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Local Metric Dimension of Certain Wheel Related Graphs

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Abstract

Let V be the vertex set and let E be the edge set of the graph $G(V, E)$. The study on the selection of minimum number of vertices in a graph which will distinguish all the adjacent vertices in terms of distance between the vertices is given by the local metric basis problem. A subset W of V is said to be a local metric basis of G if, for any two adjacent vertices $u, v \in V \backslash W$, there exists a vertex $w \in W$ such that $d(w, u) \neq d(w, v)$. The minimum cardinality of local metric basis is called the local metric dimension of G and is denoted by $\beta_l(G)$. In this paper, we calculate the local metric dimension of wheel graph and certain wheel related graphs.

1 Introduction

A graph structured framework can be used to study robot navigation. How do we fix minimum number of points in a graph so that the movement of robot can be calculated distinctly from the points of the graph? The study of minimum number of points in a graph which will distinguish all the vertices in terms of distance between the vertices is given by the metric basis problem.

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Definition 1.1. *[4] Let $G(V, E)$ be a graph. Then a minimum subset W of V is said to be a metric basis if for any two vertices $u, v \in V/W$ there exists a vertex $w \in W$ such that $d(w, u) \neq d(w, v)$. The cardinality of a metric basis is said to be the metric dimension of the graph and is denoted by $\beta(G)$.*

The metric dimension problem is to identify a metric basis. The metric dimension of graphs was introduced in 1970s, independently by Harary and Melter [4] and by Slater [13]. Bharati *et al.* determined the metric dimension of directed graphs [1].

The study of the minimum number of points in a graph which will distinguish all the adjacent vertices in terms of distance between the vertices is given by the local metric basis problem.

Definition 1.2. *[12] Let $G(V, E)$ be a graph. Then a minimum subset W_l of V is said to be a local metric basis if, for any two adjacent vertices $u, v \in V/W_l$, there exists a vertex $w \in W_l$ such that $d(w, u) \neq d(w, v)$. The cardinality of a local metric basis is said to be the local metric dimension of the graph and is denoted by $\beta_l(G)$.*

The local metric dimension problem is to identify a local metric basis of a graph.

In 2010, Okamoto et al. [12] introduced the concept of local metric basis and local metric dimension of a graph and discovered many results pertaining to it. Cynthia and Fancy [3, 5, 6, 7] calculated the local metric dimension of generalized Petersen graph, de Bruijn digraph, Kautz digraph and undirected circulant graph. Cynthia and Ramya [8, 9, 10, 11] investigated the local metric dimension of cyclic split graph, mesh related architectures, cube connected cycle derived networks and torus networks.

Chithra et al. [2] calculated equitable coloring parameters of certain wheel related graphs. Motivated by the local metric dimension problem and the wheel related graph architecture, we have determined local metric dimension of wheel graphs, helm graphs and sunflower graphs.

2 Local Metric Dimension of Wheel Graph

In this section, we solve the local metric dimension problem of the wheel graph W_n, $n \geq 5$.

Definition 2.1. *A wheel graph W_n, $n \geq 4$ is a graph formed by connecting a single central vertex v to all the vertices $v_i, i = 1, 2, \ldots, n,$ of a cycle of length n.*

Figure 1 is the wheel graph W_n, $n = 8$.

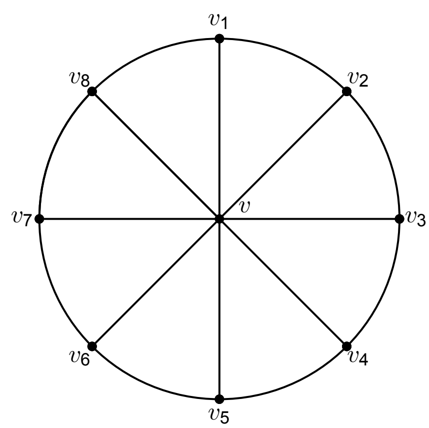

Figure 1: Wheel graph W_8

Notation: The vertices v_1 to v_n on the cycle of wheel graph are denoted by 1 to n, and we denote the central vertex v as c.

The following Figure 2 illustrates the wheel graph with 10 vertices on the cycle labeled 1 to 10 and a central vertex labeled c.

Note

1. The vertex labels are taken modulo n.
2. The value of $2n(\bmod n)$ and $n(\bmod n)$ are taken as n for computational purpose.

Structural Properties of Wheel Graph

- The wheel graph W_n has $n + 1$ vertices and $2n$ edges.
- The degree of the vertices on the cycle of W_n is 3 and that of the central vertex is n.

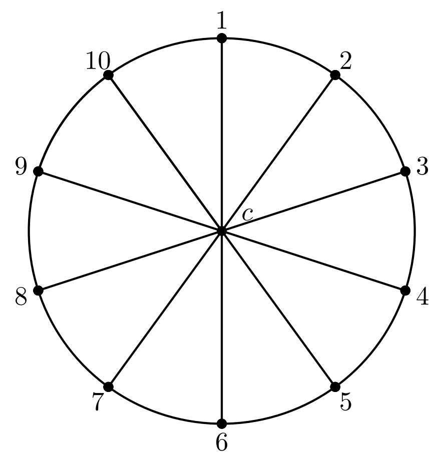

Figure 2: Illustration for labeling the vertices of the wheel graph

- Each vertex x where $1 \leq x \leq n$ has 3 pairs of adjacent vertices namely $\{x, l\}$, $\{x, l'\}$ and $\{x, c\}$ where $l = (x+1)(\mathrm{mod}\, n)$ and $l' = (x+n-1)(\mathrm{mod}\, n)$.
- The diameter of W_n is 2.

Lemmas 2.2 and 2.3 give the upper and lower bounds of the wheel graph respectively.

Lemma 2.2. *Let G be the wheel graph W_n, $n \geq 5$. Then $\beta_l(G) \leq \lceil \frac{n}{4} \rceil$.*

Proof.
Let W_n be the wheel graph denoted by G, where $n \geq 5$. Let x_1 be a member of the local metric basis such that $1 \leq x_1 \leq n$. Then the vertices l, l' and c, where $l = (x_1+1)(\mathrm{mod}\, n)$ and $l' = (x_1+n-1)(\mathrm{mod}\, n)$, form adjacent pairs of vertices $\{c, l\}$ and $\{c, l'\}$ which are at a distance one from x_1. Also, there are $n-3$ vertices on the cycle which are pairwise adjacent and are at a distance two from x_1.

To resolve this, let us choose the vertex $x_2 = (x_1+4)(\mathrm{mod}\, n)$to be the second member of the local metric basis. Then $d(x_2, c) \neq d(x_2, l)$ and $d(x_2, c) \neq d(x_2, l')$. Also, the vertices $(x+6)(\mathrm{mod}\, n)$, $(x+7)(\mathrm{mod}\, n)$, ..., $(x+n-2)(\mathrm{mod}\, n)$ on the cycle will be at the same distance from x_2, which forms a total of $n-7$ vertices and $n-8$ pairs of adjacent vertices which are at an equal distance from x_2.

To resolve this, choose the vertex $x_3 = (x_1 + 8)(\bmod n)$ to be the next member of the local metric basis. The vertex x_3 resolves three more vertices, thus leaving $n - 11$ vertices on the cycle which are at equal distance when measured from x_3, of which $n - 12$ pairs of adjacent vertices are at equal distances.

Proceeding like this, we get the local metric basis to be

$$\Big\{x_i,\ i = 1, 2, \ldots, \Big\lceil \frac{n}{4} \Big\rceil : x_i = x_1 + 4i - 4\Big\}.$$

Thus the total number of vertices in a local metric basis is $\lceil \frac{n}{4} \rceil$.

Consequently, $\beta_l(G) \leq \lceil \frac{n}{4} \rceil$.

Lemma 2.3. *Let G be the wheel graph W_n, $n \geq 5$. Then $\beta_l(G) \geq \lceil \frac{n}{4} \rceil$.*

Proof.

Let x_1 be any vertex on the cycle such that $1 \leq x_1 \leq n$. Then there exists pairs of adjacent vertices on the cycle, given by $\{x, l\}$, $\{x, l'\}$, $\{l, t\}$, $\{l', t'\}$, where $l = (x + 1)(\bmod n)$, $l' = (x + n - 1)(\bmod n)$, $t = (x + 2)(\bmod n)$, $t' = (x + n - 2)(\bmod n)$, such that $d(x_1, l) \neq d(x_1, t)$ and $d(x_1, l') \neq d(x_1, t')$, leaving $n - 4$ pairs of adjacent vertices on the cycle which are at equal distance from x_1. In addition, the n pairs of adjacent vertices $\{c, y\}$, where $0 \leq y \leq n$ give $d(x_1, c) = d(x_1, l)$ and $d(x_1, c) = d(x_1, l')$, leaving $n - 2$ pairs of adjacent vertices at distinct distances when measured from the vertex x_1.

Thus, choosing x_1 as the local metric basis, we get $n-2$ pairs of adjacent vertices at equal distance when measured from x_1.

The second member of the local metric basis x_2 is chosen as follows:

Case (i): Let $x_2 = (x + 3)(\bmod n)$.
We have $d(x_2, c) \neq d(x_2, l)$ and $d(x_2, c) \neq d(x_2, l')$, leaving $n - 7$ pairs of adjacent vertices at equal distances when measured from x_2. Also, choosing x_2 resolves 3 pairs of adjacent vertices on the cycle to have distinct distance from x_2.

Case (ii): Let $x_2 = (x + 4)(\bmod n)$.
We have $d(x_2, c) \neq d(x_2, l)$ and $d(x_2, c) \neq d(x_2, l')$, leaving $n - 8$ pairs of adjacent vertices at equal distances when measured from x_2. Also, x_2 will resolve 4 pairs of adjacent vertices on the cycle to have distinct distances when measured from x_2.

Case (iii): Let $x_2 = (x+5)(\bmod n)$.

We have $d(x_2, c) \neq d(x_2, l)$ and $d(x_2, c) \neq d(x_2, l')$, leaving $n-8$ pairs of adjacent vertices at equal distances when measured from x_2. Also, x_2 will resolve 4 pairs of adjacent vertices on the cycle to have distinct distances when measured from x_2. Choosing the basis as x_2 will leave adjacent vertices $\{h, k\}$, where $h = (x_2 + n - 2)(\bmod n)$ and $k = (x_2 + n - 3)(\bmod n)$ such that $d(x_2, h) = d(x_2, k)$, where $x_1 \leq h$, $k \leq x_2$.

From the above cases, we choose $x_2 = (x+4)(\bmod n)$ to be the second member of the local metric basis such that it resolves the adjacent pair of vertices on the cycle taken in order.

Thus each vertex resolves 4 pairs of adjacent vertices on the cycle and the cycle has n pairs of such adjacent vertices. Therefore, it is necessary to choose at least $\lceil \frac{n}{4} \rceil$ vertices to be the member of a local metric basis.

Thus $\beta_l(G) \geq \lceil \frac{n}{4} \rceil$.

Theorem 2.4. *Let G be the wheel graph W_n, $n \geq 5$. Then $\beta_l(G) = \lceil \frac{n}{4} \rceil$.*

Proof.

From Lemma 2.2, we have $\beta_l(G) \leq \lceil \frac{n}{4} \rceil$. By Lemma 2.3 we have $\beta_l(G) \geq \lceil \frac{n}{4} \rceil$. Therefore, $\beta_l(G) = \lceil \frac{n}{4} \rceil$, $n \geq 5$.

3 Local Metric Dimension of Helm Graph

This section demonstrates finding the local metric dimension of the helm graph H_n, $n \geq 5$.

Definition 3.1. *A helm graph H_n, $n \geq 4$ is a graph obtained from a wheel graph W_n by adjoining a pendent edge at each vertex on the cycle C_n.*

Note.

1. The vertices on the cycle are labeled $1, 2, \ldots, n$ and the central vertex is labeled c.

2. The end vertices of the pendent edges are labeled $n+1, n+2, \ldots, 2n$ such that the pendent edges are $\{x(x+n),\ x = 1, 2, \ldots, n\}$ (see Figure 3).

3. The vertex labels on the cycle are taken modulo n.
4. The value of $2n(\bmod n)$ and $n(\bmod n)$ are taken as n for computational purposes.

Structural Properties of Helm Graph

- The graph H_n has $2n+1$ vertices and $3n$ edges.
- The degree of the vertices on the cycle of H_n is 4, the end vertices are of degree 1 and that of the central vertex is n.
- Each vertex x, where $1 \leq x \leq n$ has 4 pairs of adjacent vertices: namely, $\{x, l\}$, $\{x, l^{'}\}$, $\{x, c\}$ and $\{x, x+n\}$, where $l = (x+1)(\bmod n)$ and $l^{'} = (x+n-1)(\bmod n)$.
- The diameter of H_n is 4.

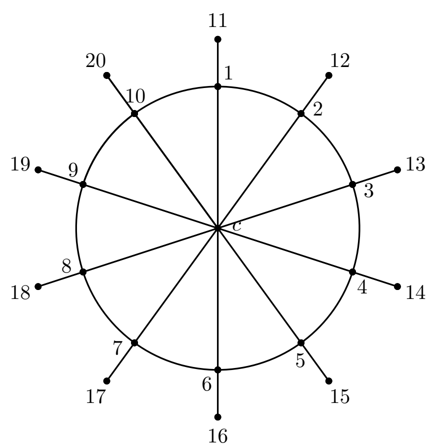

Figure 3: Helm graph H_{10}

Theorem 3.2. *Let G be the helm graph H_n, $n \geq 5$. Then $\beta_l(G) = \lceil \frac{n}{4} \rceil$.*

Proof.
Let H_n be the helm graph denoted by G, where $n \geq 5$. The helm graph has the same topological structure as the wheel graph except for the pendent edges. Let x_1 be a member of the local metric basis such that $1 \leq x_1 \leq n$. Then the end vertices $\{x+n, x = 1, 2, \ldots, n\}$ on the pendent edges $\{x(x+n), x = 1, 2, \ldots, n\}$ are such that $d(x_1, x) \neq d(x_1, x+n)$. Thus the adjacent vertex to the end vertices of the pendent edges are resolved with distinct distance from x_1. Proceeding in a similar way to the wheel graph given in Theorem 2.4, we get $\beta_l(G) = \lceil \frac{n}{4} \rceil$.

4 Local Metric Dimension of Sunflower Graph

In this section we solve the local metric dimension problem of the sunflower graph SF_n, $n \geq 7$.

Definition 4.1. *A Sunflower graph SF_n consists of a central vertex c, n-cycle $v_1, v_2, \ldots, v_n$ and additional n vertices $w_1, w_2, \ldots, w_n$, where w_i is joined by edges to the vertices v_i, v_{i+1} for $i = 1, 2, \ldots, n$.*

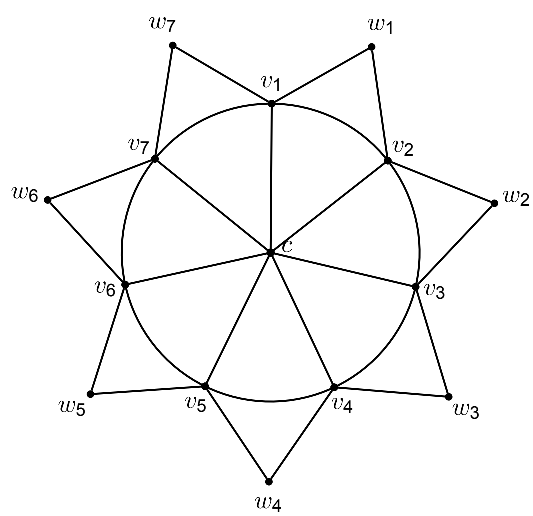

Figure 4: Sunflower graph SF_7

Structural Properties of Sunflower Graph

- The graph SF_n has $2n + 1$ vertices and $4n$ edges (see Figure 4).
- The degree of the vertices v_i, $i = 1, 2, \ldots, n$ is 5, w_i, $i = 1, 2, \ldots, n$ is 2 and that of c is n.
- The diameter of SF_n is 4.

Note

1. The vertices v_i are labeled as i, vertices w_i are labeled as $n + i$ for $i = 1, 2, \ldots, n$ and the central vertex as c.

2. The vertices on the cycle are taken modulo n and we take $2n(\bmod n) = n(\bmod n) = n$.

The following Figure 5 gives the sunflower graph SF_8 with vertices labeled 1 to 16 and we label the central vertex as c.

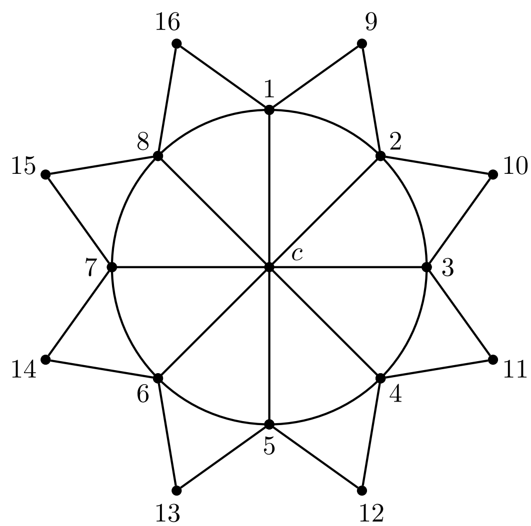

Figure 5: Labeling the vertices of sunflower graph SF_8

Lemma 4.2. *Let G be the sunflower graph SF_n, where $n \geq 7$. Then $\beta_l(G) \leq \lceil \frac{n}{4} \rceil$.*

Proof.
Let G be the sunflower graph SF_n, $n \geq 7$. Let x_1 be the first member of the local metric basis such that $1 \leq x_1 \leq n$. Then there exists the following pairs of adjacent vertices which are at equal distances from the vertex x_1:

$$\{c,\, (x_1+1)(\bmod n)\},$$
$$\{c,\, (x_1+n-1)(\bmod n)\},$$
$$\{(x_1+j)(\bmod n),\, (x_1+j+1)(\bmod n)\} \quad \text{where } j = 2, 3, \ldots, n-3,$$
$$\{(x_1+1)(\bmod n),\, x_1+n\},$$
$$\{(x_1+2)(\bmod n),\, x_1+n+1\},$$
$$\{(x_1+n-1)(\bmod n),\, (x_1+n-1)(\bmod n)+n\},$$
$$\{(x_1+n-2)(\bmod n),\, (x_1+n-2)(\bmod n)+n\}$$

Let x_2 be the second member of the local metric basis, where $x_2 = (x_1 + 4)(\bmod n)$. Then

$$\begin{aligned}
&d(x_2, c) \neq d(x_2, (x_1+1)(\bmod n)),\\
&d(x_2, c) \neq d(x_2, (x_1+n-1)(\bmod n)),\\
&d(x_2, (x_1+1)(\bmod n)) \neq d(x_2, x_1+n),\\
&d(x_2, (x_1+2)(\bmod n)) \neq d(x_2, x_1+n+1),\\
&d(x_2, (x_1+n-1)(\bmod n)) \neq d(x_2, (x_1+n-1)(\bmod n)+n),\\
&d(x_2, (x_1+n-2)(\bmod n)) \neq d(x_2, (x_1+n-2)(\bmod n)+n),\\
&d(x_2, (x_1+j)(\bmod n)) \neq d(x_2, (x_1+j+1)(\bmod n)) \quad \text{for } j = 2, 3, 4, 5.
\end{aligned}$$

Thus the second metric basis x_2 gives $n-8$ pairs of adjacent vertices which are equidistant when measured from x_2. All these $n-8$ pairs of adjacent vertices are on the cycle given by $(x_1+j)(\bmod n)$, $j = 6, 7, \ldots, n-2$.

Thus one vertex on the cycle resolves 4 pairs of adjacent vertices on the cycle. In addition, the vertices x_1 and x_2 resolve 8 pairs of adjacent vertices on the cycle to distinct distances when measured from x_1 or x_2. Therefore, we need $\lceil \frac{n-8}{4} \rceil$ vertices to resolve the remaining $n-8$ pairs of adjacent vertices on the cycle.
Thus the total number of vertices selected for the local metric basis is $\lceil \frac{n}{4} \rceil$ and the vertices are given by

$$\left\{ x_i : \ x_i = (x_1 + 4i - 4)(\bmod n),\ i = 1, 2, \ldots, \left\lceil \frac{n}{4} \right\rceil \right\}$$

Thus $\beta_l(G) \leq \lceil \frac{n}{4} \rceil$.
Note: The vertex x_1 is any vertex on the cycle such that $1 \leq x_1 \leq n$.

Lemma 4.3. *Let G be the sunflower graph $SF_n, n \geq 7$. Then $\beta_l(G) \geq \lceil \frac{n}{4} \rceil$.*

Proof.
Let G be the sunflower graph $SF_n,\ n \geq 7$. Choose the first member of the local metric basis namely x_1 as follows:

Case I: Let $x_1 = c$

The vertices on the cycle are all at a distance one when measured from x_1. The vertices $y,\ n+1 \leq y \leq 2n$ are all at a distance two

when measured from x_1. Thus the vertices on the cycle alone form the adjacent pairs of vertices which are equidistant from x_1. Proceeding as in wheel graphs, we take $\lceil \frac{n}{4} \rceil$ vertices to resolve all the vertices on the cycle.
Therefore, the total number of vertices selected for the basis is $\lceil \frac{n}{4} \rceil$.

Case II: Let $x_1 = x + n,\ 1 \le x \le n$
The following pairs of adjacent vertices are equidistant from x_1:

$$\begin{aligned}
&\{x,\ (x+1)(\mathrm{mod}\, n)\},\ \{c,\ (x+2)(\mathrm{mod}\, n)\}\\
&\{c,\ (x+n-1)(\mathrm{mod}\, n)\},\ \{(x+2)(\mathrm{mod}\, n),\ (x+1)(\mathrm{mod}\, n)+n\},\\
&\{(x+3)(\mathrm{mod}\, n),\ (x+2)(\mathrm{mod}\, n)+n\},\\
&\{(x+n-1)(\mathrm{mod}\, n),\ (x+n-1)(\mathrm{mod}\, n)+n\},\\
&\{(x+n-2)(\mathrm{mod}\, n),\ (x+n-2)(\mathrm{mod}\, n)+n\},\\
&\{(x+j)(\mathrm{mod}\, n),\ (x+j+1)(\mathrm{mod}\, n)\},\ j = 3, 4, \ldots, n-3.
\end{aligned}$$

Out of $4n$ pairs of adjacent vertices, the vertex x_1 has $n+2$ pairs of adjacent verices which are equidistant from x_1. Let x_2 be the second member of the local metric basis. Consider:

- Let $x_2 = x_1 + 1$, $n+1 \le x_2 \le 2n$. Then the vertex x_2 gives $n-3$ pairs of adjacent vertices which are equidistant from x_2, out of the $n+2$ pairs of adjacent vertices.
- Let $x_2 = x_1 + 2,\ n+1 \le x_2 \le 2n$. Then among $n+2$ pairs of adjacent vertices we get $n-7$ pairs of adjacent vertices on the cycle, which are equidistant from x_2.

  Case (i): From the above, choose $x_2 = x_1 + 2$, $n+1 \le x_2 \le 2n$. Select the alternate vertices which lie outside the cycle given by $x + n + 2i,\ i = 0, 1, 2, \ldots, \lfloor \frac{n-2}{2} \rfloor, 1 \le x \le n$. Then the total number of vertices selected for the basis is $\lfloor \frac{n}{2} \rfloor$.

  Case (ii): If we choose $x_2 = x_1 + 2,\ n+1 \le x_2 \le 2n$, then the $n-7$ pairs of adjacent vertices on the cycle which are equidistant from x_2. Proceeding as in the wheel graph, we select one vertex on the cycle which will resolve 4 pairs of adjacent vertices on the cycle into distinct distance when measured from that vertex. Hence we select $\lceil \frac{n-7}{4} \rceil$ vertices to be the member of local metric basis.

Thus the total number of vertices selected for the local metric basis is $\lceil \frac{n-7}{4} \rceil + 2$.

Case III: Let x_1 be the vertex on the cycle such that $1 \leq x_1 \leq n$.

There are $n+2$ pairs of adjacent vertices which are equidistant from x_1. Let $x_2 = (x_1 + 4)(\bmod n)$ be the next vertex selected for the basis, resulting in $(n-8)$ pairs of adjacent vertices on the cycle which are equidistant when measured from x_2.

By applying Lemma 2.3, for every four pairs of adjacent vertices, choose one vertex.

Thus the total number of vertices selected for the basis is $\lceil \frac{n-8}{4} \rceil + 2 = \lceil \frac{n}{4} \rceil$.

From the above Cases I to III, we have $\beta_l(G) \geq \lceil \frac{n}{4} \rceil$.

Theorem 4.4. *Let G be the Sunflower graph SF_n, $n \geq 7$. Then $\beta_l(G) = \lceil \frac{n}{4} \rceil$.*

Proof.
By Lemmas 4.2 and 4.3, $\beta_l(G) = \lceil \frac{n}{4} \rceil$.

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