International Journal of Mathematics and
Computer Science, **16**(2021), no. 4, 1831–1838

A Classification of Totally Umbilical Semi-invariant Submanifolds of Riemannian Product Manifolds

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(Received March 3, 2021, Accepted June 19, 2021)

Abstract

This paper is dedicated to the study of totally umbilical semi-invariant submanifolds in the setting of Riemannian product manifolds. Moreover, by using curvature tensor, we obtain a classification for these submanifolds.

1 Introduction

In 1978, Bejancu introduced the notion of CR-submanifolds of a Kaehler manifold [1] which generalizes the holomorphic and totally real submanifolds. In fact, this provided a single setting to study the holomorphic and totally real submanifolds. Later, Chen [3] studied CR-submanifolds in the setting of Kaehler manifolds. The geometric point of view of CR-submanifolds was taken up by a number of differential geometrs and several papers have appeared in the literature (for instance, [7], [8], [9]). However, the contact version of the CR-submanifolds was explored by Bejancu and Papaghuic [2]; more precisely, they studied contact CR-submanifolds with the name of Semi-invariant submanifolds in the frame of Sasakian manifolds. Moreov,er

Key words and phrases: Semi-invariant submanifolds, totally umbilical submanifolds, Riemannian product manifolds.
AMS (MOS) Subject Classifications: 53C25, 53C40, 53C42.
ISSN 1814-0432, 2021, http://ijmcs.future-in-tech.net

Chen [4] classified totally umbilical CR-submanifolds of the Kaehler manifolds. Furthermore, Deshmukh and Husain [9] studied totally umbilical CR-submanifolds of a Kaehler manifold. In 2003, Sahin [6] studied Semi-invariant submanifolds of Riemannian product manifolds. Later, Atceken [11] extended the study of Semi-invariant submanifolds in the setting of Riemannian product manifolds. In the present paper, our aim is to study the totally umbilical Semi-invariant submanifolds of the Riemannian manifolds. In fact, we obtain a classification for these submanifolds.

2 Preliminaries

Let (M_1, g_1) and (M_2, g_2) be the Riemannian manifolds with dimensions m_1 and m_2 respectively and let $M_1 \times M_2$ be the Riemannian product manifold of M_1 and M_2. We denote the projection mapping of $T(M_1 \times M_2)$ onto TM_1 and TM_2 by $\sigma_\star$ and $\pi_\star$ respectively. Then we have $\sigma_\star + \pi_\star = I$, $\sigma_\star^2 = \sigma_\star$ $\pi_\star^2 = \pi_\star$ and $\sigma_\star \circ \pi_\star = \pi_\star \circ \sigma_\star = 0$, with $\star$ denoting mean derivatives.

The Riemannian metric of the Riemannian product manifold $M = M_1 \times M_2$ is defined by

$$g(X, Y) = g_1(\sigma_\star X, \sigma_\star Y) + g_2(\pi_\star X, \pi_\star Y),$$

for any $X, Y \in T\bar{M}$. If we set $F = \sigma_\star - \pi_\star$, then $F^2 = I$, $F \neq I$ and g satisfies the condition

$$g(FX, Y) = g(X, FY),$$

for any $X, Y \in T\bar{M}$. Thus F defines an almost Riemannian product structure on $\bar{M}$. We denote the Levi-Civita connection on $\bar{M}$ by $\bar{\nabla}$. Then the covariant derivative of F is defined as

$$(\bar{\nabla}_X F)Y = \bar{\nabla}_X FY - F\bar{\nabla}_X Y,$$

for any $X, Y \in TM$. We say that F is parallel with respect to the connection $\bar{\nabla}$ if we have $(\bar{\nabla}_X F)Y = 0$. From [10], we know that F is parallel; that is, F is Riemannian product structure.

Let $\bar{M}$ be a Riemannian product manifold with Riemannian product structure F and M be an immersed submanifold of $\bar{M}$. We also let g be the induced metric tensor on M as well as on $\bar{M}$. If $\bar{\nabla}$ is the Levi-civita connection on $\bar{M}$, then the Gauss and Weingarten formulas are given, respectively, as

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y) \tag{2.1}$$

$$\bar{\nabla}_X V = -A_V X + \nabla^{\perp}_X V, \tag{2.2}$$

for any $X, Y \in TM$ and $V \in T^{\perp}M$, where ∇ is the connection on M and $\nabla^{\perp}$ is the connection in the normal bundle, h is the second fundamental form of M and A_V the shape operator of M. The second fundamental form h and the shape operator A_V are related by

$$g(A_V X, Y) = g(h(X, Y), V). \tag{2.3}$$

For any $X \in TM$, we can write

$$FX = fX + \omega X, \tag{2.4}$$

where fX and ωX are the tangential and normal components of FX, respectively and for $V \in T^{\perp}M$

$$FV = tV + nV, \tag{2.5}$$

where tV and nV are the tangential and normal components of FV. The submanifold M is said to be invariant if ω is identically zero. On the other hand, M is said to be an anti-invariant submanifold if f is identically zero.

The covariant derivatives of f, ω, t and n are defined as

$$(\bar{\nabla}_X f)Y = \nabla_X fY - f\nabla_X Y \tag{2.6}$$

$$(\bar{\nabla}_X \omega)Y = \nabla^{\perp}_X \omega Y - \omega\nabla_X Y \tag{2.7}$$

$$(\bar{\nabla}_X t)V = \nabla_X tV - t\nabla_X V \tag{2.8}$$

$$(\bar{\nabla}_X n)V = \nabla^{\perp}_X nV - n\nabla^{\perp}_X Y. \tag{2.9}$$

Using (2.1),(2.2) (2.4) and (2.6), we get

$$(\bar{\nabla}_X f)Y = A_{\omega Y} X + th(X, Y). \tag{2.10}$$

Now, we define the Semi-invariant submanifold of a Riemannian product manifold $\bar{M}$ as follows:

Definition 2.1.[6] A Submanifold M of a Riemannian product manifold $\bar{M}$ is said to be semi-invariant submanifold if there exist two orthogonal complimentary distributions D and $D^{\perp}$ satisfying

(i) $TM = D \oplus D^{\perp}$

(ii) D is a invariant distribution; i.e., $FD \subseteq D$.

(iii) $D^{\perp}$ is anti-invariant distribution; i.e., $FD^{\perp} \subseteq T^{\perp}M$.

If μ is invariant subspace under F of the normal bundle $T^{\perp}M$, then, in the case of semi-invariant submanifold, the normal bundle $T^{\perp}M$ can be decomposed as

$$T^{\perp}M = \mu \oplus \omega D^{\perp}.$$

A semi-invariant submanifold M is said to be totally umbilical if

$$h(X,Y) = g(X,Y)H, \tag{2.11}$$

where $H = \frac{1}{n}(\text{trace } h)$ is called the mean curvature vector. For the totally umbilical semi-invariant submanifold M, the equations (2.1)and (2.2) take the form

$$\bar{\nabla}_X Y = \nabla_X Y + g(X,Y)H, \tag{2.12}$$

$$\bar{\nabla}_X V = -g(H,V)X + \nabla^{\perp}_X V. \tag{2.13}$$

The Riemannian curvature tensor is defined as

$$R(X,Y)Z = \bar{\nabla}_X \bar{\nabla}_Y Z - \bar{\nabla}_Y \bar{\nabla}_X Z - \bar{\nabla}_{[X,Y]} Z. \tag{2.14}$$

The equation of Coddazi for totally umbilical semi-invariant submanifold M is given by

$$\bar{R}(X,Y,Z,V) = g(Y,Z)g(\nabla^{\perp}_X H, V) - g(X,Z)g(\nabla^{\perp}_Y H, V), \tag{2.15}$$

where $\bar{R}(X,Y,Z,V) = g(\bar{R}(X,Y)Z,V)$ and X, Y, Z are vector fields on M and $V \in T^{\perp}M$.

It is easy to see that the Riemannian curvature tensor for Riemannian product manifolds satisfies the following properties

$$(a)\bar{R}(FX,FY)Z = \bar{R}(X,Y)Z \qquad (b)F\bar{R}(X,Y)Z = \bar{R}(X,Y)FZ. \tag{2.16}$$

By an extrinsic sphere we mean a submanifold of an arbitrary Riemannian manifold which is totally umbilical and has a nonzero parallel mean curvature vector [9].

3 Totally Umbilical semi-invariant submanifolds of Riemannian product manifolds

In this section, we will study a special class of semi-invariant submanifolds which are totally umbilical. Throughout the section, we consider M as a totally umbilical semi-invariant submanifold of a Riemannian product manifold. Now we have the following theorem:

Theorem 3.1. Let M be a totally umbilical semi-invariant submanifold of a Riemannian product manifold $\bar{M}$ such that the mean curvature vector $H \in \mu$. Then one of the following statement is true

(i) M is totally geodesic,

(ii) M is anti-invariant.

Proof. For $N \in FD^{\perp}$ and $X \in D$, we have

$$\bar{\nabla}_X FN = F\bar{\nabla}_X N. \tag{3.17}$$

Using equations (2.12) and (2.13) the above equation becomes

$$\nabla_X FN = -FXg(N, H) + F\nabla^{\perp}_X N.$$

By the assumption that $H \in \mu$, the above equation reduces to

$$\nabla_X FN = F\nabla^{\perp}_X N,$$

which implies that $\nabla^{\perp}_X N \in FD^{\perp}$, for any $N \in FD^{\perp}$. Also, we have $g(N, H) = 0$ for any $N \in FD^{\perp}$. Then

$$g(\nabla^{\perp}_X N, H) = -g(N, \nabla^{\perp}_X H) = 0.$$

The above equation gives $\nabla^{\perp}_X H \in \mu$.

Now, for any $X \in D$, we have

$$\bar{\nabla}_X FH = F\bar{\nabla}_X H.$$

Using Gauss and Weingarten formulae and the assumption that M is a totally umbilical, the above equation takes the form

$$Xg(H, FH) + \nabla^{\perp}_X FH = -FXg(H, H) + F\nabla^{\perp}_X H,$$

or

$$\nabla^{\perp}_X FH = -FXg(H,H) + F\nabla^{\perp}_X H.$$

Comparing tangential parts

$$fXg(H,H) = 0.$$

From the above equation, it is evident that $fX = 0$ or $H = 0$; i.e., M is either totally geodesic or anti-invariant.

Now, for any $Z \in D^{\perp}$, using equation (2.10)

$$-f\nabla_Z Z = A_{\omega Z}Z + th(Z,Z).$$

Taking the inner product with $W \in D^{\perp}$, the above equation takes the form

$$-g(f\nabla_Z Z, W) = g(A_{\omega Z}Z, W) + g(th(Z,Z), W).$$

As M is a totally umbilical semi-invariant submanifold, the above equation becomes

$$g(Z,W)g(H,FZ) + g(tH,W)\|Z\|^2 = 0, \tag{3.18}$$

which has a solution if either $H \in \mu$ or $\dim D^{\perp} = 1$.

If, in addition, $H \notin \mu$, then we have

Theorem 3.2. Let M be a totally umbilical semi-invariant submanifold of a Riemannian product manifold $\bar{M}$ such that the dimension of invariant distribution ≥ 4. Then M is an extrinsic sphere.

Proof. Since $\dim D \geq 4$, we can choose a set of orthogonal vectors $X, Y \in D$ such that $g(X,Y) = 0$ and $g(X,FY) = 0$. Now, from equation (2.16)(b), we have

$$F\bar{R}(X,Y)Z = \bar{R}(X,Y)FZ.$$

Replacing Z by FY

$$F\bar{R}(X,Y)FY = \bar{R}(X,Y)F^2Y.$$

$$F\bar{R}(X,Y)FY = \bar{R}(X,Y)Y.$$

Taking the inner product with $V \in T^{\perp}M$, we get

$$g(F\bar{R}(X,Y)FY, V) = g(\bar{R}(X,Y)Y, V),$$

or

$$\bar{R}(X, Y, FY, nV) = \bar{R}(X, Y, Y, V).$$

Now, from Coddazi equation

$$g(Y, FY)g(\nabla^{\perp}_{X}H, V) - g(X, FY)g(\nabla^{\perp}_{Y}H, V) = g(Y, Y)g(\nabla^{\perp}_{X}H, V) - g(X, Y)g(\nabla^{\perp}_{Y}H, V),$$

or

$$\|Y\|^2 g(\nabla^{\perp}_{X}H, V).$$

From the above equation it is evident that $g(\nabla^{\perp}_{X}H, V) = 0$; i.e., $\nabla^{\perp}_{X}H = 0$ $\forall X \in D$. If we take $X \in D^{\perp}$ and $Y \in D$, then, by similar calculation, we obtain $\nabla^{\perp}_{X}H = 0$, $\forall X \in D^{\perp}$. Hence, the submanifold is an extrinsic sphere.

Theorem 3.3. Let M be a totally umbilical semi-invariant submanifold of a Riemannian product manifold $\bar{M}$. Then M is either

(i) totally geodesic,

(ii) anti-invariant,

(iii) dim $D^{\perp} = 1$,
or

(iv) an Extrinsic sphere.

(Case (iv) holds if dim $M \geq 5$.)

Proof. If $H \in \mu$, then, by Theorem 3.1, M is either totally geodesic or anti-invariant which is case (i) or case (ii). If $H \notin \mu$, then equation (3.18) has a solution if dim $D^{\perp} = 1$ which is case (iii). Moreover, if $H \notin \mu$ and the dimension D of the invariant distribution is at least 4, then, by Theorem 3.2, M is an extrinsic sphere. The proof of the theorem is now complete.

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