a) \star G(b)$ is a bi-ideal of S, for all $(a, b) \in A \times B$. Consequently, $(F, A) \star (G, B)$ is a soft bi-ideal over S.

Proposition 2.10. *Let (F, A) and (G, B) be generalized soft bi-ideals over S. Then $(F, A) \diamond (G, B)$ is a generalized soft bi-ideal over S.*

Proof. By definition 2.1, we can write $(F, A) \diamond (G, B) = (FG, C)$. Let $x, y \in FG(c)$ and $z \in S$. Then

$$\begin{aligned} xzy \in FG(c)SFG(c) &= F(c)G(c)SF(c)G(c) \\ &\subseteq F(c)G(c)SSG(c) \\ &\subseteq F(c)G(c)SG(c) \subseteq F(c)G(c) = FG(c). \end{aligned}$$

Thus $FG(c)$ is a generalized bi-ideal of S for all $c \in A \cap B$. Therefore, the product $(F, A) \diamond (G, B)$ is a generalized soft bi-ideal over S.

3 Regular and intra-regular semigroups

A semigroup S is called regular (intra-regular) if for all $a \in S$ there exists $x \in S$ (there are $x, y \in S$)such that $a = axa$ ($a = xa^2y$)[3]. The following result shows that the concepts of soft bi-ideals and generalized soft bi-ideals over a regular semigroup coincide.

Theorem 3.1. *Let (F, A) be a soft set over a regular semigroup. Then the following statements are equivalent*

(a) (F, A) is a soft bi-ideal over S,

(a) (F, A) is a soft generalized bi-ideal over S.

Proof. $(a) \Rightarrow (b)$ is straightforward by the definition. Suppose (F, A) is a soft generalized bi-ideal over S. Hence $F(a)$ is a generalized bi-ideal of S, for all $a \in A$. By Definition 2.4, (F, A) is a soft semigroup over S. Since S is regular, for $x, y \in F(a)$, there exist $u, v \in S$ such that $x = xux$ and $y = yvy$. Hence $xy = xuxyvy = x(uxyv)y \in F(a)$; that is, $F(a)F(a) \subseteq F(a)$. Thus (F, A) is a soft semigroup over S. Consequently, (F, A) is a soft bi-ideal over S.

Theorem 3.2. *Let (F, A) be a soft generalized bi-ideal over a semigroup S. Then S is regular if and only if*

$$(F, A) \diamond (S, A) \diamond (F, A) = (F, A)$$

Proof. First, suppose that (F, A) is a soft generalized bi-ideal over S and $x \in F(a)$, for $a \in A$. Then there exists $z \in S$ such that $x = xzx$, as S is regular. So $x = xzx$ is an element of $F(a)SF(a)$ and hence $F(a)SF(a) \supseteq F(a)$ since $F(a)SF(a) \subseteq F(a)$. Thus, for all $a \in A$, $F(a)SF(a) = F(a)$; that is, $(F, A) \diamond (S, A) \diamond (F, A) = (F, A)$. For the other direction, consider the soft set (G, S) over S defined by $G(x) = xS^1$, for all $x \in S$. Then (G, S) is a soft quasi-ideal [3] and consequently is a soft generalized bi-ideal over S. By hypothesis, we obtain $G(x) = G(x)SG(x)$. From Theorem 3.1.2 in [8], it follows that S is regular.
Using soft bi-ideals and generalized soft bi-ideals over a semigroup S, we give a characterization of regular semigroups.

Theorem 3.3. *A semigroup S is regular $\Leftrightarrow$*

$$(G, A) \sqcap (F, B) \sqsubseteq (G, A) \diamond (F, B)$$

for every generalized soft bi-ideal (G, A) and soft left ideal (F, B) over S, where $A \cap B \neq \phi$.

Proof. ($\Rightarrow$) We write $(G, A) \sqcap (F, B) = (H, A \cap B)$ and $(G, A) \diamond (F, B) = (GF, A \cap B)$ such that

$$H(c) = G(c) \cap F(c), \quad GF(c) = G(c)F(c) \quad \forall c \in A \cap B.$$

Now, Let $z \in G(c) \cap F(c)$. Then $z \in G(c)$ and $z \in F(c)$. Since S is regular, there exists $w \in S$ such that $z = zwz \in G(c)SF(c) \subseteq G(c)F(c) = GF(c)$. Hence $(G, A) \sqcap (F, B) \sqsubseteq (G, A) \diamond (F, B)$.
($\Leftarrow$) Assume that $A = B = S$ and the soft sets (G, S) and (F, S) are defined by $G(x) = xS \cup x = xS^1$ and $F(x) = Sx \cup x = S^1x$. Then (G, S) is a generalized soft bi-ideal since it is a soft right ideal and (F, S) is a soft left ideal over S. By hypothesis,

$$x \in G(x) \cap F(x) \subseteq G(x)F(x) = xS^1S^1x \subseteq xS^1x.$$

That is, for all $x \in S$, there exists $y \in S$ such that $x = xyx$. This means S is a regular semigroup.

Theorem 3.4. *A semigroup S is regular* $\Leftrightarrow$

$$(G, A) \sqcap (F, B) = (G, A) \diamond (F, B) \diamond (G, A), \tag{3.1}$$

for every soft interior ideal (F, B) and a generalized soft bi-ideal (G, A) over S.

Proof. ($\Rightarrow$) Assume that $G(c)$ and $F(c)$ are generalized bi-ideal and interior ideal of S, respectively, for all $c \in A \cap B$. Then

$$G(c)F(c)G(c) \subseteq G(c)SG(c) \subseteq G(c)$$

and

$$G(c)F(c)G(c) \subseteq SF(c)S \subseteq F(c)$$

Thus $G(c)F(c)G(c) \subseteq F(c) \cap G(c)$, for all $c \in A \cap B$. Hence

$$(G, A) \diamond (F, B) \diamond (G, A) \sqsubseteq (G, A) \sqcap (F, B).$$

Now, let $z \in G(c) \cap F(c)$. Since S is regular, there exists $w \in S$ such that $z = zwz$. Hence

$$z = (zwz)wz = z(wzw)z \in G(c)F(c)G(c).$$

Thus $G(c) \cap F(c) \subseteq G(c)F(c)G(c)$, for all $c \in A \cap B$. Therefore,

$$(G, A) \sqcap (F, B) = (G, A) \diamond (F, B) \diamond (G, A).$$

($\Leftarrow$) Suppose that condition (1) holds. Then

$$(G, A) = (G, A) \sqcap (S, A) = (G, A) \diamond (S, A) \diamond (G, A).$$

By Theorem 3.2, S is a regular semigroup.

Theorem 3.5. *A semigroup S is regular and intra-regular* $\Leftrightarrow$

$$(F, A) \sqcap (G, B) \sqsubseteq (F, A) \diamond (G, B),$$

for every generalized soft bi-ideals (F, A) and (G, B) over S, where $A \cap B \neq \phi$.

Proof. Assume that ($\Rightarrow$) holds and (F, A) and (G, B) are generalized soft bi-ideals over S. Let $z \in F(c) \cap G(c) \quad \forall c \in A \cap B$. Since S is regular, there exists $w \in S$ such that $z = zwz$. In addition, since S is intra-regular, there are $x, y \in S$ such that $z = xz^2y$. Hence

$$z = zwz = zwzwz = zw(xz^2y)wz = (zwxz)(zywz) \in F(c)G(c) = FG(c).$$

Thus $F(c) \cap G(c) \subseteq FG(c)$ for all $c \in A \cap B$. That is, $(F, A) \sqcap (G, B) \sqsubseteq (F, A) \diamond (G, B)$.
Let (H, C) be any soft bi-ideal over S. Then (H, C) is a soft semigroup over S and so

$$(H, C) \diamond (H, C) \sqsubseteq (H, C).$$

Since (H, C) is a generalized soft bi-ideal over S, by hypotheses, we obtain

$$(H, C) \diamond (H, C) \sqsupseteq (H, C).$$

Thus $(H, C) \diamond (H, C) = (H, C)$ and theorem 166 in [2] implies that S is a regular and an intra-regular semigroup. □

References

[1] M. Akram, N. Yaqoob, Intuitionistic fuzzy soft ordered ternary semigroups, International Journal of Pure and Applied Mathematics, **84,** no. 2, (2013), 93–107.

[2] I. M. Ali, Some contributions to soft semigroups and related structures, Ph. D Thesis, Quaid-i-Azam University, 2009.

[3] I. M. Ali, M. Shabir, P. K. Shum, On soft ideals over semigroups, Southeast Asian Bulletin of Mathematics, **34,** (2010), 595–610.

[4] T. Changphas, B. Thongkam, On soft Γ-semigroups, Ann. Fuzzy Math. Inf., **4,** no. 2, (2012), 217–223.

[5] M. J. Howie, An Introduction to Semigroup Theory, Academic Press, 1976.

[6] B. Y. Jun, J. K. Lee, A. Khan, Soft ordered semigroups, Math. Log. Quart., **56,**, no. 1, (2010), 42–50.

[7] A. D. Molodtsov, Soft set theory first results, Comput. Math, Appl.,**37,**(1999), 19–31.

[8] N. J. Mordeson, S. D. Malik, N. Kuroki, Fuzzy Semigroups, Springer-Verlag, Berlin Heidelberg, 2003.

[9] A. Ramadan, A. Halil, E. Hamouda, A. Seif, Soft ideals over a semigroup generated by a soft set, J. New Theory, **17,** (2017), 94–102.

[10] A. Ramadan, E. Hamouda, A. Seif, Soft interior ideals over semigroups, Ital. J. Pure Appl. Math.,(2020), (accepted).

[11] P. Ramasamy, T. Manikantan, Soft bi-ideals of soft near-rings, International Journal of Mathematics and Computer Science, **15,** (2020), no. 1, 223–239

[12] M. Shabir, A. Ahmad, On soft ternary semigroups, Ann. Fuzzy Math. Inf., **3,** no.1, (2012), 39–59.