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Some norm inequalities of Davidson-Power type

Fadi Alrimawi

Department of Basic Sciences
Faculty of Arts and Sciences
Al-Ahliyyah Amman University
Amman, Jordan

email: f.rimawi@ammanu.edu.jo

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Abstract

Recently, Al-natoor, Audeh, and Kittaneh proved unitarily invariant norm inequalities involving concave increasing functions. They generalized an inequality of Kittaneh which improved an earlier inequality of Davidson and Power. In this paper, we discuss some of these unitarily invariant norm inequalities involving convex increasing functions.

1 Introduction

Let $\mathbb{M}_n(\mathbb{C})$ be the algebra of all $n \times n$ complex matrices. For a matrix $A \in \mathbb{M}_n(\mathbb{C})$, let $s_1(A), s_2(A), ..., s_n(A)$ denote the singular values of A (i.e., the eigenvalues of $|A| = (A^*A)^{1/2}$) arranged in decreasing order and repeated according to multiplicity.

A matrix $A \in \mathbb{M}_n(\mathbb{C})$ is called Hermitian if $A = A^*$. The positive semidefinite matrix $A \in \mathbb{M}_n(\mathbb{C})$, written as $A \geq 0$, is a Hermitian matrix with $x^*Ax \geq 0$ for all $x \in \mathbb{C}^n$.

On $\mathbb{M}_n(\mathbb{C})$, a unitarily invariant norm $|||\cdot|||$ is a norm that satisfies the invariance property $|||UAV||| = |||A|||$ for every $A \in \mathbb{M}_n(\mathbb{C})$ and every unitary matrices $U, V \in \mathbb{M}_n(\mathbb{C})$. Some typical examples of unitarily invariant

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norms are the spectral norm $\|\cdot\|$ and the the Schatten p-norms $\|\cdot\|_p$ for $p \geq 1$. Recall that, for $A \in \mathbb{M}_n(\mathbb{C})$, $\|A\| = s_1(A)$ and $\|A\|_p = \left(\sum_{j=1}^{n} s_j^p(A)\right)^{1/p}$ for $p \geq 1$.

Davidson and Power [7] proved that if $A, B \in \mathbb{M}_n(\mathbb{C})$ are positive semidefinite, then

$$\|A+B\| \leq \max(\|A\|, \|B\|) + \|AB\|^{1/2}. \tag{1.1}$$

Kittaneh [8] proved that if $A, B \in \mathbb{M}_n(\mathbb{C})$ are positive semidefinite, then

$$\|A+B\| \leq \max(\|A\|, \|B\|) + \left\|A^{1/2}B^{1/2}\right\|, \tag{1.2}$$

which gives a stronger version of the inequality (1.1). An equivalent formulation of the inequality (1.2) has been given in [5]. For more refinements of the inequality (1.2), one can refer to [9].

Also, in [8], Kittaneh proved that if $A, B \in \mathbb{M}_n(\mathbb{C})$ are positive semidefinite and $p \geq 1$, then

$$\|A+B\|_p \leq \left(\|A\|_p^p + \|B\|_p^p\right)^{1/p} + 2^{1/p}\left\|A^{1/2}B^{1/2}\right\|_p. \tag{1.3}$$

Recently, in [1], Al-natoor, Audeh, and Kittaneh proved that for every unitarily invariant norm and every nonnegative concave function f on $[0, \infty)$, if $A, B, X \in \mathbb{M}_n(\mathbb{C})$ such that A and B are positive semidefinite, then

$$\begin{aligned} |||f(|(AX+XB)\oplus 0|)||| \leq{} & \left|\left|\left| f\left(\frac{1}{2}A\right) \oplus f\left(\frac{1}{2}XBX^*\right)\right|\right|\right| + \\ & \left|\left|\left| f\left(\frac{1}{2}X^*AX\right) \oplus f\left(\frac{1}{2}B\right)\right|\right|\right| + \\ & |||f\left(\left|A^{1/2}XB^{1/2}\right|\right) \oplus f\left(\left|A^{1/2}XB^{1/2}\right|\right)||| \end{aligned} \tag{1.4}$$

In particular,

$$\|AX+XB\| \leq \frac{1}{2}\max(\|A\|, \|XBX^*\|) + \frac{1}{2}\max(\|X^*AX\|, \|B\|) + \left\|A^{1/2}XB^{1/2}\right\| \tag{1.5}$$

and

$$\begin{aligned} \|AX+XB\|_p \leq{} & \left(\left\|\frac{1}{2}A\right\|_p^p + \left\|\frac{1}{2}XBX^*\right\|_p^p\right)^{1/p} + \left(\left\|\frac{1}{2}B\right\|_p^p + \left\|\frac{1}{2}X^*AX\right\|_p^p\right)^{1/p} \\ & + 2^{1/p}\left\|A^{1/2}XB^{1/2}\right\|_p. \end{aligned} \tag{1.6}$$

The inequalities (1.5) and (1.6) generalize the inequalities (1.2) and (1.3), respectively.

In [1], unitarily invariant norm inequalities involving concave increasing functions were discussed. In this paper, we will discuss some of the unitarily invariant norm inequalities, given in [1], involving convex increasing functions. We show that our inequalities generalize the inequalities (1.2) and (1.3).

2 Main results

In this section, we introduce unitarily invariant norm inequalities involving convex increasing functions. We need the following lemmas in our work.

The first lemma is a consequence of the spectral theorem (see, e.g., [2, p. 5]):

Lemma 2.1. *Let $A \in \mathbb{M}_n(\mathbb{C})$ and let f be a nonnegative increasing function on $[0, \infty)$. Then*

$$s_j\left(f\left(|A|\right)\right) = f\left(s_j\left(A\right)\right)$$

for $j = 1, 2, ..., n$.

The second lemma follows (For the proof, we refer the reader to [3]):

Lemma 2.2. *Let $A, B \in \mathbb{M}_n(\mathbb{C})$. Then*

$$s_j(AB^*) \le \frac{1}{2} s_j\left(A^*A + B^*B\right)$$

for $j = 1, 2, ..., n$.

The following is the third lemma (For the proof, we refer the reader to [6]):

Lemma 2.3. *Let $A, B \in \mathbb{M}_n(\mathbb{C})$ be Hermitian. Then*

$$|A + B| \le \frac{1}{2}\left(U\left(|A| + |B|\right)U^* + V\left(|A| + |B|\right)V^*\right)$$

for some unitary matrices $U, V \in \mathbb{M}_n(\mathbb{C})$.

In [4], we can find part (a) and part (c) of the following lemma. Also, we can refer the reader to [11] and [6] for the proofs of parts (b) and (d), respectively:

Lemma 2.4. *Let $A_1, ..., A_n \in \mathbb{M}_n(\mathbb{C})$ be positive semidefinite and let $\alpha_1, ..., \alpha_n \in [0,1]$ such that $\sum_{i=1}^{n} \alpha_i = 1$. Then, for every unitarily invariant norm, we have*

(a) $|||f(\sum_{i=1}^{n} \alpha_i A_i)||| \leq |||\sum_{i=1}^{n} \alpha_i f(A_i)|||$ *for every non-negative convex function f on $[0,\infty)$.*

(b) $|||\sum_{i=1}^{n} f(A_i)||| \leq |||f(\sum_{i=1}^{n} A_i)|||$ *for every non-negative convex function f on $[0,\infty)$ with $f(0) = 0$.*

(c) $|||\sum_{i=1}^{n} \alpha_i f(A_i)||| \leq |||f(\sum_{i=1}^{n} \alpha_i A_i)|||$ *for every non-negative concave function f on $[0,\infty)$.*

(d) $|||f(\sum_{i=1}^{n} A_i)||| \leq |||\sum_{i=1}^{n} f(A_i)|||$ *for every non-negative concave function f on $[0,\infty)$.*

Throughout this paper, we assume that all functions are continuous.

Our first main result is the following:

Theorem 2.5. *Let $A, B, X \in \mathbb{M}_n(\mathbb{C})$ such that A and B are positive semidefinite and let f be a nonnegative convex function on $[0,\infty)$. Then,*

$$\begin{aligned} |||f(|(AX+XB)\oplus 0|)||| \leq\ & \frac{1}{4}|||f(2A)\oplus f(2XBX^*)||| + \\ & \frac{1}{4}|||f(2X^*AX)\oplus f(2B)||| + \\ & \frac{1}{2}\left|\left|\left|f\left(2\left|A^{1/2}XB^{1/2}\right|\right)\oplus f\left(2\left|A^{1/2}XB^{1/2}\right|\right)\right|\right|\right|, \end{aligned}$$

for every unitarily invariant norm.

Proof. Let $S = \begin{bmatrix} A^{1/2} & XB^{1/2} \\ 0 & 0 \end{bmatrix}$ and $T = \begin{bmatrix} A^{1/2}X & 0 \\ B^{1/2} & 0 \end{bmatrix}$.

Then, for $j=1,2,...,2n$, we have

$$\begin{aligned} s_j\left(f\left(\left|(AX+XB)\oplus 0\right|\right)\right) &= s_j\left(f\left(\left|ST^*\right|\right)\right) \\ &= f\left(s_j\left(ST^*\right)\right) \quad \text{(by lemma 2.1)} \\ &\le f\left(\frac{1}{2}s_j\left(S^*S+T^*T\right)\right) \quad \text{(by lemma 2.2)} \\ &= f\left(s_j\left(\begin{array}{c} \frac{1}{2}\begin{bmatrix} A & A^{1/2}XB^{1/2} \\ B^{1/2}X^*A^{1/2} & B^{1/2}\left|X\right|^2B^{1/2}\end{bmatrix} + \\ \frac{1}{2}\begin{bmatrix} A^{1/2}\left|X^*\right|^2A^{1/2} & A^{1/2}XB^{1/2} \\ B^{1/2}X^*A^{1/2} & B\end{bmatrix}\end{array}\right)\right) \\ &= f\left(s_j\left(K+L+Y\right)\right) \\ &= s_j\left(f\left(\left|K+L+Y\right|\right)\right), \end{aligned}$$

where

$K=\frac{1}{2}\begin{bmatrix} A & 0 \\ 0 & B^{1/2}\left|X\right|^2B^{1/2}\end{bmatrix}$,

$L=\frac{1}{2}\begin{bmatrix} A^{1/2}\left|X^*\right|^2A^{1/2} & 0 \\ 0 & B\end{bmatrix}$,

and

$Y=\begin{bmatrix} 0 & A^{1/2}XB^{1/2} \\ B^{1/2}X^*A^{1/2} & 0\end{bmatrix}$.

Using the fact that every unitarily invariant norm is an increasing function of singular values, it follows that

$$|||f\left(\left|(AX+XB)\oplus 0\right|\right)|||$$

$$
\begin{aligned}
&\le |||f(|K+L+Y|)||| \\
&\le \left|\left|\left|f\left(\frac{1}{2}\left(U\left(|K+L|+|Y|\right)U^* + V\left(|K+L|+|Y|\right)V^*\right)\right)\right|\right|\right| \quad \text{(by Lemma 2.3)} \\
&\le \left|\left|\left|\frac{f\left(U\left(|K+L|+|Y|\right)U^*\right) + f\left(V\left(|K+L|+|Y|\right)V^*\right)}{2}\right|\right|\right| \quad \text{(by Lemma 2.4(a))} \\
&\le \frac{1}{2}\left(|||f\left(U\left(|K+L|+|Y|\right)U^*\right)||| + |||f\left(V\left(|K+L|+|Y|\right)V^*\right)|||\right) \\
&= |||f\left((K+L)+|Y|\right)||| \\
&\le \left|\left|\left|\frac{1}{4}f(4K) + \frac{1}{4}f(4L) + \frac{1}{2}f(2|Y|)\right|\right|\right| \quad \text{(by Lemma 2.4(a))} \\
&\le \left|\left|\left|\frac{1}{4}f(4K)\right|\right|\right| + \left|\left|\left|\frac{1}{4}f(4L)\right|\right|\right| + \left|\left|\left|\frac{1}{2}f(2|Y|)\right|\right|\right| \\
&= \frac{1}{4}\left|\left|\left|f\left(2\begin{bmatrix} A & 0 \\ 0 & B^{1/2}|X|^2B^{1/2}\end{bmatrix}\right)\right|\right|\right| + \frac{1}{4}\left|\left|\left|f\left(2\begin{bmatrix} A^{1/2}|X^*|^2A^{1/2} & 0 \\ 0 & B\end{bmatrix}\right)\right|\right|\right| \\
&\quad + \frac{1}{2}\left|\left|\left|f\left(2\left|\begin{bmatrix} 0 & A^{1/2}XB^{1/2} \\ B^{1/2}X^*A^{1/2} & 0\end{bmatrix}\right|\right)\right|\right|\right| \\
&= \frac{1}{4}|||f(2A) \oplus f\left(2B^{1/2}|X|^2B^{1/2}\right)||| + \frac{1}{4}|||f\left(2A^{1/2}|X^*|^2A^{1/2}\right) \oplus f(2B)||| \\
&\quad + \frac{1}{2}\left|\left|\left|f\left(\begin{bmatrix} 2\left|B^{1/2}X^*A^{1/2}\right| & 0 \\ 0 & 2\left|A^{1/2}XB^{1/2}\right|\end{bmatrix}\right)\right|\right|\right| \\
&= \frac{1}{4}|||f(2A) \oplus f\left(2B^{1/2}|X|^2B^{1/2}\right)||| + \frac{1}{4}|||f\left(2A^{1/2}|X^*|^2A^{1/2}\right) \oplus f(2B)||| \\
&\quad + \frac{1}{2}|||f\left(2\left|B^{1/2}X^*A^{1/2}\right|\right) \oplus f\left(2\left|A^{1/2}XB^{1/2}\right|\right)||| \\
&= \frac{1}{4}|||f(2A) \oplus f(2XBX^*)||| + \frac{1}{4}|||f(2X^*AX) \oplus f(2B)||| \\
&\quad + \frac{1}{2}|||f\left(2\left|A^{1/2}XB^{1/2}\right|\right) \oplus f\left(2\left|A^{1/2}XB^{1/2}\right|\right)|||,
\end{aligned}
$$

as required. □

By specializing Theorem 2.5 for certain types of unitarily invariant norms, we get the following two corollaries:

Corollary 2.6. *Let $A, B, X \in \mathbb{M}_n(\mathbb{C})$ such that A and B are positive semidefinite. Then*

$$
\|AX + XB\| \le \frac{1}{2}\max(\|A\|, \|XBX^*\|) + \frac{1}{2}\max(\|X^*AX\|, \|B\|) + \left\|A^{1/2}XB^{1/2}\right\|. \tag{2.7}
$$

Proof. This follows by considering the spectral norm and taking $f(t) = t, t \geq 0$, in Theorem 2.5. $\square$

Corollary 2.7. *Let $A, B, X \in \mathbb{M}_n(\mathbb{C})$ such that A and B are positive semidefinite, $p \geq 1$. Then*

$$\|AX + XB\|_p \leq \left(\left\|\frac{1}{2}A\right\|_p^p + \left\|\frac{1}{2}XBX^*\right\|_p^p\right)^{1/p} + \left(\left\|\frac{1}{2}B\right\|_p^p + \left\|\frac{1}{2}X^*AX\right\|_p^p\right)^{1/p} + 2^{1/p}\left\|A^{1/2}XB^{1/2}\right\|_p. \tag{2.8}$$

Proof. Consider the Schatten $p-$norms and take $f(t) = t, t \geq 0$, in Theorem 2.5. $\square$

By letting $X = I$ in the previous two corollaries, we obtain the inequalities (1.2) and (1.3), respectively.

Remark 2.8. *The inequalities (2.7) and (2.8) coincide with the inequalities (1.5) and (1.6) given in [1]. This occurs because the function $f(t) = t, t \geq 0$ can be considered as a convex and concave function simultaneously.*

Specializing Theorem 2.5 for certain types of functions gives the following two corollaries.

Corollary 2.9. *Let $A, B, X \in \mathbb{M}_n(\mathbb{C})$ such that A and B are positive semidefinite. Then*

$$\begin{aligned} \left|\left|\left|e^{(|AX+XB|)} - I\right|\right|\right| \leq\ & \frac{1}{4}\left|\left|\left|\left(e^{(2A)} - I\right) \oplus \left(e^{(2XBX^*)} - I\right)\right|\right|\right| + \\ & \frac{1}{4}\left|\left|\left|\left(e^{(2X^*AX)} - I\right) \oplus \left(e^{(2B)} - I\right)\right|\right|\right| + \\ & \frac{1}{2}\left|\left|\left|\left(e^{\left(2\left|A^{1/2}XB^{1/2}\right|\right)} - I\right) \oplus \left(e^{\left(2\left|A^{1/2}XB^{1/2}\right|\right)} - I\right)\right|\right|\right| \end{aligned}$$

for every unitarily invariant norm.

Proof. This follows by taking $f(t) = e^t - 1, t \geq 0$, in Theorem 2.5. $\square$

Corollary 2.10. *Let $A, B, X \in \mathbb{M}_n(\mathbb{C})$ such that A and B are positive semidefinite. Then, for $r \geq 1$, we have*

$$\begin{aligned} \left|\left|\left||AX + XB|^r\right|\right|\right| \leq\ & 2^{r-2}\left|\left|\left|A^r \oplus (XBX^*)^r\right|\right|\right| + \\ & 2^{r-2}\left|\left|\left|B^r \oplus (X^*AX)^r\right|\right|\right| + \\ & 2^{r-1}\left|\left|\left|\left|A^{1/2}XB^{1/2}\right|^r \oplus \left|A^{1/2}XB^{1/2}\right|^r\right|\right|\right| \end{aligned}$$

for every unitarily invariant norm.

Proof. Take $f(t) = t^r, r \geq 1$, in Theorem 2.5. $\square$

Using an argument similar to that used in the proof of Theorem 2.5, we have our second main result:

Theorem 2.11. *Let $A, B, X \in \mathbb{M}_n(\mathbb{C})$ such that A and B are positive semidefinite and let f be a nonnegative convex function on $[0, \infty)$. Then,*

$$|||f(|(AX - XB) \oplus 0|)||| \leq \frac{1}{2} |||f(A) \oplus f(XBX^*)||| + \frac{1}{2} |||f(X^*AX) \oplus f(B)|||$$

for every unitarily invariant norm.

Proof. Let $S = \begin{bmatrix} A^{1/2} & XB^{1/2} \\ 0 & 0 \end{bmatrix}$ and $M = \begin{bmatrix} A^{1/2}X & 0 \\ -B^{1/2} & 0 \end{bmatrix}$. Then, for $j = 1, 2, ..., 2n$, we have

$$\begin{aligned}
s_j(f(|(AX - XB) \oplus 0|)) &= s_j(f(|SM^*|)) \\
&= f(s_j(SM^*)) \quad \text{(by lemma 2.1)} \\
&\leq f\left(\frac{1}{2} s_j(S^*S + M^*M)\right) \quad \text{(by lemma 2.2)} \\
&= f\left(s_j\left(\begin{array}{l} \frac{1}{2}\begin{bmatrix} A & A^{1/2}XB^{1/2} \\ B^{1/2}X^*A^{1/2} & B^{1/2}|X|^2B^{1/2} \end{bmatrix} + \\ \frac{1}{2}\begin{bmatrix} A^{1/2}|X^*|^2A^{1/2} & -A^{1/2}XB^{1/2} \\ -B^{1/2}X^*A^{1/2} & B \end{bmatrix} \end{array}\right)\right) \\
&= f\left(s_j\left(\begin{array}{l} \begin{bmatrix} \frac{1}{2}A & 0 \\ 0 & \frac{1}{2}B^{1/2}|X|^2B^{1/2} \end{bmatrix} + \\ \begin{bmatrix} \frac{1}{2}A^{1/2}|X^*|^2A^{1/2} & 0 \\ 0 & \frac{1}{2}B \end{bmatrix} \end{array}\right)\right) \\
&= s_j\left(f\left(\begin{array}{l} \begin{bmatrix} \frac{1}{2}A & 0 \\ 0 & \frac{1}{2}B^{1/2}|X|^2B^{1/2} \end{bmatrix} + \\ \begin{bmatrix} \frac{1}{2}A^{1/2}|X^*|^2A^{1/2} & 0 \\ 0 & \frac{1}{2}B \end{bmatrix} \end{array}\right)\right).
\end{aligned}$$

Using the fact that every unitarily invariant norm is an increasing function of singular values, it follows that

$$|||f(|(AX - XB) \oplus 0|)|||$$

$$
\begin{aligned}
&\leq \left|\left|\left|f\left(\frac{1}{2}\begin{bmatrix} A & 0 \\ 0 & B^{1/2}|X|^2B^{1/2}\end{bmatrix}+\frac{1}{2}\begin{bmatrix} A^{1/2}|X^*|^2A^{1/2} & 0 \\ 0 & B\end{bmatrix}\right)\right|\right|\right| \\
&\leq \frac{1}{2}\left|\left|\left|f\left(\begin{bmatrix} A & 0 \\ 0 & B^{1/2}|X|^2B^{1/2}\end{bmatrix}\right)+f\left(\begin{bmatrix} A^{1/2}|X^*|^2A^{1/2} & 0 \\ 0 & B\end{bmatrix}\right)\right|\right|\right| \text{(by Lemma 2.4(a))} \\
&\leq \frac{1}{2}\left|\left|\left|f\left(\begin{bmatrix} A & 0 \\ 0 & B^{1/2}|X|^2B^{1/2}\end{bmatrix}\right)\right|\right|\right|+\frac{1}{2}\left|\left|\left|f\left(\begin{bmatrix} A^{1/2}|X^*|^2A^{1/2} & 0 \\ 0 & B\end{bmatrix}\right)\right|\right|\right| \\
&= \frac{1}{2}|||f(A)\oplus f\left(B^{1/2}|X|^2B^{1/2}\right)|||+\frac{1}{2}|||f\left(A^{1/2}|X^*|^2A^{1/2}\right)\oplus f(B)||| \\
&= \frac{1}{2}|||f(A)\oplus f(XBX^*)|||+\frac{1}{2}|||f(X^*AX)\oplus f(B)|||,
\end{aligned}
$$

as required. □

Now, we have the following corollary:

Corollary 2.12. *Let $A, B \in \mathbb{M}_n(\mathbb{C})$ such that A and B are positive semidefinite. Then,*

$$|||(A-B)\oplus 0||| \leq |||A\oplus B||| \tag{2.9}$$

for every unitarily invariant norm.

Proof. The result follows by taking $f(t)=t, t\geq 0$, and $X=I$ in Theorem 2.11. □

The previous corollary was proved by Bhatia and Kittaneh in [3]. One can refer to [10] and [12] for extensions of the inequality (2.9).

Specializing Theorem 2.11 for certain types of functions gives the following two corollaries:

Corollary 2.13. *Let $A, B, X \in \mathbb{M}_n(\mathbb{C})$ such that A and B are positive semidefinite. Then*

$$
\begin{aligned}
|||e^{(|AX-XB|)}-I||| \;\leq\; & \frac{1}{2}|||\left(e^{(A)}-I\right)\oplus\left(e^{(XBX^*)}-I\right)|||+ \\
& \frac{1}{2}|||\left(e^{(X^*AX)}-I\right)\oplus\left(e^{(B)}-I\right)|||
\end{aligned}
$$

for every unitarily invariant norm.

Proof. This result follows by taking $f(t)=e^t-1, t\geq 0$, in Theorem 2.11. □

Corollary 2.14. *Let $A, B, X \in \mathbb{M}_n(\mathbb{C})$ such that A and B are positive semidefinite. Then, for $r \geq 1$, we have*

$$\left|\left|\left|\left|AX - XB\right|^r\right|\right|\right| \leq \frac{1}{2}\left|\left|\left|A^r \oplus (XBX^*)^r\right|\right|\right| + \frac{1}{2}\left|\left|\left|(X^*AX)^r \oplus B^r\right|\right|\right|$$

for every unitarily invariant norm.

Proof. This result follows by taking $f(t) = t^r, r \geq 1$, in Theorem 2.11. □

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References

[1] A. Al-natoor, W. Audeh, F. Kittaneh, Norm inequalites of Davidson-Power type, Math. Ineq. Appl., **23,** (2020), 689–697.

[2] R. Bhatia, Matrix Analysis, Springer, New York, 1997.

[3] R. Bhatia and F. Kittaneh, On the singular values of a product of operators, SIAM J. Matrix Anal. Appl. 11 (1990), 272–277.

[4] J.-C. Bourin and E.-Y. Lee, Unitary orbits of Hermitian operators with convex or concave functions, Bull. London Math. Soc. 44 (2012), 1085–1102.

[5] J.-C. Bourin and A. Mhanna, Positive block matrices and numerical ranges, C.R. Acad. Sci. Paris 355 (2017), 1077–1081.

[6] J.-C. Bourin and M. Uchiyama, A matrix subadditivity inequality for $f(A+B)$ and $f(A)+f(B)$, Linear Algebra Appl. 423 (2007), 512–518.

[7] K. Davidson and S.C. Power, Best approximation in C^*-algebras, J. Reine Angew. Math. 368 (1986), 43–62.

[8] F. Kittaneh, Norm inequalities for certain operator sums, J. Funct. Anal. 143 (1997), 337–348.

[9] F. Kittaneh, Norm inequalities for sums of positive operators, J. Operator Theory 48 (2002), 95–103.

[10] F. Kittaneh, Inequalities for commutators of positive operators, J. Funct. Anal. 250 (2007), 132–143.

[11] T. Kosem, Inequalities between $f(A + B)$ and $f(A) + f(B)$, Linear Algebra Appl. 418 (2006), 153–160.

[12] X. Zhan, Singular values of differences of positive semidefinite matrices, SIAM J. Matrix Anal. Appl. 22 (2002), 819–823.