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Hilbert-Schmidt numerical radius inequalities for 2×2 operator matrices

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Abstract

Let A and C be operators in the Hilbert-Schmidt class. Then

$$\max(\sqrt{2}w_2(A), \sqrt{2}w_2(C)) \leq w_2\left(\begin{bmatrix} A & C \\ -C & -A \end{bmatrix}\right) \leq \sqrt{2}\left(w_2(A) + w_2(C)\right).$$

Several Hilbert-Schmidt numerical radius inequalities are given.

1 Introduction

Let $B(H)$ denote the algebra of all bounded linear operators on a complex separable Hilbert space H. For $A \in H$, let $\|A\|_p, \|A\|$, and $\|A\|_2$ denote, respectively, the Schatten $p-$norms, and in particular the spectral norm and the Hilbert-Schmidt norm of A. Note that the Schatten $p-$norms for $1 \leq p < \infty$ of the operator $A \in B(H)$ is defined by $\|A\|_p = (tr\,|A|^p)^{1/p} = \left(\sum_{j=1}^{n} s_j^p(A)\right)^{1/p}$, where $s_1(A) \geq s_2(A) \geq ...$ are the singular values of A. For recent studies about singular value inequalities, we refer the reader to [3], [4], [5] and [6]. The Schatten $p-$classes, for $0 < p < \infty$, are denoted by C_p which

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consist of all operators A for which $\|A\|_p$ is finite. In particular, if $p=\infty$ we denote the norm by $\|A\|$ and that is called the spectral norm and if $p=2$ we denote the norm by $\|A\|_2$ and it is called Hilbert-Schmidt norm. It is known that for $A \in B(H)$, the numerical radius of A is given by

$$w(A)=\sup_{\theta\in\mathbb{R}}\left\|Re\left(e^{i\theta}A\right)\right\|.$$

The generalization of the numerical radius is defined by

$$w_N(A)=\sup_{\theta\in\mathbb{R}}N\left(Re\left(e^{i\theta}A\right)\right),$$

for every $A \in B(H)$, where $N(.)$ is a norm in $B(H)$. In particular, if $N=\|.\|_2$, then $w_2(A)=\sup_{\theta\in\mathbb{R}}\left\|Re\left(e^{i\theta}A\right)\right\|_2$ is called the Hilbert-Schmidt numerical radius. If A is self-adjoint, then $w_N(A)=N(A)$. It is known that the Hilbert Schmidt numerical radius and, in particular, the numerical radius define norms on $B(H)$.

For more details and proofs about numerical radius, we refer the reader to [7], [8], [9], and [11].

In [1], Aldalabih and Kittaneh have given a refinement of the triangle inequality for the Hilbert-Schmidt norm:

$$\|B+C\|_2 \le \sqrt{2}w_2\begin{bmatrix} 0 & B \\ C^* & 0 \end{bmatrix} \le \|B\|_2+\|C\|_2. \tag{1.1}$$

The authors in [2] have provided a refinement of triangle inequality for the Schatten $p-$norms:

If $B, C \in C_p$, $p \ge 1$, then

$$\|B+C\|_p \le 2^{1-\frac{1}{p}}w_p\begin{bmatrix} 0 & B \\ C^* & 0 \end{bmatrix} \le \|B\|_p+\|C\|_p. \tag{1.2}$$

In particular, when $p=2$ the inequality (1.1) follows.

The authors in [1] have been provided the following properties for the Hilbert-Schmidt numerical radius of 2×2 operator matrices:

1. Let $A, B, C, D \in C_2$. Then

$$w_2\begin{bmatrix} A & B \\ C & D \end{bmatrix} \le \sqrt{w_2^2(A)+w_2^2(D)+\|B\|_2^2+\|C\|_2^2}. \tag{1.3}$$

2. Let $A, B, C, D \in C_2$. Then

$$w_2 \begin{bmatrix} A & B \\ C & D \end{bmatrix} \geq w_2 \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}, \tag{1.4}$$

and

$$w_2 \begin{bmatrix} A & B \\ C & D \end{bmatrix} \geq w_2 \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}. \tag{1.5}$$

3. Let $A, D \in C_2$. Then

$$w_2 \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} \leq \sqrt{w_2^2(A) + w_2^2(D)}, \tag{1.6}$$

$$w_2 \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} \leq \frac{w_2(B+C) + w_2(B-C)}{\sqrt{2}}, \tag{1.7}$$

and

$$w_2 \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} = \sqrt{w_2^2(A) + \frac{1}{2}\|B\|_2^2}. \tag{1.8}$$

We provide several new Hilbert-Schmidt numerical radius inequalities for 2×2 operator matrices.

2 Main Results

We begin our study by the following Hilbert-Schmidt numerical radius inequality for 2×2 operator matrices.

Theorem 2.1. *Let $A, B, C \in C_2$. Then*

$$w_2 \begin{bmatrix} A & B \\ B & D \end{bmatrix} \leq \sqrt{w_2^2(A) + w_2^2(D)} + \sqrt{2} w_2(B). \tag{2.9}$$

Proof. Throughout the proof of this theorem, let

$$K = \begin{bmatrix} Re\left(e^{i\theta}A\right) & 0 \\ 0 & Re\left(e^{i\theta}D\right) \end{bmatrix}, L = \begin{bmatrix} 0 & Re\left(e^{i\theta}B\right) \\ Re\left(e^{i\theta}B\right) & 0 \end{bmatrix},$$

$$X = \left[\sqrt{\|Re\,(e^{i\theta}A)\|_2^2 + \|Re\,(e^{i\theta}D)\|_2^2} + \sqrt{2}\left\|\left(Re\left(e^{i\theta}B\right)\right)\right\|_2\right],$$

$$Y = \sup_{\theta\in\mathbb{R}} \left[\sqrt{\|Re\,(e^{i\theta}A)\|_2^2 + \|Re\,(e^{i\theta}D)\|_2^2} \right],$$

and

$$Z = \sqrt{\sup_{\theta\in\mathbb{R}} \|Re\,(e^{i\theta}A)\|_2^2 + \sup_{\theta\in\mathbb{R}} \|Re\,(e^{i\theta}D)\|_2^2}.$$

Now,

$$\begin{aligned}
\left\| Re e^{i\theta} \begin{bmatrix} A & B \\ B & D \end{bmatrix} \right\|_2 &= \frac{1}{2} \left\| \begin{bmatrix} e^{i\theta}A & e^{i\theta}B \\ e^{i\theta}B & e^{i\theta}D \end{bmatrix} + \begin{bmatrix} e^{-i\theta}A^* & e^{-i\theta}B^* \\ e^{-i\theta}B^* & e^{-i\theta}D^* \end{bmatrix} \right\|_2 \\
&= \frac{1}{2} \left\| \begin{bmatrix} e^{i\theta}A + e^{-i\theta}A^* & e^{i\theta}B + e^{-i\theta}B^* \\ e^{i\theta}B + e^{-i\theta}B^* & e^{i\theta}D + e^{-i\theta}D^* \end{bmatrix} \right\|_2 \\
&= \left\| \begin{bmatrix} Re\left(e^{i\theta}A\right) & Re\left(e^{i\theta}B\right) \\ Re\left(e^{i\theta}B\right) & Re\left(e^{i\theta}D\right) \end{bmatrix} \right\|_2 \\
&= \|K + L\|_2 \\
&\le \|K\|_2 + \|L\|_2 \\
&= X.
\end{aligned}$$

Thus,

$$\begin{aligned}
w_2 \begin{bmatrix} A & B \\ B & D \end{bmatrix} &= \sup_{\theta\in\mathbb{R}} \left\| Re e^{i\theta} \begin{bmatrix} A & B \\ B & D \end{bmatrix} \right\|_2 \\
&= \sup_{\theta\in\mathbb{R}} [X] \\
&\le Y + \sup_{\theta\in\mathbb{R}} \left[\sqrt{2} \left\| \left(Re\left(e^{i\theta}B\right)\right) \right\|_2 \right] \\
&\le Z + \sup_{\theta\in\mathbb{R}} \left[\sqrt{2} \left\| \left(Re\left(e^{i\theta}B\right)\right) \right\|_2 \right] \\
&= \sqrt{w_2^2(A) + w_2^2(D)} + \sqrt{2}w_2(B).
\end{aligned}$$

□

The following result is Hilbert-Scmidt numerical radius inequality for 2×2 operator matrices of the form $\begin{bmatrix} A & C \\ -C & -A \end{bmatrix}$. Numerical radius inequalities for 2×2 operator matrices for this form of matrices have been given in [10].

Theorem 2.2. *Let* $A, C \in C_2$. *Then*

$$w_2 \left(\begin{bmatrix} A & C \\ -C & -A \end{bmatrix} \right) \le \sqrt{2} \left(w_2(A) + w_2(C) \right). \tag{2.10}$$

Proof.

$$\begin{aligned} w_2\begin{bmatrix} A & C \\ -C & -A \end{bmatrix} &= w_2\left(\begin{bmatrix} A & 0 \\ 0 & -A \end{bmatrix} + \begin{bmatrix} 0 & C \\ -C & 0 \end{bmatrix}\right) \\ &\leq w_2\begin{bmatrix} A & 0 \\ 0 & -A \end{bmatrix} + w_2\begin{bmatrix} 0 & C \\ -C & 0 \end{bmatrix} \\ &\leq \sqrt{2w_2^2(A)} + \sqrt{2}w_2(C) \\ &= \sqrt{2}w_2(A) + \sqrt{2}w_2(C) = \sqrt{2}\left(w_2(A) + w_2(C)\right). \end{aligned}$$

□

By making use of inequalities (1.4) and (1.5), we provide the following finding.

Theorem 2.3. *Let $A, C \in C_2$. Then*

$$w_2\left(\begin{bmatrix} A & C \\ -C & -A \end{bmatrix}\right) \geq \max(\sqrt{2}w_2(A), \sqrt{2}w_2(C)). \tag{2.11}$$

Proof. Applying inequalities (1.4) and (1.5), we get inequality (2.11). □

Remark 2.4. *Combining inequalities (2.10) and (2.11) leads to the following inequality.*

$$\max(\sqrt{2}w_2(A), \sqrt{2}w_2(C)) \leq w_2\left(\begin{bmatrix} A & C \\ -C & -A \end{bmatrix}\right) \leq \sqrt{2}\left(w_2(A) + w_2(C)\right). \tag{2.12}$$

Corollary 2.5. *Let $A \in C_2$. Then*

$$w_2(A) \leq \frac{1}{\sqrt{2}}w_2\left(\begin{bmatrix} A & A \\ -A & -A \end{bmatrix}\right) \leq 2w_2(A). \tag{2.13}$$

Proof. Replacing C by A in inequality (2.12) gives inequality (2.13). □

Using inequalities (1.6) and (1.7) we get the following result.

Theorem 2.6. *Let $A, B, C, D \in C_2$. Then*

$$w_2\begin{bmatrix} A & B \\ C & D \end{bmatrix} \leq \sqrt{w_2^2(A) + w_2^2(D)} + \frac{w_2(B+C) + w_2(B-C)}{\sqrt{2}}. \tag{2.14}$$

Proof.

$$\begin{aligned} w_2 \begin{bmatrix} A & B \\ C & D \end{bmatrix} &= w_2 \left(\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} + \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} \right) \\ &\leq w_2 \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} + w_2 \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} \\ &\leq \sqrt{w_2^2(A) + w_2^2(D)} + \frac{w_2(B+C) + w_2(B-C)}{\sqrt{2}}. \end{aligned}$$

which is inequality (2.14). □

To reach our next finding, we need the following two lemmas, the first one is well-known, the second one can be given by using the same procedure used to prove Theorem 1(b) in [1].

Lemma 2.7. *Let $A \in C_2$. Then*

$$\frac{1}{\sqrt{2}} \|A\|_2 \leq w_2(A) \leq \|A\|_2 . \tag{2.15}$$

Lemma 2.8. *Let $A, B \in C_2$. Then*

$$w_2 \begin{bmatrix} 0 & 0 \\ C & D \end{bmatrix} = \sqrt{w_2^2(D) + \frac{1}{2} \|C\|_2^2}. \tag{2.16}$$

The following theorem is an application of inequalities (1.8), (2.15) and (2.16).

Theorem 2.9. *Let $A, B, C, D \in C_2$. Then*

$$w_2 \begin{bmatrix} A & B \\ C & D \end{bmatrix} \leq w_2(A) + w_2(B) + w_2(D) + w_2(C). \tag{2.17}$$

Proof.

$$\begin{aligned} w_2 \begin{bmatrix} A & B \\ C & D \end{bmatrix} &= w_2 \left(\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ C & D \end{bmatrix} \right) \\ &\leq w_2 \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} + w_2 \begin{bmatrix} 0 & 0 \\ C & D \end{bmatrix} \\ &= \sqrt{w_2^2(A) + \frac{1}{2} \|B\|_2^2} + \sqrt{w_2^2(D) + \frac{1}{2} \|C\|_2^2}, \\ &\quad \text{(by applying inequalities (1.8) and (2.16)).} \\ &\leq w_2(A) + \frac{1}{\sqrt{2}} \|B\|_2 + w_2(D) + \frac{1}{\sqrt{2}} \|C\|_2 \\ &\leq w_2(A) + w_2(B) + w_2(D) + w_2(C). \\ &\quad \text{(by making use of inequality (2.15)).} \end{aligned}$$

□

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