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Generalized Lower Order and Approximation of Entire Transcendental Functions of Several Complex Variables in Some Banach Spaces

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Abstract

We obtain coefficients characterizations of generalized lower order $\lambda_m(f;\alpha,\beta)$ of entire transcendental functions f of m several complex variables $m \geq 2$ for generalized growth in terms of the sequence of best polynomial approximations of f in the Hardy Banach spaces $H_q(U^m)$ and in the Banach spaces $B_m(p,q,\eta)$. The presented results are extensions of investigations made by Vakarchuk and Zhir [17-21] and by Goldberg and Shah [4,13] to the multidimensional case.

1 Introduction

We denote an m-dimensional complex space by $\mathbb{C}^m$ as

$$\mathbb{C}^m = \{\mathbf{z} = (\mathbf{z_1}, \ldots, \mathbf{z_m}) : \mathbf{z_j} \in \mathbb{C}, \mathbf{j} = \overline{\mathbf{1}, \mathbf{m}}\}$$

where $\mathbb{C}$ is the space of complex numbers $z = x + iy$. Let f be an entire function of m complex variables $\mathbf{z} = (\mathbf{z_1}, \ldots, \mathbf{z_m}) \in \mathbb{C}^\mathbf{m}$ and let $\{D_R\} \in$

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$\mathbb{C}^m, R > 0$ be a family of complete m-circular domains depending on the parameter R and such that $\mathbf{z} \in \mathbf{D_R}$ if and only if

$$\frac{\mathbf{z}}{R} = (\frac{z_1}{R}, \ldots, \frac{z_m}{R}) \in D, D = D_1.$$

Suppose that $M_{f,D}(R) = \max\{|f(\mathbf{z})| : \mathbf{z} \in \mathbf{D_R}\}$. Goldberg [4] and Fuks [2] introduced the order and type of growth of $f(\mathbf{z})$ as:

$$\rho_D = \limsup_{R\to\infty} \frac{\log\log M_{f,D}(R)}{\log R}, \tag{1.1}$$

for $0 < \rho_D < \infty$

$$T_D = \limsup_{R\to\infty} \frac{\log M_{f,D}(R)}{R^{\rho_D}}. \tag{1.2}$$

Analogous to order, we define lower order as

$$\lambda_D = \liminf_{R\to\infty} \frac{\log\log M_{f,D}(R)}{\log R}, \tag{1.3}$$

for $0 < \lambda_D \leq \rho_D < \infty$. The quantities ρ_D, λ_D and T_D are called the D-order, D-lower order and D-type of the entire function f, respectively.

We now write the expansion of the entire function f as a Taylor series

$$f(\mathbf{z}) = \sum_{|\mathbf{k}|=\mathbf{0}}^{\infty} \mathbf{c_k(f) z^k},$$

where $\mathbf{k} = (\mathbf{k_1} \ldots, \mathbf{k_m}) \in \mathbb{Z}_+^{\mathbf{m}}, |\mathbf{k}| = \mathbf{k_1} + \cdots + \mathbf{k_m}; \mathbf{z^k} = \mathbf{z_1^{k_1}} \ldots \mathbf{z_m^{k_m}}$ and $c_{\mathbf{k}}(f) = c_{k_1,\ldots,k_m}(f)$ are the Taylor coefficients of f. Goldberg [4] established the following relationship between ρ and T with the moduli of the coefficients $|c_{\mathbf{k}}(f)|$:

$$\rho = \limsup_{|\mathbf{k}|\to\infty} \frac{|\mathbf{k}|\log|\mathbf{k}|}{-\log|c_{\mathbf{k}}(f)|}, \tag{1.4}$$

and

$$T = \limsup_{|\mathbf{k}|\to\infty} \frac{|\mathbf{k}|}{e\rho|c_{\mathbf{k}}(f)|^{-\frac{\rho}{|\mathbf{k}|}}}. \tag{1.5}$$

Similarly, we define

$$\lambda = \liminf_{|\mathbf{k}|\to\infty} \frac{|\mathbf{k}|\log|\mathbf{k}|}{-\log|c_{\mathbf{k}}(f)|}. \tag{1.6}$$

Sheremeta [14,15] generalized Goldberg's results with the help of general functions as follows:

Let L denote the class of functions h satisfying the following conditions:
(i) $h(x)$ is defined on $[a, \infty)$ and is positive, strictly increasing, differentiable and tends to ∞ as $x \to \infty$,
(ii) for any function $\varphi(x)$ such that $\varphi(x) \to \infty$ as $x \to \infty$,

$$\lim_{x\to\infty} \frac{h[(1+\frac{1}{\varphi(x)})x]}{h(x)} = 1.$$

Let Δ denote the class of functions h satisfying condition (i) and

$$\lim_{x\to\infty} \frac{h(cx)}{h(x)} = 1$$

for every $c > 0$; that is, $h(x)$ is slowly increasing.

Sheremeta [14] introduced the following notions of generalized order of growth for entire functions of several complex variables:

$$\rho_m(f;\alpha,\beta) = \limsup_{R\to\infty} \frac{\alpha(\log M_{f,D}(R))}{\beta(\log R)}, \tag{1.7}$$

for $\alpha(x) \in \Delta$ and $\beta(x) \in L$. Similarly, we define the generalized lower order of growth as

$$\lambda_m(f;\alpha,\beta) = \liminf_{R\to\infty} \frac{\alpha(\log M_{f,D}(R))}{\beta(\log R)}. \tag{1.8}$$

Setting $\alpha(x) = \log x$ and $\beta(x) = x$ in (1.7) and (1.8), we obtain the Goldberg definitions of classical order (1.4) and lower order (1.6) of entire functions.

Sheremeta established the relationship between the generalized order of growth (1.7) of an entire function f and its Taylor coefficients. For $m = 1$, Shah [13, Thm. 2] obtained the coefficient characterization of the generalized lower order of growth (1.8) of f in terms of Taylor coefficients. Following the technique of Sheremeta and Shah we can obtain the following theorem:

Theorem A. Let $f(\mathbf{z}) = \sum_{|\mathbf{k}|=\mathbf{0}}^{\infty} \mathbf{c_k}(\mathbf{f})\mathbf{z^k}$ be an entire function of m complex variables with generalized lower order of growth $\lambda_m(f;\alpha,\beta)$, where $\alpha \in \Delta$ and $\beta \in L$ for $c = 1$, a function $F(x,1) = F(x) = \beta^{-1}(\alpha(x))$, where β^{-1} is a function inverse to β, satisfies the condition
(i) For some function $\varphi(x) \to \infty$ (even slowly) as $x \to \infty$,

$$\frac{\beta(x\varphi(x))}{\beta(e^x)} \to 0 \quad \text{as} \quad x \to \infty,$$

(ii)

$$\frac{dF(x)}{d(\log x)} = O(1) \quad \text{as} \quad x \to \infty,$$

(iii) $|\frac{c_{\mathbf{k}}(f)}{c_{\mathbf{k+1}}(f)}|$ is ultimately a non decreasing function of k. Then

$$\lambda_m(f;\alpha,\beta) = \liminf_{|\mathbf{k}|\to\infty} \frac{\alpha(|\mathbf{k}|)}{\beta(-|\mathbf{k}|^{-1}\log|\mathbf{c_k(f)}|)}. \tag{1.9}$$

Recently Vakarchuk and Zhir [21] studied the best polynomial approximations of entire transcendental functions of several complex variables in Banach spaces. Thus, let $U^m = \{\mathbf{z} \in \mathbb{C}^{\mathbf{m}} : |\mathbf{z_j}| < \mathbf{1}, \mathbf{j} = \overline{\mathbf{1},\mathbf{m}}\}$ be a unit polydisk in $\mathbb{C}^m$ and let $\Gamma^m = \{\mathbf{z} \in \mathbb{C}^{\mathbf{m}} : |\mathbf{z_j}| = \mathbf{1}, \mathbf{j} = \overline{\mathbf{1},\mathbf{m}}\}$ be its skeleton. Also, let $\mathbb{R}^m$ be an m-dimensional real space. Moreover, by

$$T^m = \{\mathbf{x} = (\mathbf{x_1},\ldots,\mathbf{x_m}) \in \mathbb{R}^{\mathbf{m}} : \mathbf{0} \le \mathbf{x_j} \le \mathbf{2}\pi, \mathbf{j} = \overline{\mathbf{1},\mathbf{m}}\}$$

and

$$\Pi^m = \{\mathbf{r} = (\mathbf{r_1},\ldots,\mathbf{r_m}) \in \mathbb{R}^{\mathbf{m}} : \mathbf{0} \le \mathbf{r_j} < \mathbf{1}, \mathbf{j} = \overline{\mathbf{1},\mathbf{m}}\},$$

we denote m-dimensional cubes in the space $\mathbb{R}^m$. Let $A(U^m)$ be the set of all functions f analytic in the set U^m. For any function $f \in A(U^m)$, we get

$$M_q(f,r) = \{\frac{1}{(2\pi)^m}\int_{T^m} |f(\mathbf{re^{it}})|^{\mathbf{q}}\mathbf{dt}\}^{\frac{\mathbf{1}}{\mathbf{q}}}, \mathbf{0} < \mathbf{q} < \infty,$$

where $f(\mathbf{re^{it}}) = \mathbf{f}(\mathbf{r_1e^{it_1}},...,\mathbf{r_me^{it_m}}), \mathbf{dt} = \mathbf{dt_1}\ldots\mathbf{dt_m}$ and

$$M_\infty(f,r) = \max\{|f(\mathbf{re^{it}})| : \mathbf{t} \in \mathbf{T^m}\}, \mathbf{r} \in \mathbf{\Pi^m}.$$

Let $H_q(U^m), 0 < q \le \infty$, denote the Hardy space of functions $f(\mathbf{z}) \in \mathbf{A}(\mathbf{U^m})$ satisfying the condition

$$\| f \|_{H_q} = \sup\{M_q(f,r) : \mathbf{r} \in \mathbf{\Pi^m}\} < \infty,$$

and let $H'_q(U^m)$ denote the Bergman space of functions $f(\mathbf{z}) \in \mathbf{A}(\mathbf{U^m})$ satisfying the condition

$$\| f \|_{H'_q} = \{\frac{1}{(2\pi)^m}\int_{T^m} |f(e^{i\mathbf{t}})|^q d\mathbf{t}\}^{\frac{1}{q}}, q > 0.$$

For $q = \infty$, let $\| f \|_{H'_\infty} = \| f \|_{H_\infty} = \sup\{|f(\mathbf{z})|, \mathbf{z} \in \mathbf{U^m}\}$. Then H_q and $H_{q'}$ are Banach spaces for $q \geq 1$. Following [21, p.1794], we say that a function $f \in A(U^m)$ is in the space $B_m(p, q, \eta)$ if

$$\| f \|_{p,q,\eta} = \int_{\Pi^m} (1 - \mathbf{r})^{\eta(\frac{1}{\mathbf{p}} - \frac{1}{\mathbf{q}}) - \mathbf{1}} \mathbf{M}_{\mathbf{q}}^{\eta}(\mathbf{f}, \mathbf{r}) \mathbf{dr}\}^{\frac{1}{\eta}} < \infty, \mathbf{0} < \eta < \infty, \tag{1.10}$$

$0 < p < q \leq \infty$, and

$$\| f \|_{p,q,\infty} = \sup\{(1 - \mathbf{r})^{(\frac{1}{\mathbf{p}} - \frac{1}{\mathbf{q}})} \mathbf{M}_{\mathbf{q}}(\mathbf{f}, \mathbf{r}) : \mathbf{r} \in \mathbf{\Pi^m}\} < \infty, \eta = \infty.$$

It is known [5] that $B_m(p, q, \eta)$ is a Banach space for $p > 0$ and $q, \eta \geq 1$; otherwise, it is a Frechet space.

Let P_n denote the subspace of algebraic polynomials of m complex variables of the form

$$P_n = \{\sum_{|\mathbf{k}|=\mathbf{0}}^{\infty} c_{\mathbf{k}} \mathbf{z}^{\mathbf{k}} : c_{\mathbf{k}} \in \mathbb{C}\},$$

where $n \in \mathbb{Z}_+$. Let X be one of the Banach spaces of analytic functions of m complex variables listed earlier. We will denote by $E_n(f, X)$ the value of the best polynomial approximation of the function $f \in X$ by elements of the subspace P_n; i.e.,

$$E_n(f, X) = \inf\{\| f - p_n \|_X : p_n \in P_n\}. \tag{1.11}$$

Several authors [1,3,8,13] have established the relationship between the order and type of an entire function and the rate of its best polynomial approximation in different domains. Kumar [9] has investigated the growth and approximation of entire function solutions of Helmholtz equation. Vakarchuk and Zhir [17-20] studied some problems of approximation of entire transcendental functions in some Banach spaces. It has been noticed that the study of growth of entire transcendental functions in terms of $E_n(f, X)$ in several complex variables has not been done so extensively [10-12,16,21,22] as in single complex variable. Vakarchuk and Zhir [21] have obtained the necessary and sufficient condition for $f \in X$ to be an entire transcendental function of the generalized order of growth $\rho_m(f; \alpha, \beta)$ in terms of $E_n(f, X)$, where X be one of the Banach spaces of functions analytic in U^m. To the best of our knowledge, the generalized lower order $\lambda_m(f; \alpha, \beta)$ has not yet been characterized in terms of $E_n(f, X)$ in m-complex variables.

2 Main Results

Theorem 2.1. Let the functions $\alpha(x)$ and $\beta(x)$ satisfy the conditions of Theorem A. Moreover, suppose that the sequence $\{\frac{E_n(f,B_m(p,q,\eta))}{E_{n+1}(f,B_m(p,q,\eta))}\}$ forms a nondecreasing function of n. Then the entire transcendental function $f(\mathbf{z}) \in \mathbf{B_m}(\mathbf{p}, \mathbf{q}, \eta)$ has generalized lower order $\lambda_m(f, \alpha, \beta)$ if and only if

$$\lambda \equiv \lambda_m(f, \alpha, \beta) = \liminf_{n\to\infty} \frac{\alpha(n)}{\beta[\log\{E_n(f, B_m(p, q, \eta))\}^{-\frac{1}{n}}]}. \tag{2.1}$$

Proof. This result will be proven in two steps. First, we consider the case where $q = 2$; i.e., the space $B_m(p, 2, \eta)$, where $0 < p < 2$ and $\eta \geq 1$. Let $f(\mathbf{z}) \in \mathbf{B_m}(\mathbf{p}, \mathbf{q}, \eta)$ be of generalized lower order λ.

Then, from the definition of generalized lower order λ, given $\varepsilon > 0$, there exists a sequence $\{R_n\}, R_n \to \infty$ as $n \to \infty$ such that

$$M_{f,D}(R) < \exp[(\alpha^{-1}\{(\lambda + \varepsilon)[\beta(\log R)]\})], R = R_n.$$

Using Cauchy inequality, we get

$$\begin{aligned}|c_{\mathbf{k}}(f)| &\leq R^{-\mathbf{k}} M_{f,D}(R)\\ &\leq R^{-|\mathbf{k}|} M_{f,D}(R)\\ &\leq \exp\{-|\mathbf{k}| \log \mathbf{R} + (\alpha^{-\mathbf{1}}\{(\lambda + \varepsilon)[\beta(\log \mathbf{R})]\})\}, \mathbf{R} = \mathbf{R_n}.\end{aligned}$$

The above inequality holds for a sequence $\{R_n\}, R_n \to \infty$, as $n \to \infty$, we can choose

$$R = R_n = \exp[\beta^{-1}\{\frac{\alpha(n)}{\overline{\lambda}}\}] = \exp[F(n, \frac{1}{\overline{\lambda}})], \overline{\lambda} = \lambda + \varepsilon.$$

Substituting the value of R_n in above inequality, we obtain

$$|c_{\mathbf{k}}(f)| \leq \exp\{-(n+1)F(n, \frac{1}{\overline{\lambda}})\} \exp[\alpha^{-1}\{\overline{\lambda}\frac{\alpha(n)}{\overline{\lambda}}\}] = \frac{1}{\exp[n\{F(n, \frac{1}{\overline{\lambda}}) - 1\}]}. \tag{2.2}$$

Let $T_n(f, \mathbf{z}) = \sum_{|\mathbf{k}|=\mathbf{0}}^{\mathbf{n}} \mathbf{c_k z^k}$ be the n^{th} partial sum of the Taylor series of the function f.

In view of [21, p. 1805], we have

$$E_n(f;B_m(p,2,\eta)) \leq \{\kappa^*_{B_m(p,2,\eta)}(n+1)\}^{\frac{-1}{\eta}}\{\sum_{|\mathbf{k}|=\mathbf{n+1}}^{\infty}|c_{\mathbf{k}}(f)|^2\}^{\frac{1}{2}}. \qquad (2.3)$$

By using (2.2) in (2.3), we obtain

$$E_n(f;B_m(p,2,\eta)) \leq \frac{\{\kappa^*_{B_m(p,2,\eta)}(n+1)\}^{\frac{-1}{\eta}}}{\exp\{n[F(n,\frac{1}{\lambda})-1]\}}\{\sum_{|\mathbf{k}|=\mathbf{n+1}}^{\infty}\Omega^2_{\mathbf{k}}(\alpha,\beta)\}^{\frac{1}{2}}, \qquad (2.4)$$

where

$$\Omega_k(\alpha,\beta) \cong \frac{\exp\{n\{\beta^{-1}[\frac{1}{\lambda}\alpha(n)]-1\}\}}{\exp\{|\mathbf{k}|\{\beta^{-\mathbf{1}}[\frac{\mathbf{1}}{\lambda}\alpha(|\mathbf{k}|)]-\mathbf{1}\}\}}.$$

Set

$$\hat{\Omega}(\alpha,\beta) \cong \exp\{-\{\beta^{-1}[\frac{1}{\lambda}\alpha(1)]-1\}\}.$$

Clearly $\hat{\Omega}(\alpha,\beta)<1$. Then, for $|\mathbf{k}| \geq \mathbf{n}$, we get

$$\Omega_{\mathbf{k}}(\alpha,\beta) \leq \exp\{(n-|\mathbf{k}|)\{\beta^{-\mathbf{1}}[\frac{\mathbf{1}}{\lambda}\alpha(\mathbf{n})]-\mathbf{1}\}\} \leq \hat{\mathbf{\Omega}}^{|\mathbf{k}|-(\mathbf{n})}(\alpha,\beta). \qquad (2.5)$$

From (2.4) and (2.5), we get

$$E_n(f;B_m(p,2,\eta)) \leq \frac{\{\kappa^*_{B_m(p,2,\eta)}(n)\}^{\frac{-1}{\eta}}}{(1-\hat{\Omega}^2(\alpha,\beta))^{\frac{m}{2}}\exp\{n\{\beta^{-1}[\frac{1}{\lambda}\alpha(n)]-1\}\}}. \qquad (2.6)$$

Now, we have

$$\overline{\lambda} \geq \frac{\alpha(n)}{\beta(n^{-1}(-\log E_n(f,B_m(p,2,\eta)))+\log\frac{\{\kappa^*_{B_m(p,2,\eta)}(n)\}^{-\frac{1}{\eta}}}{(1-\hat{\Omega}^2(\alpha,\beta))^{\frac{m}{2}}})}. \qquad (2.7)$$

Observe that $\lim_{|\mathbf{k}|\to\infty}[\kappa^*_{B_m(p,2,\eta)}(|\mathbf{k}|)]^{\frac{\mathbf{1}}{|\mathbf{k}|}}=\mathbf{1}$. Proceeding to the limit as $n\to\infty$ on the right hand side of (2.7), since $\varepsilon>0$ was arbitrarily, we get

$$\lambda_m(f,\alpha,\beta) \geq \liminf_{n\to\infty}\frac{\alpha(n)}{\beta[\log\{E_n(f,B_m(p,2,\eta))\}^{-\frac{1}{n}}]}. \qquad (2.8)$$

To obtain the reverse inequality in (2.8), assume that the sequence $\{\frac{E_n(f,B_m(p,2,\eta))}{E_{n+1}(f,B_m(p,2,\eta))}\}$ forms a nondecreasing function of n. Consider the function

$$\xi(\mathbf{z}) = \sum_{\mathbf{n=1}}^{\infty} \mathbf{E_n}(\mathbf{f}, \mathbf{B_m}(\mathbf{p}, \mathbf{2}, \eta))\kappa^*_{\mathbf{B_m}(\mathbf{p},\mathbf{2},\eta)}(\mathbf{n})\mathbf{z}^{\mathbf{n}}.$$

$$\{\kappa^*_{B_m(p,2,\eta)}(n)\}^{-1} = \max_{|\mathbf{k}|=\mathbf{n}}\{\kappa^*_{B_m(p,2,\eta)}(n)\}^{-1},$$

where

$$\kappa^*_{B_m(p,2,\eta)}(k) = (\Pi_{j=1}^{m} B_{k_j,p,2,\eta})^{-1}$$

is a constant depending on the space $B_m(p, 2, \eta)$ and k and does not depend on f.
From [21, eq. 44], we have

$$\lim_{n\to\infty} [E_n(f, B_m(p, 2, \eta))]^{\frac{1}{n}} = 0.$$

It follows that $\xi(\mathbf{z})$ represents an entire function of m complex variables $\mathbf{z}$. If we denote the coefficients of the Taylor series of $\xi(\mathbf{z})$ by

$$c_{\mathbf{k}}(\xi) = E_n(f, B_m(p, 2, \eta))\kappa^*_{B_m(p,2,\eta)}(k),$$

then

$$\frac{c_{\mathbf{k}}(\xi)}{c_{\mathbf{k+1}}(\xi)} = \frac{E_n(f, B_m(p, 2, \eta))\kappa^*_{B_m(p,2,\eta)}(k)}{E_{n+1}(f, B_m(p, 2, \eta))\kappa^*_{B_m(p,2,\eta)}(k + 1)}.$$

Hence, under the assumption of the theorem, $|\frac{c_{\mathbf{k}}(\xi)}{c_{\mathbf{k+1}}(\xi)}|$ forms a nondecreasing function of n. Now, using Theorem A, for the limit as $n \to \infty (i.e., |\mathbf{k}| \to \infty)$, we get

$$\lambda_m(\xi, \alpha, \beta) = \liminf_{|\mathbf{k}|\to\infty} \frac{\alpha(|\mathbf{k}|)}{\beta[\log\{|c_{\mathbf{k}}(\xi)\}^{-\frac{1}{n}}]}.$$

Now

$$-\frac{1}{|\mathbf{k}|} \log |c_{\mathbf{k}}(\xi)| = -\frac{1}{n} \log E_n(f, B_m(p, 2, \eta)) + O(1)$$

Since $\beta \in L$, we get

$$\begin{aligned} \lambda_m(\xi, \alpha, \beta) &= \liminf_{n\to\infty} \frac{\alpha(n)}{\beta[\log\{E_n(f, B_m(p,2,\eta))\kappa^*_{B_m(p,2,\eta)}(k)\}^{-\frac{1}{n}}]}. \\ &= \liminf_{n\to\infty} \frac{\alpha(n)}{\beta[\log\{E_n(f, B_m(p,2,\eta))\}^{-\frac{1}{n}}]}. \end{aligned} \tag{2.9}$$

Now using the relation [21, eq. 45]

$$|c_{\mathbf{k}}(f)| \le E_n(f, B_m(p,2,\eta))\kappa^*_{B_m(p,2,\eta)}(k)$$

in (2.9) we see that

$$\lambda_m(f, \alpha, \beta) \le \lambda_m(\xi, \alpha, \beta). \tag{2.10}$$

Combining (2.8) and (2.10), we get required result.

Next, we consider the spaces $B_m(p, q, \eta)$ for $0 < p < q, q \neq 2$, and $q, \eta \ge 1$. Note that [7, p.103], for $p \ge p_1, q \le q_1$ and $\eta \le \eta_1$, if at least one of the inequalities is strict, then the strict inclusion $B_m(p, q, \eta) \subset B_m(p_1, q_1, \eta_1)$ holds and for any function $f \in B_m(p, q, \eta)$, the following relation is true:

$$\| f \|_{p_1,q_1,\eta_1} \le C_{p_1,q_1,\eta_1;p,q,\eta} \| f \|_{p,q,\eta}, \tag{2.11}$$

where $C_{p_1,q_1,\eta_1;p,q,\eta}$ is a positive constant that depends only on the indicated subscripts and is independent of f. In view of the definition of best polynomial approximation, the inequality (2.11) yields

$$E_n(f; B_m(p_1, q_1, \eta_1)) \le C_{p_1,q_1,\eta_1;p,q,\eta} E_n(f; B_m(p, q, \eta)), \tag{2.12}$$

where $f \in B_m(p, q, \eta)$ and $C_{p,q,\eta;p_1,q_1,\eta_1}$ is a constant independent of f and n.

For the general case $B_m(p, q, \eta), q \neq 2$, we prove the necessity of condition (2.1). Let $f \in B_m(p, q, \eta)$ be an entire transcendental function whose generalized lower order $\lambda_m(f; \alpha, \beta)$ is defined by (2.1). We set

$$\{\kappa^*_{B_m(p,q,\eta)}(n+1)\}^{-1} = \max_{|\mathbf{k}|=\mathbf{n+1}} \{\kappa^*_{B_m(p,q,\eta)}(\mathbf{k})\}^{\mathbf{-1}}$$

and write [21, p.1807]

$$E_n(f; B_m(p, q, \eta)) \le \{\kappa^*_{B_m(p,q,\eta)}(n+1)\}^{-\frac{1}{\eta}} \{ \sum_{|\mathbf{k}|=\mathbf{n+1}}^{\infty} |c_{\mathbf{k}}(f)| \}. \tag{2.13}$$

Now, using the relation (2.2) and (2.13) and same reasoning as in deducing relations (2.3)-(2.5), we obtain

$$E_n(f;B_m(p,q,\eta)) \leq \frac{\{\kappa^*_{B_m(p,q,\eta)}(n)\}^{\frac{-1}{\eta}}}{(1-\hat{\Omega}^2(\alpha,\beta))^{\frac{m}{2}}\exp\{n\{\beta^{-1}[\frac{1}{\lambda}\alpha(n)]-1\}\}}, \tag{2.14}$$

or

$$\lambda_m(f;\alpha,\beta)+\varepsilon \geq \frac{\alpha(n)}{\beta(n^{-1}(-\log E_n(f,B_m(p,q,\eta)))+\log\frac{\{\kappa^*_{B_m(p,q,\eta)}(n)\}^{-\frac{1}{\eta}}}{(1-\hat{\Omega}^2(\alpha,\beta))^{\frac{m}{2}}})}. \tag{2.15}$$

Now $\lim_{|\mathbf{k}|\to\infty}[\kappa^*_{B_m(p,q,\eta)}(|\mathbf{k}|)]^{\frac{1}{|\mathbf{k}|}} = \mathbf{1}$. Taking the limit as $n\to\infty$, since $\varepsilon>0$ was arbitrarily, we get

$$\lambda_m(f,\alpha,\beta) \geq \liminf_{n\to\infty}\frac{\alpha(n)}{\beta[\log\{E_n(f,B_m(p,q,\eta))\}^{-\frac{1}{n}}]}. \tag{2.16}$$

To prove the reverse inequality, let $0<p<q<2$ and $\eta,q\geq 1$. By using the relation (2.1) with $p_1=p, q_1=2$ and $\eta_1=\eta$ and relation (2.8) already established for $q=2$, we get

$$\liminf_{n\to\infty}\frac{\alpha(n)}{\beta[\log\{E_n(f,B_m(p,q,\eta))\}^{-\frac{1}{n}}]}$$
$$\geq \liminf_{n\to\infty}\frac{\alpha(n)}{\beta[\log\{E_n(f,B_m(p,2,\eta))\}^{-\frac{1}{n}}]} = \lambda_m(f,\alpha,\beta).$$

Now let $0<p\leq 2<q$. Since

$$M_2(f;\mathbf{r}) \leq \mathbf{M_q(f;r)},$$

where $\mathbf{r}\in\mathbf{\Pi^m}$, from [21, p. 1808] we get

$$E_n(f;B_m(p,q,\eta)) \geq |c_{\mathbf{k}}(f)|\{\kappa^*_{B_m(p,q,\eta)}(\mathbf{k})\}^{-\frac{1}{\eta}}, \tag{2.17}$$

where $|\mathbf{k}|=\mathbf{n+1}$. Then for sufficiently large n, we have

$$\frac{\alpha(n)}{\beta[\log\{E_n(f,B_m(p,q,\eta))\}^{-\frac{1}{n}}]}$$
$$\geq \frac{\alpha(n)}{\beta[-|\mathbf{k}|^{-\mathbf{1}}(\log|\mathbf{c_k(f)}|+\log\{\kappa^*_{\mathbf{B_m(p,q,\eta)}}(\mathbf{k})\}^{\frac{1}{\eta}})]}.$$

Now proceeding to limits and using (1.9), we obtain

$$\liminf_{n\to\infty} \frac{\alpha(n)}{\beta[\log\{E_n(f, B_m(p,q,\eta))\}^{-\frac{1}{n}}]}$$
$$\geq \liminf_{|\mathbf{k}|\to\infty} \frac{\alpha(|\mathbf{k}|)}{\beta[-|\mathbf{k}|^{-\mathbf{1}}\log|\mathbf{c_k(f)}|]} = \lambda_m(f,\alpha,\beta).$$

Let $2 \leq p < q$. Setting $q_1 = q, \eta_1 = \eta$ and $p_1 \in (0,2)$ in inequality (2.11), where p_1 is a fixed number, we get

$$E_n(f; B_m(p,q,\eta)) \geq \frac{|c_{\mathbf{k}}(f)|}{C_{p,q,\eta;p_1,q,\eta}}\{\kappa^*_{B_m(p,q,\eta)}(k)\}^{\frac{1}{\eta}}. \tag{2.18}$$

Substituting $p_1 = p$ in (2.17), we obtain the following relation

$$E_n(f; B_m(p_1,q,\eta)) \geq |c_{\mathbf{k}}(f)|\{k^*_{B_m(p_1,q,\eta)}(\mathbf{k})\}^{-\frac{\mathbf{1}}{\eta}}. \tag{2.19}$$

Now combining (2.18) and (2.19), we get

$$E_n(f; B_m(p,q,\eta)) \geq \frac{|c_{\mathbf{k}}(f)|}{C_{p,q,\eta;p_1,q,\eta}}\{k^*_{B_m(p_1,q,\eta)}(\mathbf{k})\}^{-\frac{\mathbf{1}}{\eta}}, \tag{2.20}$$

where $|\mathbf{k}| = \mathbf{n}+\mathbf{1}$ and $C_{p,q,\eta;p_1,q,\eta}$ is a constant independent of n and f. Using (2.20) and applying the same analogy as in the previous case $0 < p \leq 2 < q$, for sufficiently large n or $|\mathbf{k}| \to \infty$, we have

$$\frac{\alpha(n)}{\beta(\{\log(|E_n(f;B_m(p,q,\eta))|^{\frac{-1}{n}})\})}$$
$$\geq \frac{\alpha(|\mathbf{k}|)}{\beta(\{\log(|c_{\mathbf{k}}(f)|^{\frac{-1}{|\mathbf{k}|}} + \log(C_{p,q,\eta;p_1,q,\eta})^{\frac{1}{|\mathbf{k}|}} + \log\{\kappa^*_{B_m(p_1,q,\eta)}(\mathbf{k})\}^{-\frac{\mathbf{1}}{|\mathbf{k}|\eta}})\})}.$$

By applying limits and using (2.1), we get

$$\limsup_{n\to\infty} \frac{\alpha(n)}{\beta(\{\log(|E_n(f;B_m(p,q,\eta))|^{\frac{-1}{n}})\})} \geq \lambda_m(f,\alpha,\beta).$$

Consequently, the proof is complete.

Theorem 2.2. Let the functions $\alpha(x)$ and $\beta(x)$ satisfy the conditions of Theorem A. Moreover, suppose that the sequence $\{\frac{E_n(f,H_q(U^m))}{E_{n+1}(f,H_q(U^m))}\}$ forms a nondecreasing function of n. Then the entire transcendental function $f(\mathbf{z}) \in \mathbf{H_q}(\mathbf{U^m})$ has a generalized lower order $l_m(f,\alpha,\beta)$ if and only if

$$l \equiv l_m(f,\alpha,\beta) = \liminf_{n\to\infty} \frac{\alpha(n)}{\beta[\log\{E_n(f,H_q(U^m))\}^{-\frac{1}{n}}]}. \tag{2.21}$$

Proof. Let $f(\mathbf{z}) = \sum_{|\mathbf{k}|=\mathbf{0}}^{\infty} \mathbf{c_k(f)z^k}$ be an entire transcendental function of m complex variables having finite generalized lower order $\lambda_m(f;\alpha,\beta)$. Since

$$\lim_{|\mathbf{k}|\to\infty} |c_{\mathbf{k}}(f)|^{-\frac{1}{|\mathbf{k}|}} = 0, \tag{2.22}$$

and $f(\mathbf{z}) \in \mathbf{B_m}(\mathbf{p},\mathbf{q},\eta)$, where $0 < p < q \le \infty$ and $q,\eta \ge 1$, in view of relation (1.10), we get

$$E_n(f;B_m(\frac{q}{2},q,q) \le E_n(f;H_q(U^m)), 1 \le q < \infty. \tag{2.23}$$

In the case of Hardy space $H_\infty(U^m)$,

$$E_n(f;B_m(p,\infty,\infty) \le E_n(f;H_\infty(U^m)), 1 \le p < \infty. \tag{2.24}$$

Using (2.23), we can write

$$\begin{aligned} l_m(f,\alpha,\beta) &= \liminf_{n\to\infty} \frac{\alpha(n)}{\beta[\log\{E_n(f,H_q(U^m))\}^{-\frac{1}{n}}]} \\ &\ge \liminf_{n\to\infty} \frac{\alpha(n)}{\beta[\log\{E_n(f,H_q(B_m(\frac{q}{2},q,q))\}^{-\frac{1}{n}}]} \\ &\ge \lambda_m(f;\alpha,\beta), 1 \le q < \infty. \end{aligned} \tag{2.25}$$

Using estimate (2.24), we have proven inequality (2.25) in the case $q=\infty$. For the reverse inequality

$$l_m \le \lambda, \tag{2.26}$$

we use inequality (2.2) which is true for any $|\mathbf{k}| > \mathbf{n_o}$, and estimate from above the finite generalized order $\lambda_m(f;\alpha,\beta)$ of f as follows. We have

$$\begin{aligned} E_n(f;H_q(U^m)) &\le \| f - T_n(f) \|_{H_q} \le \sum_{|\mathbf{k}|=\mathbf{n+1}}^{\infty} |c_{\mathbf{k}}(f)| \\ &\le \frac{1}{\exp\{n\{\beta^{-1}[\frac{1}{\lambda}\alpha(n)] - 1\}\}} \sum_{|\mathbf{k}|=\mathbf{n+1}}^{\infty} \hat{\Omega}^{|\mathbf{k}|-\mathbf{n}-\mathbf{1}}(\alpha,\beta) \\ &\le \frac{1}{(1-\hat{\Omega}(\alpha,\beta))[\exp\{n\{\beta^{-1}[\frac{1}{\lambda}\alpha(n)] - 1\}\}]}, \end{aligned}$$

or

$$\frac{1}{E_n(f;H_q(U^m))} \geq (1-\hat{\Omega}(\alpha,\beta))[\exp\{n\{\beta^{-1}[\frac{1}{\lambda}\alpha(n)]-1\}\}].$$

This yields

$$\lambda_m(f,\alpha,\beta)+\varepsilon \geq \limsup_{n\to\infty} \frac{\alpha(n)}{\beta[\log(|E_n(f,H_q(U^m))|^{-\frac{1}{n+1}})+\log((1-\hat{\Omega}(\alpha,\beta))^{-\frac{1}{n+1}})]}. \tag{2.27}$$

Since $\hat{\Omega}(\alpha,\beta)<1$, by using the properties of α,β and proceeding to the limit as $n\to\infty$, we obtain inequality (2.26). Finally, we get

$$l_m = \lambda_m(f,\alpha,\beta). \tag{2.28}$$

As a result, the proof is complete.
Note. We can find an analog of Theorem 2.2 for the Bergman spaces by using (1.10) for $1\leq q<\infty$ and for $q=\infty$ from Theorem 2.1.

References

[1] M. Z. Dveirin, On the relationship between the rate of polynomial approximation of an entire function and its order and type, Ukr. Mat. J., **67,** No. 10, (2015), 1498–1514.

[2] B. A. Fuks, Introduction to Theory of Analytic Functions of Many Complex Variables (in Russian), Fizmatgiz, Moscow, 1962.

[3] A. Giroux, Approximation of entire functions over bounded domains, J. Approx. Theory, **28,** No. 1, (1980), 45–53.

[4] A. A. Goldberg, Elementary remarks on the formulas for the determination of the order and type of entire functions of many complex variables, Dokl. Akad. Nauk Arm. SSSR, **29,** No. 4, (1959), 145–151.

[5] M. I. Gvaradze, Factors of one class of analytic functions defined on a polydisk, Tr. Tbilis. Mat. Inst., **66,** (1988), 15–21.

[6] M. I. Gvaradze, On One Class of Spaces of Analytic Functions (in Russian), Candidate Degree Thesis, Tbilisi, 1975.

[7] M. I. Gvaradge, On one class of spaces of analytic functions, Mat. Zametki, **21,** No. 2, (1977), 141–150.

[8] I. I. Ibragimov, N. I. Shikhaliev, On the best polynomial approximation of analytic functions in a space of analytic functions, Dokl. Akad. Nauk SSSR, **227,** No. 2, (1976), 280–283.

[9] Devendra Kumar, Generalized growth and approximation of entire function solution of Helmholtz equation in Banach spaces, Annali Dell. Univ. Ferrara, **62,** No. 1, (2016), 83–97.

[10] Devendra Kumar, On the growth and approximation of transcendental entire functions on algebraic varieties, Int. J. Anal. Appl., **12,** No. 1, (2016), 22–29.

[11] Devendr Kumar, On the growth and best polynomial approximation of entire functions in $\mathbb{C}^m (m \geq 2)$ in some Banach spaces, The Journal of Analysis, (2017).

[12] V. M. Muradov, On the relationship between the best polynomial approximation of analytic functions of many complex variables and the Faber coefficients. In Special Problems of the Theory of Functions, Proc. of the Institute of Mathematics and Mechanics, Academy of Sciences of Azerb. SSR., Issue 3, (1986), 195–212.

[13] S. M. Shah, Polynomial approximation of an entire function and generalized orders, J. Approx. Theory, **19,** No. 4, (1977), 315–324.

[14] M. N. Sheremeta, On the relationship between the growth of the maximum modulus of an entire function and the moduli of coefficients of its power expansions, Izv. Vyssh. Uchebn. Zaved. Ser. Mat., No. 2 (1967), 100–108.

[15] M. N. Sheremeta, On the relationship between the growth of entire or analytic functions of zero order in a disk and the coefficients of their power expansions, Izv. Vyssh. Uchebn. Zaved. Ser. Mat., No. 6, (1968), 115–121.

[16] S. B. Vakarchuk, On the best approximation by generalized polynomials in one space of analytic functions of two complex variables, Izv. Vyssh. Uchebn. Zaved. Ser. Mat., No. 7, (1991), 14–25.

[17] S. B. Vakarchuk, On the best polynomial approximation of functions analytic in the unit circle in Banach spaces, Mat. Zametki, **55,** No. 4, (1994), 6–14.

[18] S. B. Vakarchuk, On the best polynomial approximation of entire transcendental functions in Banach spaces. I, Ukr. Mat. Zh., **47,** No. 9, (1994), 1123–1133.

[19] S. B Vakarchuk, On the best polynomial approximation of entire transcendental functions in Banach spaces. II, Ukr. Mat. Zh., **47,** No. 10, (1994), 1318–1322.

[20] S. B. Vakarchuk, S. I. Zhir, On some problems of polynomial approximation of entire transcendental functions, Ukr. Mat. J., **54,** No. 9, (2002), 1393–1401.

[21] S. B. Vakarchuk, S. I. Zhir, On the best polynomial approximations of entire transcendental functions of many complex variables in some Banach spaces, Ukran. Math. J., **66,** No. 12, (2015), 1763–1811.

[22] S. I. Zhir, S. B. Vakarchuk, Some problems of the best polynomial approximation of entire transcendental functions of one and many complex variables. In: Abstr. of the Internet. Conf. "Approximation Theory of Functions and its Applications" on the occasion of the 70th Birthday of O. I. Stepanets'(Kam'Yanets'-Podil's'kyi, May 28-June 3, 2012), Institute of Mathematics, Ukrainian National Academy of Science, Kiev, (2012), 44–45.