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Commutativity and Prime Ideals with Proposed Algebraic Identities

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Abstract

In modern algebra, the correlation between the derivations and the algebraic structures of quotient rings has become an exciting subject during the last few decades. In particular, the derivations and their generalizations play a major role in mathematics, economics, quantum physics, and biology such as in chemotherapy. The main purpose of this article is to study the commutativity of a quotient ring over a prime ideal. The objectives used to achieve this goal are by proposing algebraic identities contained in a prime ideal that concern generalized reverse derivation and multiplicative left centralizer mappings. The proposed identities describe mostly the two types of mappings on an associative ring. If the identity $\Delta(\omega_1\omega_2)\pm\mu(\omega_1)\xi(\omega_2)\pm[\omega_1,\omega_2]$ belongs to a prime ideal for all elements ω_1, ω_2 of an associative ring, then one of the following is fulfilled: (1) $[\omega_1, \mu(\omega_1)]$ is contained in the prime ideal for all ω_1 in the ring that the multiplicative left centralizer μ is defined on, (2) the quotient ring over a prime ideal is commutative, and (3) a reverse derivation maps the ring into the prime ideal.

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1 Introduction

Throughout this article, $C(S)$ refers to the center of an associative ring S. A proper ideal Λ of S is said to be prime if $\omega_1 S\omega_2 \subseteq \Lambda$ for all elements ω_1, ω_2 of S yields either $\omega_1 \in \Lambda$ or $\omega_2 \in \Lambda$. So the ring S is prime if and only if the zero ideal is prime of S. For any elements ω_1, ω_2 of S, the symbols $[\omega_1, \omega_2]$ and $\omega_1 \circ \omega_2$ indicate to the commutator $\omega_1\omega_2 - \omega_2\omega_1$ and the anti-commutator $\omega_1\omega_2 + \omega_2\omega_1$ respectively. The following identities of the commutator and the anti-commutator have a vital role throughout this paper by implementing several simplifications and calculations to fulfill the main theorems: $[\omega_1\omega_2, \omega_3] = \omega_1[\omega_2, \omega_3] + [\omega_1, \omega_3]\omega_2$, $[\omega_1, \omega_2\omega_3] = [\omega_1, \omega_2]\omega_3 + \omega_2[\omega_1, \omega_3]$, and $\omega_1 \circ \omega_2\omega_3 = (\omega_1 \circ \omega_2)\omega_3 - \omega_2[\omega_1, \omega_3] = \omega_2(\omega_1 \circ \omega_3) + [\omega_1, \omega_2]\omega_3$ for all elements $\omega_1, \omega_2, \omega_3 \in S$. Recall that a ring S is semiprime if for all element $\omega_1 \in S$, $\omega_1 S\omega_1 = 0$ infers that $\omega_1 = 0$. A map $\delta : S \to S$ that preserves the addition operation (an additive map) satisfying $\delta(\omega_1\omega_2) = \delta(\omega_1)\omega_2 + \omega_1\delta(\omega_2)$ for all $\omega_1, \omega_2 \in S$ is said to be derivation (D, for short). There are different kinds of derivations that can be found in the literature; for instance, [1] and [2]. In recent decades, several mathematicians have generalized the notion of derivation in different manners. In 1991, the concept of generalized derivation (GD, for short) has been raised by Brešar [3] as one of these ways of expansions to be an additive mapping $\xi : S \to S$ that fulfills $\xi(\omega_1\omega_2) = \xi(\omega_1)\omega_2 + \omega_1\delta(\omega_2)$ for all $\omega_1, \omega_2 \in S$, where $\delta : S \to S$ is a D mapping. In one of the early articles of Herstein [4], the concept of a reverse derivation (RD, for short) originated as follows: an additive mapping $\delta : S \to S$ that satisfying $\delta(\omega_1\omega_2) = \delta(\omega_2)\omega_1 + \omega_2\delta(\omega_1)$ for all $\omega_1, \omega_2 \in S$. For a commutative ring, the concept of D and the concept of RD are the same. According to Aboubakr and González [5], the term of generalized reverse derivation (GRD, for short) is defined as an additive map $\xi : S \to S$ such that $\xi(\omega_1\omega_2) = \xi(\omega_2)\omega_1 + \omega_2\delta(\omega_1)$ is valid for all elements $\omega_1, \omega_2 \in S$, where $\delta : S \to S$ is an RD and it is symbolized by (ξ, δ). Clearly, every RD is a GRD but the converse need not be true in general. Zalar [6] presented the notion of the left centralizer (LC, for short) on a ring S as an additive mapping $\mu : S \to S$ that satisfies $\mu(\omega_1\omega_2) = \mu(\omega_1)\omega_2$ for all $\omega_1, \omega_2 \in S$. Obviously, the notion of GD includes each of the concept of D and the concept of LC when $\delta = 0$. In the definition of LC, if μ is not assumed to be additive, then μ is called multiplicative left centralizer (MLC, for short) which is presented by Tammam et al. [7]. Clearly, the concept of MLC is stronger than LC. The investigation of the mappings that preserve commutativity has been played an effective role in ring theory, matrix theory, and operator theory [8,9]. Herstein proved in [10] that a prime ring S with characteristic not equal to two and possessing a nonzero D mapping δ satisfying $[\delta(\omega_1), \delta(\omega_2)] = 0$ for all elements $\omega_1, \omega_2 \in S$ must be commutative. In [11], Bell and Daif showed that a prime ring S possesses a nonzero D mapping δ must be commutative that fulfills $\delta([\omega_1, \omega_2]) = 0$ for all ω_1, ω_2 in some distinguished subset of S. Many results related to commutativity of prime and semiprime rings have been obtained by using appropriate mappings such as derivations, generalized derivations, and generalized reverse derivation that acting on suitable subsets of the rings [12–15]. Moreover, many authors have studied identities that include MLC or LC with other types of generalization of mappings [2,16–18]. Recently, in [19–21], algebraic identities were considered with derivations contained in prime ideal without using the primeness or semiprimeness hypotheses on the studied ring.

This article presents many algebraic identities that include GRD and MLC mappings as $\xi([\omega_1, \omega_2]) \pm \mu(\omega_1\omega_2) \in \Lambda$, $\xi(\omega_1)\mu(\omega_2) \pm \mu(\omega_1) \circ \omega_2 \in \Lambda$, and $\Delta(\omega_1\omega_2) \pm \mu(\omega_1)\xi(\omega_2) \pm [\omega_1, \omega_2] \in \Lambda$ to study the behavior of GRD mapping fulfilling algebraic identities. Besides, discussing identities that contain only GRD mapping included in a prime ideal.

The article is organized as follows. Section 2 deals with GRD mappings and prime ideals. In Section 3, many theorems of the commutativity of a quotient ring over a prime ideal are proved by investigating several algebraic identities involving GRD and MLC mappings that are contained in the prime ideal. Several acronyms are used for ease of expression.

2 On Generalized Reverse Derivations (GRD) Mappings and Prime Ideals

In the following proposition, the GRD mapping is adopted to prove the commutativity of a quotient ring under some conditions.

Theorem 2.1. *Let Λ be a prime ideal of a ring S and (ξ, δ) be a GRD mapping on S such that the condition $\xi([\omega_1, \omega_2]) \in \Lambda$ is valid for all elements ω_1, ω_2 of S. Then either S/Λ is commutative or $\delta(S) \subseteq \Lambda$.*

Proof.

By hypothesis,

$$\xi([\omega_1, \omega_2]) \in \Lambda \text{ for all } \omega_1, \omega_2 \in S. \tag{2.1}$$

Substituting $\omega_1\omega_2$ in place of ω_2 in (2.1) and using (2.1) implies that

$$\xi([\omega_1, \omega_2])\omega_1 + [\omega_1, \omega_2]\delta(\omega_1) = [\omega_1, \omega_2]\delta(\omega_1) \in \Lambda \text{ for all } \omega_1, \omega_2 \in S. \tag{2.2}$$

For $\omega_3 \in S$, putting $\omega_2 = \omega_2\omega_3$ in (2.2) and applying (2.2) yields $[\omega_1, \omega_2]\omega_3\delta(\omega_1) \in \Lambda$ for all $\omega_1, \omega_2, \omega_3 \in S$. The primeness property of Λ gives

$$\text{either } [\omega_1, \omega_2] \in \Lambda \text{ or } \delta(\omega_1) \in \Lambda \text{ for all } \omega_1, \omega_2 \in S. \tag{2.3}$$

Assume that $\Gamma_1 = \{\omega_1 \in S : [\omega_1, \omega_2] \in \Lambda \text{ for all } \omega_2 \in S\}$ and $\Gamma_2 = \{\omega_1 \in S : \delta(\omega_1) \in \Lambda\}$. Clearly, S is a union of the additive subgroups Γ_1 and Γ_2 and this is not true because a group cannot be the union of its subgroups and hence either $S = \Gamma_1$ or $S = \Gamma_2$. Suppose that $S \neq \Gamma_2$ (this means that $\delta(S) \not\subseteq \Lambda$), then there exists an element ω_1 of S such that $\delta(\omega_1) \notin \Lambda$ and according to (2.3), $[\omega_1, \omega_3] \in \Lambda$ for all $\omega_3 \in S$ which means that $\omega_1 + \Lambda \in C(S/\Lambda)$. Assume that $\omega_2 \in S$ such that $\omega_2 + \Lambda \notin C(S/\Lambda)$ so $[\omega_2, \kappa] \notin \Lambda$ for some $\kappa \in S$. Thus, by (2.3), $\delta(\omega_2) \in \Lambda$ and because of the additivity of δ and $\delta(\omega_1) \notin \Lambda$ this implies that $\delta(\omega_1 + \omega_2) \notin \Lambda$. As an application of (2.3), $[\omega_1 + \omega_2, \omega_3] = [\omega_1, \omega_3] + [\omega_2, \omega_3] \in \Lambda$ which means that $[\omega_2, \omega_3] \in \Lambda$, a contradiction to the assumption that $\omega_2 + \Lambda \notin C(S/\Lambda)$. □

The next corollary is immediate from Theorem 2.1 by considering $\xi = \delta$.

Corollary 2.2. *Let Λ be a prime ideal of a ring S and δ be a nonzero RD mapping on S such that the condition $\delta([\omega_1, \omega_2]) \in \Lambda$ is valid for all elements ω_1, ω_2 of S. Then S/Λ is a commutative ring.*

Bell and Daif [11, Theorem 3] proved that a prime ring S admitting a nonzero D mapping that satisfies $\delta([\omega_1, \omega_2]) = 0$ for all elements ω_1, ω_2 of nonzero ideal $\mathcal{I}$ of S must be commutative. Analogously, if Λ is the zero prime ideal in Corollary 2.2, then we have the following:

Corollary 2.3. *Let $\Lambda = (0)$ be a prime ideal and δ be a nonzero RD mapping on S such that the condition $\delta([\omega_1, \omega_2]) = 0$ is valid for all elements ω_1, ω_2 of S. Then S is a commutative ring.*

If Λ is the zero ideal in the condition $\xi([\omega_1, \omega_2]) \in \Lambda$ that used in Theorem 2.1 and S is a semiprime ring, then the following result is obtained.

Proposition 2.4. *Let (ξ, δ) be a GRD mapping on a semiprime ring S such that the statement $\xi([\omega_1, \omega_2]) = 0$ is satisfied for all elements ω_1, ω_2 of S. Then $[\delta(\omega_1), \omega_1] = 0$ for all $\omega_1 \in S$.*

Proof.

Since S is semiprime, by [22] there is a family Δ of prime ideals Λ_i whose intersection is the set containing the zero element only. Hence,

$$\xi([\omega_1, \omega_2]) \in \Lambda_i \text{ for all } \omega_1, \omega_2 \in S \text{ and } \Lambda_i \in \Delta. \tag{2.4}$$

Replacing ω_2 by $\omega_1\omega_2$ in (2.4) yields

$$\xi([\omega_1,\omega_2])\omega_1 + [\omega_1,\omega_2]\delta(\omega_1) \in \Lambda_i \text{ for all } \omega_1,\omega_2 \in S \text{ and } \Lambda_i \in \Delta. \tag{2.5}$$

Multiplying (2.4) by ω_1 from the right and subtracting the result from (2.5) implies that $[\omega_1,\omega_2]\delta(\omega_1) \in \Lambda_i$ for all $\omega_1,\omega_2 \in S$ and $\Lambda_i \in \Delta$, this means $[\omega_1,\omega_2]\delta(\omega_1) \in \bigcap_{\Lambda_i\in\Delta} = (0)$ for all $\omega_1,\omega_2 \in S$. Putting $\omega_2 = \delta(\omega_1)\omega_2$ in the last equation gives

$$[\omega_1,\delta(\omega_1)]\omega_2\delta(\omega_1) = 0 \text{ for all } \omega_1,\omega_2 \in S. \tag{2.6}$$

First, multiplying (2.6) by ω_1 from the right and secondly replacing ω_2 with $\omega_2\omega_1$, we can find the difference between the two equations. Therefore, by semiprimeness of S, one can conclude that $[\delta(\omega_1),\omega_1] = 0$ for all $\omega_1 \in S$. □

The following Lemma is used in the main results of the next section.

Lemma 2.5. *Let Λ be a prime ideal of a ring S and (ξ,δ) be a GRD mapping on S such that the condition $[\omega_1,\xi(\omega_2)] \in \Lambda$ is valid for all elements ω_1,ω_2 of S. Then either S/Λ is commutative or $\delta(S) \subseteq \Lambda$.*

Proof.
By assumption,

$$[\omega_1,\xi(\omega_2)] \in \Lambda \text{ for all } \omega_1,\omega_2 \in S. \tag{2.7}$$

In (2.7), substituting $\omega_2\omega_3$ instead of ω_2, where $\omega_3 \in S$ and using (2.7) for all $\omega_1,\omega_2,\omega_3 \in S$ gives

$$[\omega_1,\xi(\omega_3)\omega_2 + \omega_3\delta(\omega_2)] = \xi(\omega_3)[\omega_1,\omega_2] + [\omega_1,\omega_3]\delta(\omega_2) + \omega_3[\omega_1,\delta(\omega_2)] \in \Lambda. \tag{2.8}$$

In particular, when $\omega_1 = \omega_2$ in (2.8), we get

$$[\omega_1,\omega_3]\delta(\omega_1) + \omega_3[\omega_1,\delta(\omega_1)] \in \Lambda \text{ for all } \omega_1,\omega_3 \in S. \tag{2.9}$$

Substituting $\omega_3\omega_2$ instead of ω_3 in (2.9) and using (2.9), one can conclude that

$$[\omega_1,\omega_3]\omega_2\delta(\omega_1) \in \Lambda \text{ for all } \omega_1,\omega_2,\omega_3 \in S. \tag{2.10}$$

Since Λ is a prime ideal, either $[\omega_1,\omega_3] \in \Lambda$ or $\delta(\omega_1) \in \Lambda$ for all $\omega_1,\omega_3 \in S$ which is the same (2.3) of Theorem 2.1. Hence, by the same argument as in in Theorem 2.1, the required result follows. □

Remark 2.6. *For Lemma 2.5, the following statements hold:*

(a) if $\xi = \delta$, then either S/Λ is commutative or $\delta(S) \subseteq \Lambda$.

(b) if $\Lambda = (0)$ is a prime ideal, then either S is commutative or $\delta = 0$.

(c) if $\Lambda = (0)$ is a prime ideal and $\xi = \delta$, then either S is commutative or $\delta = 0$.

3 Main Results

This section focuses on the fundamental results of this work using the two types of mappings GRD and MLC.

In the following theorem, three statements are proved using specific conditions concerning ξ and μ mappings with prime ideals.

Theorem 3.1. *Let Λ be a prime ideal of a ring S, (ξ, δ) be a GRD mapping on S and μ be an MLC mapping on S. For all $\omega_1, \omega_2 \in S$, if $\xi([\omega_1, \omega_2]) \pm \mu(\omega_1\omega_2) \in \Lambda$ or $\xi(\omega_1)\mu(\omega_2) \pm \mu(\omega_1) \circ \omega_2 \in \Lambda$ is satisfied, then one of the following is fulfilled:*

(a) $[\omega_1, \mu(\omega_1)] \in \Lambda$ for all $\omega_1 \in S$.

(b) S/Λ is commutative.

(c) $\delta(S) \subseteq \Lambda$.

Proof.
Suppose that

$$\xi([\omega_1, \omega_2]) \pm \mu(\omega_1\omega_2) \in \Lambda \text{ for all } \omega_1, \omega_2 \in S. \tag{3.11}$$

Substituting $\omega_1\omega_2$ rather than ω_2 in (3.11) yields

$$\xi([\omega_1, \omega_2])\omega_1 + [\omega_1, \omega_2]\delta(\omega_1) \pm \mu(\omega_1)\omega_1\omega_2 \in \Lambda \text{ for all } \omega_1, \omega_2 \in S. \tag{3.12}$$

Multiplying (3.11) by ω_1 from the right and subtracting the result from (3.12) implies that

$$[\omega_1, \omega_2]\delta(\omega_1) \pm \mu(\omega_1)[\omega_1, \omega_2] \in \Lambda \text{ for all } \omega_1, \omega_2 \in S. \tag{3.13}$$

In (3.13), replacing ω_2 by $\omega_2\omega_3$, for $\omega_3 \in S$ implies that for all $\omega_1, \omega_2, \omega_3 \in S$

$$[\omega_1, \omega_2]\omega_3\delta(\omega_1) + \omega_2[\omega_1, \omega_3]\delta(\omega_1) \pm \mu(\omega_1)[\omega_1, \omega_2]\omega_3 \pm \mu(\omega_1)\omega_2[\omega_1, \omega_3] \in \Lambda. \tag{3.14}$$

Using right multiplication by ω_3 to (3.13) and subtracting the result from (3.14), then for all elements $\omega_1, \omega_2, \omega_3 \in S$ the following is true

$$[\omega_1, \omega_2][\omega_3, \delta(\omega_1)] + \omega_2[\omega_1, \omega_3]\delta(\omega_1) \pm \mu(\omega_1)\omega_2[\omega_1, \omega_3] \in \Lambda. \tag{3.15}$$

Replacing ω_2 with ω_3 in (3.13) and then multiplying the result by ω_2 from left side, one can get $\omega_2[\omega_1, \omega_3]\delta(\omega_1) \pm \omega_2\mu(\omega_1)[\omega_1, \omega_3] \in \Lambda$ for all $\omega_1, \omega_2, \omega_3 \in S$. Subtracting the last relation from (3.15) implies that

$$[\omega_1, \omega_2][\omega_3, \delta(\omega_1)] \pm [\mu(\omega_1), \omega_2][\omega_1, \omega_3] \in \Lambda \text{ for all } \omega_1, \omega_2, \omega_3 \in S. \tag{3.16}$$

Putting $\omega_2 = \mu(\omega_1)$ in (3.16) satisfies that

$$[\omega_1, \mu(\omega_1)][\omega_3, \delta(\omega_1)] \in \Lambda \text{ for all } \omega_1, \omega_2, \omega_3 \in S. \tag{3.17}$$

Substituting $\omega_3\omega_4$ for $\omega_4 \in S$ instead of ω_3 in (3.17) and in light of (3.17), one can get

$$[\omega_1, \mu(\omega_1)]\omega_3[\omega_4, \delta(\omega_1)] \in \Lambda \text{ for all } \omega_1, \omega_3, \omega_4 \in S. \tag{3.18}$$

Since Λ is prime, (3.18) implies that either $[\omega_1, \mu(\omega_1)] \in \Lambda$ or $[\omega_4, \delta(\omega_1)] \in \Lambda$ for all $\omega_1, \omega_4 \in S$. Applying Remark 2.6(a), the second case gives either S/Λ is commutative or $\delta(S) \subseteq \Lambda$.

Now assume that

$$\xi(\omega_1)\mu(\omega_2) \pm \mu(\omega_1) \circ \omega_2 \in \Lambda \text{ for all } \omega_1, \omega_2 \in S. \tag{3.19}$$

In (3.19), substituting $\omega_2\omega_3$ for $\omega_3 \in S$ instead of ω_2 leads to for all elements $\omega_1, \omega_2, \omega_3 \in S$

$$\begin{aligned}\xi(\omega_1)\mu(\omega_2\omega_3) \pm \mu(\omega_1) \circ \omega_2\omega_3 = \xi(\omega_1)\mu(\omega_2)\omega_3 \pm ((\mu(\omega_1) \circ \omega_2)\omega_3 - \\ \omega_2[\mu(\omega_1), \omega_3]) = (\xi(\omega_1)\mu(\omega_2) \pm \mu(\omega_1) \circ \omega_2)\omega_3 \mp \omega_2[\mu(\omega_1), \omega_3].\end{aligned} \tag{3.20}$$

Using (3.19) and by the hypothesis of Λ, (3.20) can be reduced to

$$\mp[\mu(\omega_1), \omega_3] \in \Lambda \text{ for all } \omega_1, \omega_3 \in S. \tag{3.21}$$

Replacing ω_1 by $\omega_1\omega_4$, for $\omega_4 \in S$ and letting $\omega_3 = \xi(\omega_2)$ in (3.21) and using (3.21) yields

$$\mu(\omega_1)[\omega_4, \xi(\omega_2)] \in \Lambda \text{ for all } \omega_1, \omega_2, \omega_4 \in S. \tag{3.22}$$

In (3.22), putting $\omega_4 = \omega_3\omega_4$ and using (3.22) one can verify that

$$\mu(\omega_1)\omega_3[\omega_4, \xi(\omega_2)] \in \Lambda \text{ for all } \omega_1, \omega_2, \omega_3, \omega_4 \in S. \tag{3.23}$$

Substituting $\omega_1\omega_3$ in place of ω_3 in (3.23) and on the other hand using left multiplication by ω_1 to (3.23) in order to subtract one from the other, we deduce that

$$[\omega_1, \mu(\omega_1)]\omega_3[\omega_4, \xi(\omega_2)] \in \Lambda \text{ for all } \omega_1, \omega_2, \omega_3, \omega_4 \in S. \tag{3.24}$$

The primeness of Λ implies that either $[\omega_1, \mu(\omega_1)] \in \Lambda$ or $[\omega_4, \xi(\omega_2)] \in \Lambda$ for all $\omega_1, \omega_2, \omega_4 \in S$. From Lemma 2.5, the commutativity of S/Λ or $\delta(S) \subseteq \Lambda$ has obtained from the second case. □

The next theorem depends on a condition that includes three different mappings ξ, Δ, and μ.

Theorem 3.2. *Let Λ be a prime ideal of a ring S, (ξ, δ) and (Δ, η) be GRD mappings on S and μ be an MLC mapping on S. For all $\omega_1, \omega_2 \in S$ if $\Delta(\omega_1\omega_2) \pm \mu(\omega_1)\xi(\omega_2) \pm [\omega_1, \omega_2] \in \Lambda$, then one of the following is fulfilled:*

(a) $[\omega_1, \mu(\omega_1)] \in \Lambda$ *for all* $\omega_1 \in S$.

(b) S/Λ *is commutative.*

(c) $\delta(S) \subseteq \Lambda$.

Proof.
By assumption,

$$\Delta(\omega_1\omega_2) \pm \mu(\omega_1)\xi(\omega_2) \pm [\omega_1, \omega_2] \in \Lambda \text{ for all } \omega_1, \omega_2 \in S. \tag{3.25}$$

Replacing ω_2 by $\omega_1\omega_2$ in (3.25) and using (3.25), for all $\omega_1, \omega_2 \in S$ one obtains

$$\Delta(\omega_1\omega_2)\omega_1 + \omega_1\omega_2\eta(\omega_1) \pm \mu(\omega_1)(\xi(\omega_2)\omega_1 + \omega_2\delta(\omega_1)) \pm \omega_1[\omega_1, \omega_2]$$

$$= (\Delta(\omega_1\omega_2) \pm \mu(\omega_1)\xi(\omega_2))\omega_1 + \omega_1\omega_2\eta(\omega_1) \pm \mu(\omega_1)\omega_2\delta(\omega_1) \pm \omega_1[\omega_1, \omega_2] \in \Lambda.$$

Multiplying (3.25) by ω_1 from the right and subtracting the result from the last equation gives

$$\mp[\omega_1,\omega_2]\omega_1+\omega_1\omega_2\eta(\omega_1)\pm\mu(\omega_1)\omega_2\delta(\omega_1)\pm\omega_1[\omega_1,\omega_2]$$

$$=\pm[\omega_1,[\omega_1,\omega_2]]+\omega_1\omega_2\eta(\omega_1)\pm\mu(\omega_1)\omega_2\delta(\omega_1)\in\Lambda \text{ for all } \omega_1,\omega_2\in S. \quad (3.26)$$

Again, replacing ω_2 by $\omega_1\omega_2$ in (3.26) implies that

$$\pm\omega_1[\omega_1,[\omega_1,\omega_2]]+\omega_1^2\omega_2\eta(\omega_1)\pm\mu(\omega_1)\omega_1\omega_2\delta(\omega_1)\in\Lambda \text{ for all } \omega_1,\omega_2\in S. \quad (3.27)$$

Using left multiplication by ω_1 to (3.26) and then subtracting the result from (3.27), one can see that

$$\mp[\omega_1,\mu(\omega_1)]\omega_2\delta(\omega_1)\in\Lambda \text{ for all } \omega_1,\omega_2\in S. \quad (3.28)$$

In (3.28), putting $\omega_2=\omega_2\omega_3$ for $\omega_3\in S$ yields

$$\mp[\omega_1,\mu(\omega_1)]\omega_2\omega_3\delta(\omega_1)\in\Lambda \text{ for all } \omega_1,\omega_2,\omega_3\in S. \quad (3.29)$$

Using right multiplication by ω_3 to (3.28) and subtracting the result from (3.29) implies that $\mp[\omega_1,\mu(\omega_1)]\omega_2[\omega_3,\delta(\omega_1)]\in\Lambda$ for all $\omega_1,\omega_2,\omega_3\in S$. By the hypothesis of Λ, one can get either $[\omega_1,\mu(\omega_1)]\in\Lambda$ or $[\omega_3,\delta(\omega_1)]\in\Lambda$ for all $\omega_1,\omega_3\in S$. According to Remark 2.6(a), the second case leads to S/Λ is commutative or $\delta(S)\subseteq\Lambda$. Therefore, the conclusion follows. □

Mamouni et al. [20, Theorem 1] studied a condition for prime ideal Λ with D mappings δ and δ_1 of a ring S that satisfies $[\delta(\omega_1),\delta_1(\omega_2)]\in\Lambda$ for all elements $\omega_1,\omega_2\in S$. The next result adopts the condition $[\delta_1(\omega_1),\xi(\omega_2)]\in\Lambda$ for a GRD mapping (ξ,δ) and an RD mapping δ_1 on S.

Theorem 3.3. *Let Λ be a prime ideal of a ring S, (ξ,δ) be a GRD mapping on S and δ_1 be an RD mapping on S such that for all $\omega_1,\omega_2\in S$ if $[\delta_1(\omega_1),\xi(\omega_2)]\in\Lambda$, then one of the following is satisfied:*

(a) $char(S/\Lambda)=2$.

(b) $\delta(S)\in\Lambda$.

(c) $\delta_1(S)\in\Lambda$.

(d) S/Λ *is a commutative ring.*

Proof.
Suppose that $char(S/\Lambda)\neq 2$ and

$$[\delta_1(\omega_1),\xi(\omega_2)]\in\Lambda \text{ for all } \omega_1,\omega_2\in S. \quad (3.30)$$

In (3.30), putting $\omega_1\omega_3$ for $\omega_3 \in S$ instead of ω_1 and using (3.30) yields $[\delta_1(\omega_3)\omega_1 + \omega_3\delta_1(\omega_1), \xi(\omega_2)]$

$$= [\delta_1(\omega_3), \xi(\omega_2)]\omega_1 + \delta_1(\omega_3)[\omega_1, \xi(\omega_2)] + \omega_3[\delta_1(\omega_1), \xi(\omega_2)] + [\omega_3, \xi(\omega_2)]\delta_1(\omega_1)$$
$$= \delta_1(\omega_3)[\omega_1, \xi(\omega_2)] + [\omega_3, \xi(\omega_2)]\delta_1(\omega_1) \in \Lambda \text{ for all } \omega_1, \omega_2, \omega_3 \in S. \tag{3.31}$$

In particular, if $\omega_1 = \delta_1(\omega_4)$ for $\omega_4 \in S$ in (3.31) and from (3.30) one can conclude that

$$[\omega_3, \xi(\omega_2)]\delta_1^2(\omega_4) \in \Lambda \text{ for all } \omega_2, \omega_3, \omega_4 \in S. \tag{3.32}$$

In (3.32), putting $\omega_3 = \omega_3\omega_5$ for $\omega_5 \in S$ and using (3.32) gives

$$[\omega_3, \xi(\omega_2)]\omega_5\delta_1^2(\omega_4) \in \Lambda. \tag{3.33}$$

By the primeness of Λ, (3.33) yields

$$\text{either } [\omega_3, \xi(\omega_2)] \in \Lambda \text{ or } \delta_1^2(\omega_4) \in \Lambda \text{ for all } \omega_2, \omega_3, \omega_4 \in S. \tag{3.34}$$

According to Lemma 2.5, the first case implies that either S/Λ is commutative or $\delta(S) \subseteq \Lambda$ while the second case leads to

$$\begin{aligned} \delta_1^2(\omega_1\omega_2) &= \delta_1(\delta_1(\omega_2)\omega_1 + \omega_2\delta_1(\omega_1)) \\ &= 2\delta_1(\omega_1)\delta_1(\omega_2) + \omega_1\delta_1^2(\omega_2) + \delta_1^2(\omega_1)\omega_2 \\ &= 2\delta_1(\omega_1)\delta_1(\omega_2) \in \Lambda \text{ for all } \omega_1, \omega_2 \in S. \end{aligned} \tag{3.35}$$

In (3.35), substituting $\omega_3\omega_1$ in place of ω_1 and using (3.35) leads us to

$$2\delta_1(\omega_1)\omega_3\delta_1(\omega_2) \in \Lambda \text{ for all } \omega_1, \omega_2, \omega_3 \in S. \tag{3.36}$$

Since Λ is prime with $char(S/\Lambda) \neq 2$ so S/Λ is a 2-torsion free ring and from (3.36), one can verify that $\delta_1(S) \subseteq \Lambda$. □

The following corollary is immediate from Theorem 3.3 when $\xi = \delta$.

Corollary 3.4. *Let Λ be a prime ideal of a ring S and δ, δ_1 be RD mappings on S such that for all $\omega_1, \omega_2 \in S$ if $[\delta(\omega_1), \delta_1(\omega_2)] \in \Lambda$, then one of the following is held:*

(a) $char(S/\Lambda) = 2$.

(b) $\delta(S) \in \Lambda$.

(c) $\delta_1(S) \in \Lambda$.

(d) S/Λ *is a commutative ring.*

The next proposition can be proved using Theorem 3.3 by considering $\Lambda = (0)$ in the case of S being a 2-torsion free prime ring.

Proposition 3.5. *Let* (ξ, δ) *be a GRD mapping on a 2-torsion free prime ring* S *such that the statement* $[\delta(\omega_1), \xi(\omega_2)] = 0$ *is satisfied for all elements* ω_1, ω_2 *of* S*. Then either* S *is commutative or* $\delta = 0$*.*

Proof.
Since S is semiprime, by [22] there is a family Δ of prime ideals Λ_i whose intersection is the set with only the zero element. Therefore,

$$[\delta(\omega_1), \xi(\omega_2)] \in \Lambda_i \text{ for all } \omega_1, \omega_2 \in S \text{ and } \Lambda_i \in \Delta. \tag{3.37}$$

The relation (3.37) is the same as (3.30) of Theorem 3.3 when $\delta = \delta_1$ and likewise the steps of the proof one can arrive to a similar argument as in (3.34). Hence, either $[\omega_3, \xi(\omega_2)] \in \Lambda_i$ or $\delta^2(\omega_4) \in \Lambda_i$ for all $\omega_2, \omega_3, \omega_4 \in S$ and $\Lambda_i \in \Delta$. The first case yields $[\omega_3, \xi(\omega_2)] = 0$ for all $\omega_2, \omega_3 \in S$ and as an application of Remak 2.6(b), it implies that either S is commutative or $\delta = 0$. For the second case, using the same steps as in (3.35) and (3.36) of Theorem 3.3 with the primeness of Λ_i one can verify that $2\delta(\omega_1) \in \Lambda_i$ for all $\omega_1 \in S$ and $\Lambda_i \in \Delta$. Hence, $2[\delta(\omega_1), \omega_2] \in \Lambda_i$ for all $\omega_1, \omega_2 \in S$ and $\Lambda_i \in \Delta$ which means that $2[\delta(\omega_1), \omega_2] = 0$ for all $\omega_1, \omega_2 \in S$. Using 2-torsion free property of S and Remak 2.6(c), the last equation yields either S is commutative or $\delta = 0$.□

For D mappings δ and δ_1 of semiprime ring S, Samman and Thaheem [23, Theorem 2.2] discussed the statement $\delta(\omega_1)\omega_2 + \omega_2\delta_1(\omega_1) = 0$ for all elements ω_1, ω_2 of a nonzero ideal $\mathcal{I}$ of S. The following proposition studies the same condition for RD mappings rather than D mappings in case it belongs to a prime ideal Λ instead of it being equal to $\{0\}$.

Proposition 3.6. *Let* Λ *be a prime ideal of a ring* S *and* δ, δ_1 *be RD mappings on* S *such that* $\delta(\omega_1)\omega_2 + \omega_2\delta_1(\omega_1) \in \Lambda$ *for all* $\omega_1, \omega_2 \in S$*. Then the following statements are satisfied:*

(a) either S/Λ *is commutative or* $\delta(S) \subseteq \Lambda$*;*

(b) either S/Λ *is commutative or* $\delta_1(S) \subseteq \Lambda$*.*

Proof.
(a) By assumption,

$$\delta(\omega_1)\omega_2 + \omega_2\delta_1(\omega_1) \in \Lambda \text{ for all } \omega_1, \omega_2 \in S. \tag{3.38}$$

Replacing ω_1 by $\omega_1\omega_2$ in (3.38) and using (3.38), then the equation

$$\delta(\omega_2)\omega_1\omega_2 + \omega_2\delta_1(\omega_2)\omega_1 \in \Lambda \text{ for all } \omega_1, \omega_2 \in S. \tag{3.39}$$

is obtained. Especially, if $\omega_1 = \omega_2$ in (3.38) yields

$$\delta(\omega_2)\omega_2 + \omega_2\delta_1(\omega_2) \in \Lambda \text{ for all } \omega_2 \in S. \tag{3.40}$$

Right multiplication by ω_1 to (3.40) and subtracting the result from (3.39) gives

$$\delta(\omega_2)[\omega_1, \omega_2] \in \Lambda \text{ for all } \omega_1, \omega_2 \in S. \tag{3.41}$$

In (3.41), for $\omega_3 \in S$ substituting $\omega_3\omega_1$ rather than ω_1 and using (3.41) implies that $\delta(\omega_2)\omega_3[\omega_1, \omega_2] \in \Lambda$ for all $\omega_1, \omega_2, \omega_3 \in S$. Hence, the primeness of Λ implies that

$$\text{either } \delta(\omega_2) \in \Lambda \text{ or } [\omega_1, \omega_2] \in \Lambda \text{ for all } \omega_1, \omega_2 \in S. \tag{3.42}$$

The relation (3.42) is the same as (2.3) and similar to a method of the proof used in Theorem 2.1 one can conclude that either S/Λ is commutative or $\delta(S) \subseteq \Lambda$.
(b) Likewise, in (3.38) putting $\omega_1 = \omega_2\omega_1$ and using (3.38) gives

$$\omega_1\delta(\omega_2)\omega_2 + \omega_2\omega_1\delta_1(\omega_2) \in \Lambda \text{ for all } \omega_1, \omega_2 \in S. \tag{3.43}$$

Left multiplication by ω_1 to (3.38) and then subtracting the result from (3.43) yields $[\omega_1, \omega_2]\delta_1(\omega_2) \in \Lambda$ for all $\omega_1, \omega_2 \in S$. Replacing ω_1 by $\omega_1\omega_3$ for $\omega_3 \in S$ in the last relation, then we have $[\omega_1, \omega_2]\omega_3\delta_1(\omega_2) \in \Lambda$ for all $\omega_1, \omega_2, \omega_3 \in S$. By the hypothesis of Λ and as the proof of part (a), either S/Λ is commutative or $\delta_1(S) \subseteq \Lambda$. □

Note that for Remark 2.6, the statement (a) can be obtained immediately using Proposition 3.6 as a special case by considering $\delta = -\delta_1$.

In the next theorem, the commutativity of a quotient ring will be studied relying on a condition that includes *GRD* mapping and its associated *RD* mapping.

Theorem 3.7. *Let Λ be a prime ideal of a ring S, (ξ, δ) be a GRD mapping on S such that for all $\omega_1, \omega_2 \in S$ if $\delta(\omega_1) \circ \xi(\omega_2) \in \Lambda$, then one of the following is satisfied:*

(a) $char(S/\Lambda) = 2$.

(b) $\delta(S) \in \Lambda$.

(c) S/Λ *is a commutative ring.*

Proof. Assume that $char(S/\Lambda) \neq 2$ and

$$\delta(\omega_1) \circ \xi(\omega_2) \in \Lambda \text{ for all } \omega_1, \omega_2 \in S. \tag{3.44}$$

Substituting $\omega_3\omega_1$ instead of ω_1 for $\omega_3 \in S$ in (3.44) gives for all $\omega_1, \omega_2, \omega_3 \in S$

$$\begin{aligned} \delta(\omega_3\omega_1) \circ \xi(\omega_2) &= (\delta(\omega_1)\omega_3 + \omega_1\delta(\omega_3)) \circ \xi(\omega_2) \\ &= \delta(\omega_1)\omega_3\xi(\omega_2) + \omega_1\delta(\omega_3)\xi(\omega_2) + \xi(\omega_2)\delta(\omega_1)\omega_3 + \xi(\omega_2)\omega_1\delta(\omega_3) \in \Lambda. \end{aligned} \tag{3.45}$$

Right multiplication by ω_3 to (3.44) and subtracting the result from (3.45) implies that

$$\delta(\omega_1)\omega_3\xi(\omega_2) + \omega_1\delta(\omega_3)\xi(\omega_2) - \delta(\omega_1)\xi(\omega_2)\omega_3 + \xi(\omega_2)\omega_1\delta(\omega_3) \in \Lambda \text{ for all } \omega_1, \omega_2, \omega_3 \in S. \tag{3.46}$$

Since the relation (3.44) is true for any element of S, then the following is also true.

$$\omega_1(\delta(\omega_3) \circ \xi(\omega_2)) \in \Lambda \text{ for all } \omega_1, \omega_2, \omega_3 \in S. \tag{3.47}$$

Subtracting (3.46) from (3.47) to get

$$-\delta(\omega_1)[\omega_3, \xi(\omega_2)] + [\omega_1, \xi(\omega_2)]\delta(\omega_3) \in \Lambda \text{ for all } \omega_1, \omega_2, \omega_3 \in S. \tag{3.48}$$

In particular, when $\omega_3 = \xi(\omega_2)$ in (3.48) then for all $\omega_1, \omega_2 \in S$ the relation

$$[\omega_1, \xi(\omega_2)]\delta(\xi(\omega_2)) \in \Lambda \text{ for all } \omega_1, \omega_2 \in S \tag{3.49}$$

is true. Putting $\omega_1 = \omega_1\omega_3$ for $\omega_3 \in S$ in (3.49) and using (3.49) gives $[\omega_1, \xi(\omega_2)]\omega_3\delta(\xi(\omega_2)) \in \Lambda$ for all $\omega_1, \omega_2, \omega_3 \in S$. Since Λ is a prime ideal, then it implies that either $[\omega_1, \xi(\omega_2)] \in \Lambda$ or $\delta(\xi(\omega_2)) \in \Lambda$ for all $\omega_1, \omega_2 \in S$.

By Lemma 2.5, the commutativity of S/Λ or $\delta(S) \subseteq \Lambda$ can be concluded from the first case whereas the second argument leads to:

$$\begin{aligned}\delta(\xi(\omega_1\omega_2)) &= \delta(\xi(\omega_2)\omega_1 + \omega_2\delta(\omega_1)) \\ &= \delta(\omega_1)\xi(\omega_2) + \omega_1\delta(\xi(\omega_2)) + \delta^2(\omega_1)\omega_2 + \delta(\omega_1)\delta(\omega_2) \\ &= \delta(\omega_1)\xi(\omega_2) + \delta^2(\omega_1)\omega_2 + \delta(\omega_1)\delta(\omega_2) \in \Lambda \text{ for all } \omega_1, \omega_2 \in S.\end{aligned} \tag{3.50}$$

Replacing ω_2 with ω_3 for $\omega_3 \in S$ in (3.50) and then multiplying the result from the right by ω_2, one can see that

$$\delta(\omega_1)\xi(\omega_3)\omega_2 + \delta^2(\omega_1)\omega_3\omega_2 + \delta(\omega_1)\delta(\omega_3)\omega_2 \in \Lambda \text{ for all } \omega_1, \omega_2, \omega_3 \in S. \tag{3.51}$$

Putting $\omega_2 = \omega_2\omega_3$ in (3.50), then for all $\omega_1, \omega_2, \omega_3 \in S$ the following is obtained

$$\delta(\omega_1)\xi(\omega_3)\omega_2 + \delta(\omega_1)\omega_3\delta(\omega_2) + \delta^2(\omega_1)\omega_2\omega_3 + \delta(\omega_1)\delta(\omega_3)\omega_2 + \delta(\omega_1)\omega_3\delta(\omega_2) \in \Lambda \tag{3.52}$$

Subtracting (3.52) from (3.51), one can find that

$$\delta^2(\omega_1)[\omega_3, \omega_2] - 2\delta(\omega_1)\omega_3\delta(\omega_2) \in \Lambda \text{ for all } \omega_1, \omega_2, \omega_3 \in S. \tag{3.53}$$

Putting $\omega_2 = \xi(\omega_2)$ in (3.53) and since $\delta(\xi(\omega_2)) \in \Lambda$, then $\delta^2(\omega_1)[\omega_3, \xi(\omega_2)] \in \Lambda$ for all $\omega_1, \omega_2, \omega_3 \in S$. Replacing ω_3 by $\omega_4\omega_3$ for $\omega_3 \in S$ and using primeness property of Λ implies that

$$\text{either } \delta^2(\omega_1) \in \Lambda \text{ or } [\omega_3, \xi(\omega_2)] \in \Lambda \text{ for all } \omega_1, \omega_2, \omega_3 \in S. \tag{3.54}$$

Clearly (3.54) is the same as (3.34), hence the proof can be completed as the proof of Theorem 3.3. □

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